

GENERATING SPECIAL ARITHMETIC FUNCTIONS BY
LAMBERT SERIES FACTORIZATIONS

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ABSTRACT. In this collaborative article, we summarize the known useful and interesting results and formulas we have discovered so far. Summarizing results from two related articles by Merca and Schmidt, we arrive at so-termed Lambert series factorization theorems. We unify the matrix representations that underlie two of our separate papers, and which commonly arise in identities involving partition functions and other functions generated by Lambert series. We provide a number of properties and open problems related to the inverse matrix entries defined in Schmidt's article and the Euler partition function $p(n)$ which we prove through our new results unifying the expansions of the Lambert series factorization theorems within this article.

1. INTRODUCTION

1.1. Lambert series factorization theorems. We consider recurrence relations and matrix equations related to *Lambert series* expansions of the form [4, §27.7] [1, §17.10]

$$(1.1) \quad \sum_{n \geq 1} \frac{a_n q^n}{1 - q^n} = \sum_{m \geq 1} b_m q^m, \quad |q| < 1,$$

for prescribed arithmetic functions $a : \mathbb{Z}^+ \rightarrow \mathbb{C}$ and $b : \mathbb{Z}^+ \rightarrow \mathbb{C}$ where $b_m = \sum_{d|m} a_d$. There are many well-known Lambert series for special arithmetic functions of the form in (1.1). Examples include the following series where $\mu(n)$ denotes the *Möbius function*, $\phi(n)$ denotes *Euler's phi function*, $\sigma_\alpha(n)$ denotes the *generalized sum of divisors function*, $\lambda(n)$ denotes *Liouville's function*, $\Lambda(n)$ denotes *von Mangoldt's function*, $\omega(n)$ defines the number of distinct primes dividing n , and $J_t(n)$ is *Jordan's totient function* for a fixed

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$t \in \mathbb{C}$ [4, §27.6 – §27.7]¹:

(1.2)

$$\begin{aligned} \sum_{n \geq 1} \frac{\mu(n)q^n}{1 - q^n} &= q, & (a_n, b_n) &:= (\mu(n), [n = 1]_\delta) \\ \sum_{n \geq 1} \frac{\phi(n)q^n}{1 - q^n} &= \frac{q}{(1 - q)^2}, & (a_n, b_n) &:= (\phi(n), n) \\ \sum_{n \geq 1} \frac{n^\alpha q^n}{1 - q^n} &= \sum_{m \geq 1} \sigma_\alpha(n)q^n, & (a_n, b_n) &:= (n^\alpha, \sigma_\alpha(n)) \\ \sum_{n \geq 1} \frac{\lambda(n)q^n}{1 - q^n} &= \sum_{m \geq 1} q^{m^2}, & (a_n, b_n) &:= (\lambda(n), [n \text{ is a positive square}]_\delta) \\ \sum_{n \geq 1} \frac{\Lambda(n)q^n}{1 - q^n} &= \sum_{m \geq 1} \log(m)q^m, & (a_n, b_n) &:= (\Lambda(n), \log n) \\ \sum_{n \geq 1} \frac{|\mu(n)|q^n}{1 - q^n} &= \sum_{m \geq 1} 2^{\omega(m)}q^m, & (a_n, b_n) &:= (|\mu(n)|, 2^{\omega(n)}) \\ \sum_{n \geq 1} \frac{J_t(n)q^n}{1 - q^n} &= \sum_{m \geq 1} m^t q^m, & (a_n, b_n) &:= (J_t(n), n^t). \end{aligned}$$

In this article, our new results and conjectures extend and unify the related Lambert series factorization theorems considered in two separate contexts in the references [3, 6]. In particular, in [2] Merca notes that

$$\sum_{n \geq 1} \frac{q^n}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \sum_{n \geq 1} (s_o(n) \pm s_e(n)) q^n,$$

where $s_o(n)$ and $s_e(n)$ respectively denote the number of parts in all partitions of n into an odd (even) number of distinct parts. More generally, Merca [3] proves that

$$(1.3) \quad \sum_{n \geq 1} \frac{a_n q^n}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \sum_{n \geq 1} \left(\sum_{k=1}^n (s_o(n, k) \pm s_e(n, k)) a_k \right) q^n,$$

where $s_o(n, k)$ and $s_e(n, k)$ are respectively the number of k 's in all partitions of n into an odd (even) number of distinct parts.

