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CYCLIC ORTHOGONAL DOUBLE COVERS OF CIRCULANTS BY DISJOINT UNIONS OF ONE CATERPILLAR AND CERTAIN NERVE CELL GRAPHS

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Abstract. In this paper, we present the definition of a nerve cell graph. We construct cyclic orthogonal double covers of some circulant graphs by the disjoint union of a caterpillar and certain nerve cell graphs.

1. Introduction

It is known that graph theory has become topic of interest for many areas of science such as biology, chemistry, physics, operations research, engineering, economics, communication, and especially computer science.

A graph labelling is an assignment of integers to the edges, vertices, or both, subject to certain conditions. There are many kinds of labelling such as graceful labelling, harmonious labelling, cordial labelling, and orthogonal labelling. For a survey on orthogonal labelling, see [1, 2, 3].

Suppose that $G = (V, E)$ is a graph, where V is the vertex set and E the edge set. An orthogonal double cover (ODC) of H by G is a collection $\mathcal{G} = \{G_v : v \in V\}$ of subgraphs of H, all isomorphic to G, such that (i) every edge in H occurs in exactly two members of $\mathcal{G}$ and (ii) if α and β are adjacent vertices in H, then G_α and G_β share one edge.

The existence problem of ODCs is known to be difficult in general as it includes some long-standing open problems such as the existence problems of biplanes. Also, the ODC problem originally stems from problems in database optimization, statistical design, and design theory.

The circulant graphs are Cayley graphs on cyclic groups. The circulant graph $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_k\})$ has vertex set $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ where $\{l_1, l_2, \ldots, l_k\}$ is a sequence of integers with $1 \le l_1 < l_2 < \ldots < l_k \le \lfloor m/2 \rfloor$ and the two vertices a and b are adjacent in $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_k\})$ when $a - b \equiv \pm l_i \pmod{m}; i \in \{1, 2, \ldots k\}$. For an edge (a, b) in $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_k\})$, the length of this edge is $\min\{|a-b|, m - |a-b|\}$. For two edges $E_1 = (a_1, b_1)$ and $E_2 = (a_2, b_2)$ with the same length l in $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_k\})$, the *rotation-distance* $r(l)$ between E_1 and E_2 is $r(l) = \min\{r_1, r_2 : ((a_1 +$

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$r_1), (b_1+r_1)) = E_2, ((a_2+r_2), (b_2+r_2)) = E_1\}$, where addition is calculated modulo m. Edges E_1 and E_2 of the same length l are adjacent if $r(l) = l$.

Orthogonal labelling is introduced by Gronau et al. [2]. Given a graph $G = (V, E)$ with $m-1$ edges, a 1–1 mapping $\Psi : V(G) \longrightarrow \mathbb{Z}_m$ is an orthogonal labelling of G if (i) G contains exactly two edges of length $l \in \{1, 2, \ldots, \lfloor (m-1)/2 \rfloor\}$, and exactly one edge of length $m/2$ if m is even, and (ii) $\{r(l) : l \in \{1, 2, \ldots, \lfloor (m-1)/2 \rfloor\}\} = \{1, 2, \ldots, \lfloor (m-1)/2 \rfloor\}$.

Gronau et al. [2] relates CODCs of K_m and the orthogonal labelling by the following theorem.

Theorem 1. *A CODC of K_m by a graph G exists if and only if there exists an orthogonal labelling of G.*

Sampathkumar et al. [3] called an orthogonal labelling an orthogonal $\{1, 2, ..., \lfloor m/2 \rfloor\}$-labelling and generalized it to an orthogonal $\{l_1, l_2, ..., l_k\}$-labelling, where $\{l_1, l_2, ..., l_k\}$ is a sequence of integers with $1 \leq l_1 < l_2 < ... < l_k \leq \lfloor m/2 \rfloor$.

Case 1: Either m is odd or m is even and $l_k \neq m/2$.

Given a subgraph G of $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_k\})$ with $2k$ edges, a labelling of G, in $\mathbb{Z}_m$, is an orthogonal $\{l_1, l_2, \ldots, l_k\}$-labelling of G if: (i) for every $l \in \{l_1, l_2, \ldots, l_k\}$, G contains exactly two edges of length l, and (ii) $\{r(l) : l \in \{l_1, l_2, \ldots, l_k\}\} = \{l_1, l_2, \ldots, l_k\}$.

Case 2: m is even and $l_k = m/2$.

Given a subgraph G of $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_{k-1}, m/2\})$ with $2k-1$ edges, a labelling of G, in $\mathbb{Z}_m$, is an orthogonal $\{l_1, l_2, \ldots, l_{k-1}, m/2\}$-labelling of G if: (i) for every $l \in \{l_1, l_2, \ldots, l_{k-1}\}$, G contains exactly two edges of length l, and G contains exactly one edge of length $m/2$, and (ii) $\{r(l) : l \in \{l_1, l_2, \ldots, l_{k-1}\}\} = \{l_1, l_2, \ldots, l_{k-1}\}$.

The following theorem of Sampathkumar et al. [3] is a generalization of Theorem 1.