1.2. Matrix equations for the arithmetic functions generated by Lambert series. We define the invertible $n \times n$ square matrices, A_n , as in Schmidt's article according to the convention from Merca's article as [6,

¹*Notation:* Iverson's convention compactly specifies boolean-valued conditions and is equivalent to the Kronecker delta function, $\delta_{i,j}$, as $[n = k]_\delta \equiv \delta_{n,k}$. Similarly, $[\text{cond} = \text{True}]_\delta \equiv \delta_{\text{cond}, \text{True}}$ in the remainder of the article.

TABLE 1. The bottom row sequences in the matrices, A_n^{-1} , in the definition of (1.6) on page 34 for $2 \leq n \leq 18$.

n	$r_{n,n-1}, r_{n,n-2}, \dots, r_{n,1}$
2	1
3	1, 1
4	2, 1, 1
5	4, 3, 2, 1
6	5, 3, 2, 2, 1
7	10, 7, 5, 3, 2, 1
8	12, 9, 6, 4, 3, 2, 1
9	20, 14, 10, 7, 5, 3, 2, 1
10	25, 18, 13, 10, 6, 5, 3, 2, 1
11	41, 30, 22, 15, 11, 7, 5, 3, 2, 1
12	47, 36, 26, 19, 14, 10, 7, 5, 3, 2, 1
13	76, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
14	90, 69, 51, 39, 28, 21, 14, 11, 7, 5, 3, 2, 1, 1
15	129, 97, 74, 55, 41, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
16	161, 124, 94, 72, 53, 40, 29, 21, 15, 11, 7, 5, 3, 2, 1, 1
17	230, 176, 135, 101, 77, 56, 42, 30, 22, 15, 11, 7, 5, 3, 2, 1, 1
18	270, 212, 163, 126, 95, 73, 54, 41, 29, 22, 15, 11, 7, 5, 3, 2, 1, 1

cf.§1.2]

$$(1.4) \quad A_n := (s_e(i, j) - s_o(i, j))_{1 \leq i, j \leq n},$$

where the entries, $s_{i,j} := s_e(i, j) - s_o(i, j)$, of these matrices are generated by [3, Cor. 4.3]

$$s_{i,j} = s_e(i, j) - s_o(i, j) = [q^i] \frac{q^j}{1 - q^j} (q; q)_\infty.$$

We then have formulas for the Lambert series arithmetic functions, a_n , in (1.1) and in the special cases from (1.2) for all $n \geq 1$ given by

$$(1.5) \quad \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = A_n^{-1} \left(\underbrace{b_{m+1} - \sum_{s=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-s}{6} \rfloor} (-1)^{k+1} b_{m+1-k(3k+s)/2}}_{:=B_{b,m}} \right)_{0 \leq m < n}.$$

In general, for all $n \geq 2$ we have recursive formulas for the inverses of the matrices defined by (1.4) expanded in the form of

$$(1.6) \quad A_{n+1}^{-1} = \left[\begin{array}{c|c} \text{---} & A_n^{-1} \\ \hline r_{n+1,n}, \dots, r_{n+1,1} & \mathbf{1} \end{array} \right],$$

where the first several special cases of the sequences, $\{r_{n,n}, r_{n,n-1}, \dots, r_{n,1}\}$, are given as in [6] by Table 1. We mention that unipotent lower triangular matrices form a group, so the inverses A_n^{-1} are lower triangular with 1 on diagonal.

Within this article we focus on the properties of the entries, $s_{i,j}^{(-1)}$, of the inverse matrices, A_n^{-1} , defined by (1.4). We prove several new recurrence relations and an expansion of an exact formula for the inverse matrices in the previous equation in the results of Section 2. In Section 3 we computationally observe and prove that

$$s_{n,k}^{(-1)} := \sum_{d|n} p(d-k)\mu(n/d)$$

where $p(n) = [q^n](q; q)_\infty^{-1}$ denotes *Euler's partition function*. This key conjecture immediately implies the results in Corollary 3.2. More precisely, the corollary provides exact finite divisor sum formulas for the special cases of (1.5) corresponding to the special arithmetic functions in (1.2).

1.3. Significance of our new results and conjectures. Questions involving divisors of an integer have been studied for millennia and they underlie the deepest unsolved problems in number theory and related fields. The study of partitions, i.e., the ways to write a positive integer as a sum of positive integers, is much younger, with Euler considered to be the founder of the subject. The history of both subjects is rich and interesting but in the interest of brevity we will not go into it here.

The two branches of number theory, additive and multiplicative, turn out to be related in many interesting ways. Even though there are a number of important results connecting the theory of divisors with that of partitions, these are somewhat scattered in their approach. There are many other apparent connections to convolutions involving these matrix sequences waiting to be discovered. We propose continuing the study of the relationship between divisors and partitions with the goal of identifying common threads and hopefully unifying the underlying theory. Moreover, it appears that, on the multiplicative number theory side, these connections can be extended to other important number theoretic functions such as Euler's totient function, Jordan's totient function, Liouville's function, the Möbius function, and von Mangoldt's function, among others.

Our goal is to obtain a unified view of convolutions involving the partition function and number theoretic functions. To our knowledge, such an approach has not been attempted yet. Convolutions have been used in nearly all areas of pure and applied mathematics. In a sense, they measure

the overlap between two functions. The idea for a unified approach for a large class of number theoretical functions has its origin in Merca's article [3] and in Schmidt's article [6].

Perhaps our most interesting and important result, which we discovered computationally with *Mathematica* and *Maple* starting from an example formula given in the *Online Encyclopedia of Integer Sequences* for the first column of the inverse matrices defined by (1.4), is stated in Theorem 3.1. The theorem provides an exact divisor sum formula for the inverse matrix entries, $s_{n,k}^{(-1)}$, involving a Möbius transformation of the shifted Euler partition function, $p(n-k)$. This result is then employed to formulate new exact finite (divisor) sum formulas for each of the Lambert series functions, a_n , from the special cases in (1.2). These formulas are important since there are rarely such simple and universal identities expressing formulas for an entire class of special arithmetic functions considered in the context of so many applications in number theory and combinatorics. Generalizations, further applications, and topics for future research based on our work in this article are suggested in Section 4.

2. EXACT AND RECURSIVE FORMULAS FOR THE INVERSE MATRICES

Proposition 2.1 (Recursive Matrix-Product-Like Formulas). *We let $s_{i,j} := s_e(i,j) - s_o(i,j)$ denote the terms in the original matrices, A_n , found in Schmidt's article and let $s_{i,j}^{(-1)}$ denote the corresponding entries in the inverse matrices, A_n^{-1} . Then we have that*

$$\begin{aligned} s_{n,j}^{(-1)} &= - \sum_{k=1}^{n-j} s_{n,n+1-k}^{(-1)} \cdot s_{n+1-k,j} + \delta_{n,j} \\ &= - \sum_{k=1}^{n-j} s_{n,n-k} \cdot s_{n-k,j}^{(-1)} + \delta_{n,j} \\ &= - \sum_{k=1}^n s_{n,k-1} \cdot s_{k-1,j}^{(-1)} + \delta_{n,j}. \end{aligned}$$

Proof. The proof follows from the fact that for any $n \times n$ invertible matrices, A_n and A_n^{-1} , with entries given in the notation above, we have the following inversion formula for all $1 \leq k, p \leq n$:

$$\sum_{j=1}^n s_{p,j} \cdot s_{j,k}^{(-1)} = \sum_{j=1}^n s_{p,j}^{(-1)} \cdot s_{j,k} = [p = k]_{\delta},$$

since the matrices A_n are lower triangular with 1 on diagonal. \square

We can use the third of the three formulas given in the proposition repeatedly to obtain the following recursively defined, and then exact sums

for the inverse matrix entries:

$$s_{i,j}^{(-1)} = - \sum_{k=1}^i \sum_{k_2=1}^{k-1} \sum_{k_3=1}^{k_2-1} s_{i,k-1} \cdot s_{k-1,k_2-1} \cdot s_{k_2-1,k_3-1} \cdot s_{k_3-1,j}^{(-1)} \\ + \sum_{k=j+2}^i s_{i,k-1} \cdot s_{k-1,j} - s_{i,j} + \delta_{i,j}.$$

By inductively extending the expansions in the previous equation and noticing that the product terms in the multiple nested sums resulting from this procedure are eventually zero, we obtain the result in the next corollary.