Theorem 2. *A CODC of* $\mathrm{Circ}(m; \{l_1, l_2, \ldots, l_k\})$ *by a graph G exists if and only if there exists an orthogonal $\{l_1, l_2, \ldots, l_k\}$-labelling of G.*

For a survey on the CODC of the circulant graphs, see [1, 4, 5]. We use the following notation: C_m to refer to a cycle with m edges, $K_{m,n}$ to a complete bipartite graph with partition sets of sizes m and n, K_1 to an isolated vertex, and $G_1 \cup G_2$ to the disjoint union of G_1 and G_2. Let $k \geq 1, n_1, n_2, \ldots, n_k$ be integers, $n_1, n_k \geq 1$ and $n_i \geq 0$ for $i \in \{2, 3, \ldots, k-1\}$, if we have the path $P_k := x_1 x_2 \ldots x_k$ and by connecting vertex x_i to n_i new vertices and $i \in \{1, 2, 3, \ldots, k\}$, then the resulting graph is called the caterpillar denoted by $C_k(n_1, n_2, \ldots, n_k)$.

Definition 3. *Let G be a tree with a unique vertex of highest degree, called the nerve vertex. The nerve cell graph $C_m \boxdot G$ is the graph obtained as a union of C_m with m copies of G, by collapsing each vertex of C_m with the nerve vertex of G.*

The main result of this paper can be stated as follows:

Theorem 4. *Let n be an integer, divisible by a prime $p > 3$ such that $n > 2p$. Then there is a CODC of* $\mathrm{Circ}(n; \{1, \dots, [n/2]\})$ *formed by the union of a caterpillar and one or two nerve cell graphs.*

Remark 5. *In Section 2, Theorem 2 will be applied to prove the above result (as a corollary of Theorem 7, together with similar constructions under stronger conditions (Theorems 6 and 8).*

2. Main Results

In Case 1 of Theorem 6, we construct a CODC of $\mathrm{Circ}(7^k n; \{1, \dots, \lfloor 7^k n/2 \rfloor\})$ by $G_1^{n,k} \cong C_3(1, 0, n-4) \cup 2(C_3 \boxdot C_3(1, 0, n-4)) \cup 2(C_{\frac{7^k-7}{2}} \boxdot C_3(1, 0, n-4))$, where n, k are integers with $n \geq 5$, the graph $G_1^{n,k}$ is shown in Figure 1. For the graph $G_1^{n,k}$ if we put $k = 1$, then we get the graph $G_1^{n,1}$ shown in Figure 2. In Case 2 of Theorem 6, we construct a CODC of $\mathrm{Circ}(mn; \{1, \dots, \lfloor mn/2 \rfloor\})$ by $G_2^{m,n}$, where n, k and m are integers with $m \equiv 1, 5 \pmod 6), m \neq 7^k$ and $n \geq 5$, the graph $G_2^{m,n}$ is shown in Figure 3.

Theorem 6. *Let n, k and m be integers with $m \equiv 1, 5 \pmod 6$ and $n \geq 5$. Then a CODC of* $\mathrm{Circ}(mn; \{1, \dots, \lfloor mn/2 \rfloor\})$ *can be obtained by*

$$G_1^{n,k} \cong C_3(1, 0, n-4) \cup 2(C_3 \boxdot C_3(1, 0, n-4)) \cup 2(C_{\frac{7^k-7}{2}} \boxdot C_3(1, 0, n-4))$$

if $m = 7^k$, and by

$$G_2^{m,n} \cong C_3(1, 0, n-4) \cup (C_{m-1} \boxdot C_3(1, 0, n-4))$$

if $m \neq 7^k$.

Proof.

Case 1: Let us define $\Psi : V(G_1^{n,k}) \to \mathbb{Z}_{7^k n}$ by $\Psi(v_i) = i; i \in \mathbb{Z}_{7^k n}, E(G_1^{n,k}) = \{(\Psi(v_{(\alpha+1)n+1}), \Psi(v_{2(\alpha+1)n-1})), (\Psi(v_{(\alpha+2)n-1}), \Psi(v_{(2\alpha+3)n-2})), (\Psi(v_{\alpha n}), \Psi(v_{2\alpha n-\beta})); \alpha \in \mathbb{Z}_{7^k}, \beta \in \mathbb{Z}_n \backslash \{0, 1, 2\}\} \cup \{(\Psi(v_{\alpha n}), \Psi(v_{2\alpha n})); \alpha \in \mathbb{Z}_{7^k} \backslash \{0\}\}$

Subcase 1.1: $n = 5$. From the edge set of $G_1^{5,k}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \dots, \lfloor 5 * 7^k/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \dots, \lfloor 5 * 7^k/2 \rfloor\}\} = \{1, \dots, \lfloor 5 * 7^k/2 \rfloor\}$ which can be proved as follows, $r(5\alpha) = 15\alpha; 1 \leq \alpha \leq \frac{7^k-1}{2}$, $r(5\alpha + 3) = 15\alpha + 9; \alpha \in \mathbb{Z}_{7^k}$, $r(5\alpha + 4) = 15\alpha + 13; \alpha \in \mathbb{Z}_{7^k}$.

Subcase 1.2: $n = 6$. From the edge set of $G_1^{6,k}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \dots, \lfloor \frac{(6*7^k)-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $6 * 7^k/2$, (ii) $\{r(l) : l \in \{1, \dots, \lfloor 6 * 7^k/2 \rfloor\}\} = \{1, \dots, \lfloor 6 * 7^k/2 \rfloor\}$ which can be proved as follows, $r(6\alpha) = 18\alpha; 1 \leq \alpha \leq \frac{7^k-1}{2}$, $r(6\alpha + 4) = 18\alpha + 11; \alpha \in \mathbb{Z}_{7^k}$, $r(6\alpha + 5) = 18\alpha + 16; \alpha \in \mathbb{Z}_{7^k}, r(6 * 7^k/2) = 6 * 7^k/2, r(6\alpha - 3)) = 18\alpha - 9; 1 \leq \alpha \leq \frac{7^k-1}{2}$.

Subcase 1.3: $n > 5$ is odd. From the edge set of $G_1^{n,k}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor 7^k n/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor 7^k n/2 \rfloor\}\} = \{1, \ldots, \lfloor 7^k n/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{7^k-1}{2}$, $r((\alpha+1)n-2) = (3(\alpha+1)-1)n-1; \alpha \in \mathbb{Z}_{7^k}$, $r((\alpha+1)n-1) = 3(\alpha+1)n-2; \alpha \in \mathbb{Z}_{7^k}, r(\alpha) = n+\alpha; \alpha \in \{3, 4, \ldots, \frac{n-1}{2}\}$, $r(n-\alpha) = 2n-\alpha; \alpha \in \{3, 4, \ldots, \frac{n-1}{2}\}, r((\alpha+1)n+\gamma) = (3(\alpha+1)+1)n+\gamma; \alpha \in \mathbb{Z}_{7^k-2}, \gamma \in \{3, 4, \ldots, \frac{n-1}{2}\}$.

Subcase 1.4: $n > 6$ is even. From the edge set of $G_1^{n,k}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor \frac{7^k n-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $7^k n/2$, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor 7^k n/2 \rfloor\}\} = \{1, \ldots, \lfloor 7^k n/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{7^k-1}{2}, r((\alpha+1)n-2) = (3(\alpha+1)-1)n-1; \alpha \in \mathbb{Z}_{7^k}$, $r((\alpha+1)n-1) = 3(\alpha+1)n-2; \alpha \in \mathbb{Z}_{7^k}, r(\alpha) = n+\alpha; \alpha \in \{3, 4, \ldots, \frac{n-2}{2}\}$, $r(n-\alpha) = 2n-\alpha; \alpha \in \{3, 4, \ldots, \frac{n-2}{2}\}, r((\alpha+1)n+\gamma) = (3(\alpha+1)+1)n+\gamma; \alpha \in \mathbb{Z}_{7^k-2}, \gamma \in \{3, 4, \ldots, \frac{n-2}{2}\}, r(7^k n/2) = 7^k n/2, r((\alpha - \frac{1}{2})n) = 3(\alpha - \frac{1}{2})n; 1 \leq \alpha \leq \frac{7^k-1}{2}$.

Case 2: Let us define $\Psi : V(G_2^{m,n}) \to \mathbb{Z}_{mn}$ by $\Psi(v_i) = i; i \in \mathbb{Z}_{mn}, E(G_2^{m,n}) = \{(\Psi(v_{(\alpha+1)n+1}), \Psi(v_{2(\alpha+1)n-1})), (\Psi(v_{(\alpha+2)n-1}), \Psi(v_{(2\alpha+3)n-2})), (\Psi(v_{\alpha n}), \Psi(v_{2\alpha n-\beta})); \alpha \in \mathbb{Z}_m, \beta \in \mathbb{Z}_n \backslash \{0, 1, 2\}\} \cup \{(\Psi(v_{\alpha n}), \Psi(v_{2\alpha n})); \alpha \in \mathbb{Z}_m \backslash \{0\}\}$.

Subcase 2.1: $n = 5$. From the edge set of $G_2^{m,5}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor 5m/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor 5m/2 \rfloor\}\} = \{1, \ldots, \lfloor 5m/2 \rfloor\}$ which can be proved as follows $r(5\alpha) = 15\alpha; 1 \leq \alpha \leq \frac{m-1}{2}, r(5\alpha+3) = 15\alpha+9; \alpha \in \mathbb{Z}_m, r(5\alpha+4) = 15\alpha+13; \alpha \in \mathbb{Z}_m$.