Corollary 2.2 (An Exact Nested Formula for the Inverse Matrices). *Let the notation for the next multiple, nested sums be defined as*

$$\Sigma_m(i, j) := \underbrace{\sum_{k_1=j+2}^i \sum_{k_2=j+2}^{k_1-1} \cdots \sum_{k_m=j+2}^{k_{m-1}-1}}_{m \text{ total sums}} s_{i,k_1-1} \cdot s_{k_1-1,k_2-1} \times \cdots \times s_{k_{m-1}-1,j}.$$

Then we may write an exact expansion for the inverse matrix entries as

$$s_{i,j}^{(-1)} = \delta_{i,j} - s_{i,j} + \Sigma_1(i, j) - \Sigma_2(i, j) + \cdots + (-1)^{i+j+1} \Sigma_{i-j}(i, j).$$

Proof. The proof is easily obtained by induction on j and repeated applications of the third recurrence relation stated in Proposition 2.1. \square

The terms in the multiple sums defined in the corollary are reminiscent of the formula for the multiplication of two or more matrices. We may thus potentially obtain statements of more productive exact results providing expansions of these inverse matrix terms by considering the nested, multiple sum formulas in Corollary 2.2 as partial matrix products, though for the most part we leave the observation of such results as a topic for future investigation on these forms. However, given the likeness of the nested sums in the previous equations and in Corollary 2.2 to sums over powers of the matrix A_n , we have computationally obtained the following related formula for the corresponding inverse matrices A_n^{-1} :

$$A_n^{-1} = \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{i+1} A_n^{i-1}, \quad n \geq 2.$$

Given that A_n is unipotent, this formula is not surprising. We do not provide the proof by induction used to formally prove this identity here due to the complexity of the forms of the powers of the matrix A_n which somewhat limit the utility of the formula at this point. We also notice that the corollary expresses the complicated inverse entry functions as a sum over products of sequences with known and comparatively simple generating functions stated in the introduction [3, cf. Cor. 4.3]. The results in Section 3 provide a more exact representation of the entries of these inverse matrices for all n obtained by a separate method of proof.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0
3	2	1	1	0	0	0	0	0	0	0	0	0
4	3	2	1	1	0	0	0	0	0	0	0	0
5	5	3	2	1	1	0	0	0	0	0	0	0
6	7	5	3	2	1	1	0	0	0	0	0	0
7	11	7	5	3	2	1	1	0	0	0	0	0
8	15	11	7	5	3	2	1	1	0	0	0	0
9	22	15	11	7	5	3	2	1	1	0	0	0
10	30	22	15	11	7	5	3	2	1	1	0	0
11	42	30	22	15	11	7	5	3	2	1	1	0
12	56	42	30	22	15	11	7	5	3	2	1	1
13	77	56	42	30	22	15	11	7	5	3	2	1
14	101	77	56	42	30	22	15	11	7	5	3	2
15	135	101	77	56	42	30	22	15	11	7	5	3
16	176	135	101	77	56	42	30	22	15	11	7	5
17	231	176	135	101	77	56	42	30	22	15	11	7
18	297	231	176	135	101	77	56	42	30	22	15	11

(i) The Divisor Sums $a'_{n,k}$

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0
4	2	1	1	1	0	0	0	0	0	0	0	0
5	4	3	2	1	1	0	0	0	0	0	0	0
6	5	3	2	2	1	1	0	0	0	0	0	0
7	10	7	5	3	2	1	1	0	0	0	0	0
8	12	9	6	4	3	2	1	1	0	0	0	0
9	20	14	10	7	5	3	2	1	1	0	0	0
10	25	18	13	10	6	5	3	2	1	1	0	0
11	41	30	22	15	11	7	5	3	2	1	1	0
12	47	36	26	19	14	10	7	5	3	2	1	1
13	76	56	42	30	22	15	11	7	5	3	2	1
14	90	69	51	39	28	21	14	11	7	5	3	2
15	129	97	74	55	41	30	22	15	11	7	5	3
16	161	124	94	72	53	40	29	21	15	11	7	5
17	230	176	135	101	77	56	42	30	22	15	11	7
18	270	212	163	126	95	73	54	41	29	22	15	11

(ii) The Divisor Sums $a''_{n,k}$

FIGURE 2. A comparison of the two experimental divisor sum variants, $a'_{n,k}$ and $a''_{n,k}$, defined on page 38. Theorem 3.1 summarizes the results shown in these two sequence plots.