Subcase 2.2: $n = 6$. From the edge set of $G_2^{m,6}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots \lfloor \frac{6m-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $6m/2$, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor 6m/2 \rfloor\}\} = \{1, \ldots, \lfloor 6m/2 \rfloor\}$ which can be proved as follows, $r(6\alpha) = 18\alpha; 1 \leq \alpha \leq \frac{m-1}{2}$, $r(6\alpha+4) = 18\alpha+11; \alpha \in \mathbb{Z}_m$, $r(6\alpha+5) = 18\alpha+16; \alpha \in \mathbb{Z}_m, r(3m) = 3m, r(6\alpha-3)) = 18\alpha-9; 1 \leq \alpha \leq \frac{m-1}{2}$.

Subcase 2.3: $n > 5$ is odd. From the edge set of $G_2^{m,n}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor mn/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor mn/2 \rfloor\}\} = \{1, \ldots, \lfloor mn/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{m-1}{2}$, $r((\alpha+1)n-2) = (3(\alpha+1)-1)n-1; \alpha \in \mathbb{Z}_m$, $r((\alpha+1)n-1) = 3(\alpha+1)n-2; \alpha \in \mathbb{Z}_m, r(\alpha) = n+\alpha; \alpha \in \{3, 4, \ldots, \frac{n-1}{2}\}$, $r(n-\alpha) = 2n-\alpha; \alpha \in \{3, 4, \ldots, \frac{n-1}{2}\}, r((\alpha+1)n+\gamma) = (3(\alpha+1)+1)n+\gamma; \alpha \in \mathbb{Z}_{m-2}, \gamma \in \{3, 4, \ldots, \frac{n-1}{2}\}$.

Subcase 2.4: $n > 6$ is even. From the edge set of $G_2^{m,n}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor \frac{mn-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $mn/2$, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor mn/2 \rfloor\}\} = \{1, \ldots, \lfloor mn/2 \rfloor\}$ which can

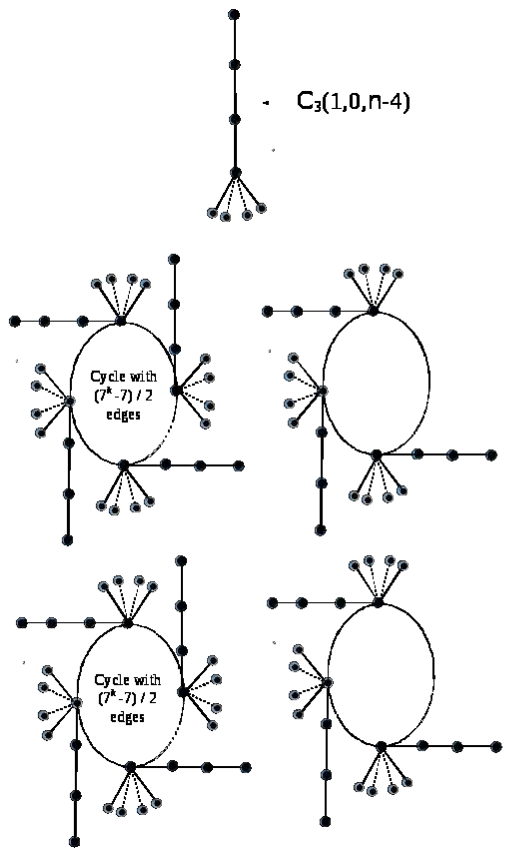

FIGURE 1. The graph $G_1^{n,k}$.

be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{m-1}{2}$, $r((\alpha+1)n-2) = (3(\alpha+1)-1)n-1; \alpha \in \mathbb{Z}_m$, $r((\alpha+1)n-1) = 3(\alpha+1)n-2; \alpha \in \mathbb{Z}_m, r(\alpha) = n+\alpha; \alpha \in \{3,4,\ldots,\frac{n-2}{2}\}$, $r(n-\alpha) = 2n-\alpha; \alpha \in \{3,4,\ldots,\frac{n-2}{2}\}, r((\alpha+1)n+\gamma) = (3(\alpha+1)+1)n+\gamma; \alpha \in \mathbb{Z}_{m-2}, \gamma \in \{3,4,\ldots,\frac{n-2}{2}\}, r(mn/2) = mn/2, r((\alpha-\frac{1}{2})n) = 3(\alpha-\frac{1}{2})n; 1 \leq \alpha \leq \frac{m-1}{2}$. □

For more illustration to Theorem 6,

(1) Let $k=2$ and $n=5$. Then the CODC of Circ$(245;\{1,\ldots,122\})$ can be obtained by $G_1^{5,2} \cong C_3(1,0,1) \cup 2(C_3 \boxdot C_3(1,0,1)) \cup 2(C_{21} \boxdot C_3(1,0,1))$.
(2) Let $k=1$ and $n=5$. Then the CODC of Circ$(35;\{1,\ldots,17\})$ can be obtained by $G_1^{5,1} \cong C_3(1,0,1) \cup 2(C_3 \boxdot C_3(1,0,1))$, see Figure 4.

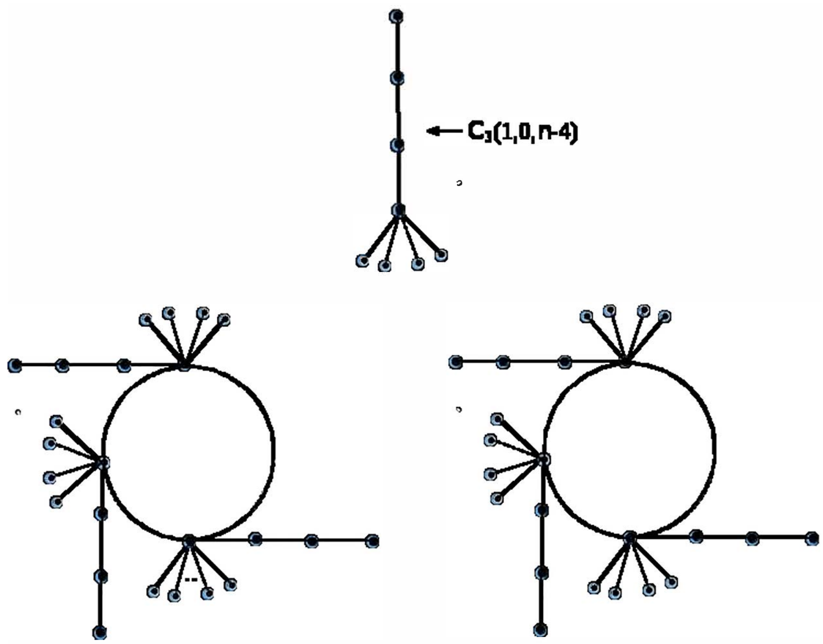

FIGURE 2. The graph $G_1^{n,1}$.

(3) Let $m = 5$ and $n = 5$. Then the CODC of $\mathrm{Circ}(25; \{1, \ldots, 12\})$ can be obtained by $G_2^{5,5} \cong C_3(1, 0, 1) \cup (C_4 \boxdot C_3(1, 0, 1))$, see Figure 5.

In Case 1 of Theorem 7, we construct a CODC of $\mathrm{Circ}(7^k n; \{1, \ldots, \lfloor 7^k n/2 \rfloor\})$ by $G_3^{n,k}$, where n, k are integers with $n > 2$, the graph $G_3^{n,k}$ is shown in Figure 6. For the graph $G_3^{n,k}$ if we set $k = 1$, then we get the graph $G_3^{n,1} \cong K_{1,n-1} \cup 2(C_3 \boxdot K_{1,n-1})$ shown in Figure 7. In Case 2 of Theorem 7, we construct a CODC of $\mathrm{Circ}(mn; \{1, \ldots, \lfloor mn/2 \rfloor\})$ by $G_4^{m,n} \cong K_{1,n-1} \cup (C_{m-1} \boxdot K_{1,n-1})$, where n and m are integers with $m \equiv 1, 5 \pmod 6$, $n > 2$ and $m \neq 7^k$, the graph $G_4^{m,n}$ is shown in Figure 8.

Theorem 7. *Let n, k and m be integers with $m \equiv 1, 5 \pmod 6$ and $n > 2$. Then a CODC of $\mathrm{Circ}(mn; \{1, \ldots, \lfloor mn/2 \rfloor\})$ can be obtained by*

$$G_3^{n,k} \cong K_{1,n-1} \cup 2(C_3 \boxdot K_{1,n-1}) \cup 2(C_{\frac{7^k-7}{2}} \boxdot K_{1,n-1})$$

if $m = 7^k$, and by

$$G_4^{m,n} \cong K_{1,n-1} \cup (C_{m-1} \boxdot K_{1,n-1})$$

if $m \neq 7^k$.

Proof.

Case 1. Let us define $\Psi : V(G_3^{n,k}) \to \mathbb{Z}_{7^k n}$ by $\Psi(v_i) = i$; $i \in \mathbb{Z}_{7^k n}$, $E(G_3^{n,k}) = \{(\Psi(v_{(\alpha+1)n+1}), \Psi(v_{(\alpha+1)n+1+i})); i = j + \alpha n, j \in \mathbb{Z}_n \backslash \{0\}, \alpha \in \mathbb{Z}_{7^k}\} \cup \{(\Psi(v_{\alpha n}), \Psi(v_{2\alpha n})); \alpha \in \mathbb{Z}_{7^k} \backslash \{0\}\}$.

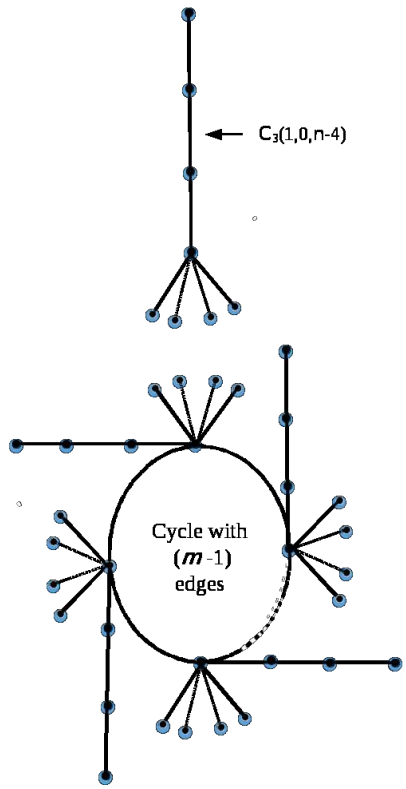

FIGURE 3. The graph $G_2^{m,n}$.

Subcase 1.1: n is odd. From the edge set of $G_3^{n,k}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor 7^k n/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor 7^k n/2 \rfloor\}\} = \{1, \ldots, \lfloor 7^k n/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; \leq \alpha \leq \frac{7^k-1}{2}, r(\alpha n + \gamma) = (3\alpha + 1)n + \gamma;\ \alpha \in \mathbb{Z}_{7^k}, \gamma \in \{1, 2, \ldots, \frac{n-1}{2}\}$.

Subcase 1.2: n is even. From the edge set of $G_3^{n,k}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor \frac{7^k n-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $7^k n/2$, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor 7^k n/2 \rfloor\}\} = \{1, \ldots, \lfloor 7^k n/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{7^k-1}{2}, r(\alpha n+\gamma) = (3\alpha+1)n+\gamma;\ \alpha \in \mathbb{Z}_{7^k}, \gamma \in \{1, 2, \ldots, \frac{n-2}{2}\}, r((\alpha + \frac{1}{2})n) = 3(\alpha + \frac{1}{2})n; 0 \leq \alpha \leq \frac{7^k-3}{2}, r(7^k n/2) = \frac{7^k n}{2}$.

Case 2. Let us define $\Psi : V(G_4^{m,n}) \to \mathbb{Z}_{mn}$ by $\Psi(v_i) = i$; $i \in \mathbb{Z}_{mn}, E(G_4^{m,n}) = \{(\Psi(v_{(\alpha+1)n+1}), \Psi(v_{(\alpha+1)n+1+i})); i = j+\alpha n, j \in \mathbb{Z}_n \backslash \{0\}, \alpha \in \mathbb{Z}_m\} \cup \{(\Psi(v_{\alpha n}), \Psi(v_{2\alpha n})); \alpha \in \mathbb{Z}_m \backslash \{0\}\}$.

Subcase 2.1: n is odd. From the edge set of $G_4^{m,n}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in$

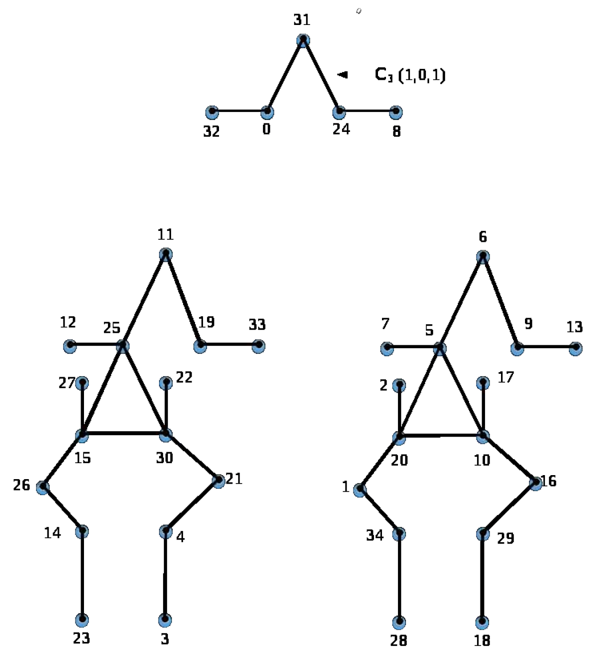

FIGURE 4. An orthogonal $\{1, \ldots, 17\}$ labelling of $G_1^{5,1}$ w.r.t. $\mathbb{Z}_{35}$.

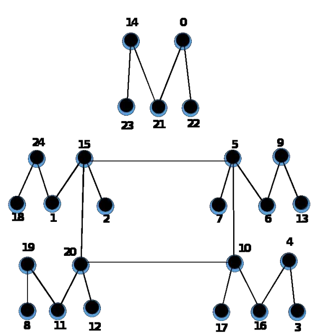

FIGURE 5. An orthogonal $\{1, \ldots, 12\}$ labelling of $G_2^{5,5}$ w.r.t. $\mathbb{Z}_{25}$.

$\{1, \ldots, \lfloor mn/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor mn/2 \rfloor\}\} = \{1, \ldots, \lfloor mn/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n$; $1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n + \gamma) = (3\alpha + 1)n + \gamma$; $\alpha \in \mathbb{Z}_m, \gamma \in \{1, 2, \ldots, \frac{n-1}{2}\}$.

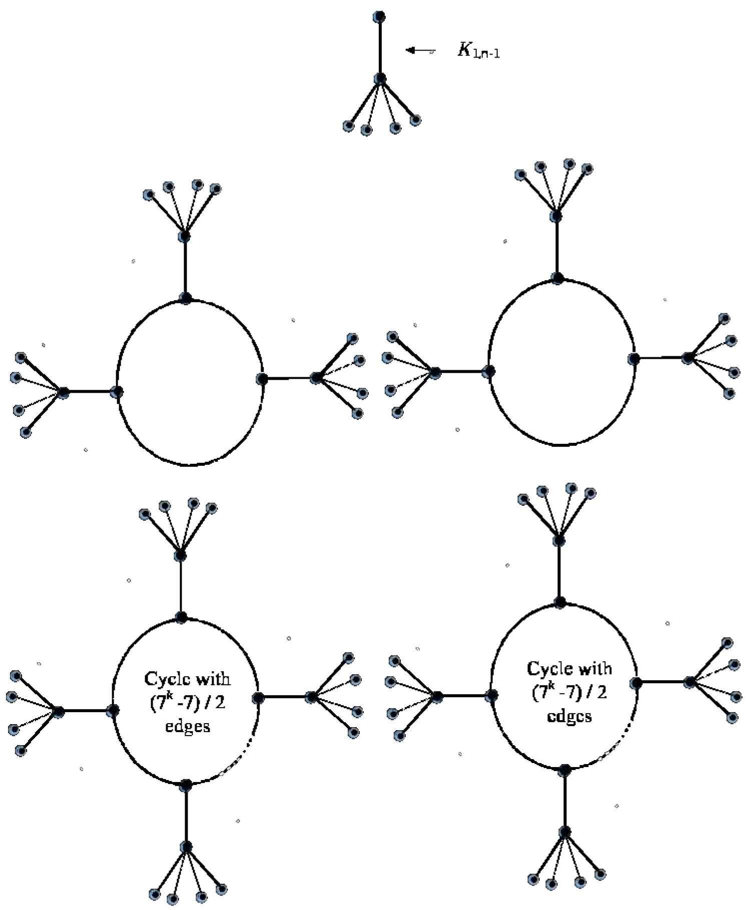

FIGURE 6. The graph $G_3^{n,k}$.

Subcase 2.2: n is even. From the edge set of $G_4^{m,n}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor \frac{mn-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $mn/2$, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor mn/2 \rfloor\}\} = \{1, \ldots, \lfloor mn/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n+\gamma) = (3\alpha+1)n+\gamma; \alpha \in \mathbb{Z}_m, \gamma \in \{1, 2, \ldots, \frac{n-2}{2}\},\ r((\alpha + \frac{1}{2})n) = 3(\alpha + \frac{1}{2})n;\ 0 \leq \alpha \leq \frac{m-3}{2}, r(mn/2) = mn/2$. □

For more illustration to Theorem 7,

(i) Let $k = 2$ and $n = 5$. Then the CODC of Circ$(245; \{1, \ldots, 122\})$ can be obtained by $G_3^{5,2} \cong K_{1,4} \cup 2(C_3 \boxdot K_{1,4}) \cup 2(C_{21} \boxdot K_{1,4})$.
(ii) Let $k = 1$ and $n = 3$. Then the CODC of Circ$(21; \{1, \ldots, 10\})$ can be obtained by $G_3^{3,1} \cong K_{1,2} \cup 2(C_3 \boxdot K_{1,2})$, see Figure 9.

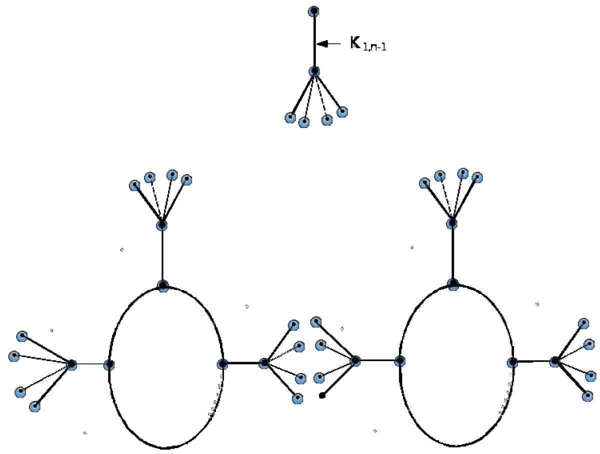

FIGURE 7. The graph $G_3^{n,1}$.

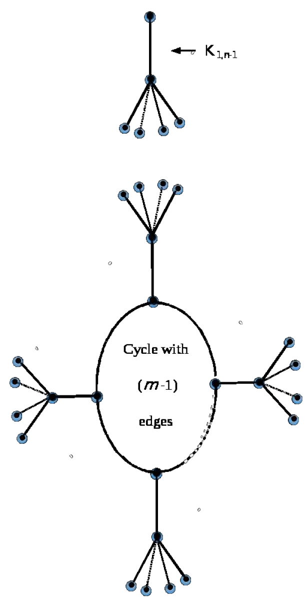

FIGURE 8. The graph $G_4^{m,n}$.

(iii) Let $m = 5$ and $n = 4$. Then the CODC of Circ$(20; \{1, \ldots, 10\})$ can be obtained by $G_4^{5,4} \cong K_{1,3} \cup (C_4 \boxdot K_{1,3})$, see Figure 10.

Let us define $\Psi : V(G_5^{m,n}) \to \mathbb{Z}_{mn}$ by $\Psi(v_i) = i$; $i \in \mathbb{Z}_{mn}$, $E(G_5^{m,n}) = \{(\Psi(v_{\alpha n}), \Psi(v_{2\alpha n})); \alpha \in \mathbb{Z}_m \backslash \{0\} \cup \{(\Psi(v_{(\alpha+1)n}), \Psi(v_{(2\alpha+1)n+2})), (\Psi(v_{(\alpha+1)n+2}), \Psi(v_{(2\alpha+1)n+2+\beta})), (\Psi(v_{4+(\alpha-(\frac{m-1}{2}))n}), \Psi(v_{(2\alpha+1)n+\gamma})); \alpha \in \mathbb{Z}_m, \beta \in \mathbb{Z}_n \backslash \{0, 2, n-2, n-1\}, \gamma \in \{2, 3\}\}$. In Theorem 8, we construct a CODC of Circ$(mn; \{1, \ldots, \lfloor mn/2 \rfloor\})$ by $G_5^{m,n}$, where n and m are integers with $m \equiv 1, 5 \pmod 6$ and $n \geq 7$.

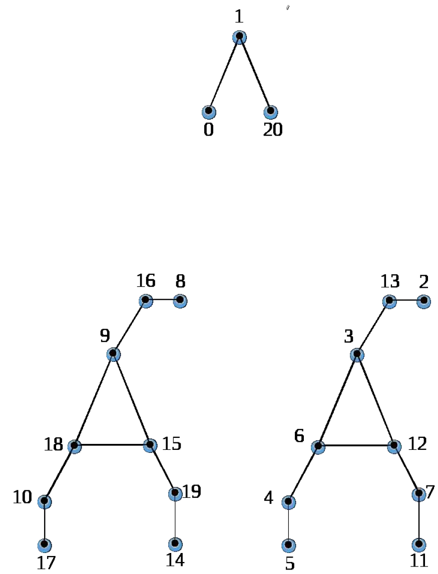

FIGURE 9. An orthogonal $\{1, \ldots, 10\}$-labelling of $G_3^{3,1}$ w.r.t. $\mathbb{Z}_{21}$.

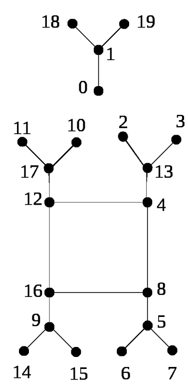

FIGURE 10. An orthogonal $\{1, \ldots, 10\}$-labelling of $G_4^{5,4}$ w.r.t. $\mathbb{Z}_{20}$.

Theorem 8. *Let n and m be integers with $m \equiv 1, 5 \pmod 6$ and $n \geq 7$. Then a CODC of* $\mathrm{Circ}(mn; \{1, \ldots, \lfloor mn/2 \rfloor\})$ *can be obtained from $G_5^{m,n}$.*

Proof.

Case 1: n is odd. From the edge set of $G_5^{m,n}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor mn/2 \rfloor\}$ are repeated twice, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor mn/2 \rfloor\}\} = \{1, \ldots, \lfloor mn/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n + 2) = (3\alpha + 1)n - 2;\ 0 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n - 2) = (3\alpha - 1)n + 2;\ 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n + 1) = (3\alpha + 1)n - 1; 0 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n - 1) = (3\alpha - 1)n + 1;\ 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n + \gamma) = (3\alpha + 1)n + \gamma; \alpha \in \mathbb{Z}_m, \gamma \in \{3, 4, \ldots, \frac{n-1}{2}\}$.

Case 2: n is even. From the edge set of $G_5^{m,n}$, the orthogonal labelling conditions of Theorem 2 are verified: (i) The edges of length $l \in \{1, \ldots, \lfloor \frac{mn-1}{2} \rfloor\}$ are repeated twice and there is only one edge of length $mn/2$, (ii) $\{r(l) : l \in \{1, \ldots, \lfloor mn/2 \rfloor\}\} = \{1, \ldots, \lfloor mn/2 \rfloor\}$ which can be proved as follows, $r(\alpha n) = 3\alpha n; 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n+2) = (3\alpha+1)n-2;\ 0 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n-2) = (3\alpha-1)n+2;\ 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n+1) = (3\alpha+1)n-1; 0 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n-1) = (3\alpha-1)n+1;\ 1 \leq \alpha \leq \frac{m-1}{2}, r(\alpha n+\gamma) = (3\alpha+1)n+\gamma; \alpha \in \mathbb{Z}_m, \gamma \in \{3, \ldots, \frac{n-2}{2}\}, r((\alpha+\frac{1}{2})n) = 3(\alpha+\frac{1}{2})n; 0 \leq \alpha \leq \frac{m-3}{2}, r(mn/2) = mn/2$. □

3. Conclusion

In conclusion, this paper gives new constructions of cyclic orthogonal double covers of certain circulant graphs by the union of a caterpillar and graphs belonging to a class that we call "nerve cell" graphs because these graphs are similar to the nerve cell in the human body.

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