We are then able to explore further with the results from this first conjecture to build tables of the following two formulas involving our sequence, $s_{n,k}^{(-1)}$, and the shifted forms of the partition function, $p(n-k)$, where we take $p(n) \equiv 0$ when $n < 0$:

$$(i) \quad a'_{n,k} := \sum_{d|n} s_{d,k}^{(-1)}$$

$$(ii) \quad a''_{n,k} := \sum_{d|n} p(d-k)\mu(n/d).$$

The results of plotting these sequences for the first few rows and columns of $1 \leq n \leq 18$ and $1 \leq k \leq 12$, respectively, are found in the somewhat surprising and lucky results given in Figure 2. From this experimental data, we arrive at the following second conjecture providing exact divisor sum formulas for the inverse matrix entries. The corollary immediately following this conjecture is implied by a proof of these results and from the formulas established in [6, §3].

Theorem 3.1 (Exact Formulas for the Inverse Matrices). *For all $n, k \geq 1$ with $1 \leq k \leq n$, we have the following formula connecting the inverse matrices and the Euler partition function:*

$$(3.1) \quad s_{n,k}^{(-1)} := \sum_{d|n} p(d-k)\mu(n/d).$$

Proof. We see that the equation (3.1) which we seek to prove is equivalent to

$$p(n-k) := \sum_{d|n} s_{d,k}^{(-1)}.$$

We next consider the variant of the Lambert series factorization theorem in (1.3) applied to the Lambert series in (1.1) with $a_n := s_{n,k}^{(-1)}$ for a fixed integer $k \geq 1$. In particular, the identity in (1.3) implies that

$$\begin{aligned} \sum_{d|n} s_{d,k}^{(-1)} &= \sum_{m=0}^n \sum_{j=1}^{n-m} (s_o(n-m, j) - s_e(n-m, j)) s_{j,k}^{(-1)} \cdot p(m) \\ &= \sum_{m=0}^n \delta_{n-k, m} \cdot p(m) \\ &= p(n-k), \end{aligned}$$

where we have by our matrix formulation in (1.4) that

$$\sum_{j=1}^m (s_o(m, j) - s_e(m, j)) s_{j,k}^{(-1)} = \delta_{m,j}.$$

Thus by Möbius inversion, we have our key formula for the inverse matrix entries given in (3.1). \square

We notice that the last equation given in the conjecture implies that we have a Lambert series generating function for the inverse matrix entries given by

$$\sum_{n \geq 1} \frac{s_{n,k}^{(-1)} q^n}{1 - q^n} = \frac{q^k}{(q; q)_\infty},$$

for fixed integers $k \geq 1$. We also note that where Merca's article [3] provides the partition function representation for the sequence $s_{n,k}$ in the matrix interpretation established in [6], the result in the theorem above effectively

provides us with an exact identity for the corresponding sequence of inverse matrix entries, $s_{n,k}^{(-1)}$, employed as in Schmidt's article to obtain the new expressions for several key special multiplicative functions.

One important and interesting consequence of the result in Theorem 3.1 is that we have now completely specified several new formulas which provide exact representations for a number of classical and special multiplicative functions cited as examples in (1.2) of the introduction. These formulas, which are each expanded in the next corollary, connect the expansions of several special multiplicative functions to sums over divisors of n involving Euler's partition function $p(n)$. In particular, we can now state several specific identities for classical number theoretic functions which connect the seemingly disparate branches of multiplicative number theory with the additive nature of the theory of partitions and special partition functions. The results in the next corollary are expanded in the following forms:

Corollary 3.2 (Exact Formulas for Special Arithmetic Functions). *For natural numbers $m \geq 0$, let the next component sequences defined in [6, §3] be defined by the formulas*

$$B_{\phi,m} = m + 1 - \frac{1}{8} \left(8 - 5 \cdot (-1)^{u_1} - 4(-2 + (-1)^{u_1} + (-1)^{u_2})m \right. \\ \left. + 2(-1)^{u_1}u_1(3u_1 + 2) + (-1)^{u_2}(6u_2^2 + 8u_2 - 3) \right)$$

$$B_{\mu,m} = [m = 0]_{\delta} + \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+25}-b}{6} \rfloor} (-1)^k [m + 1 - k(3k + b)/2 = 1]_{\delta}$$

$$B_{\lambda,m} = [\sqrt{m+1} \in \mathbb{Z}]_{\delta} \\ - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} [\sqrt{m+1 - k(3k+b)/2} \in \mathbb{Z}]_{\delta}$$

$$B_{\Lambda,m} = \log(m+1) - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} \log(m+1 - k(3k+b)/2)$$

$$B_{|\mu|,m} = 2^{\omega(m+1)} - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} 2^{\omega(m+1 - k(3k+b)/2)}$$

$$B_{J_t,m} = (m+1)^t - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} (m+1 - k(3k+b)/2)^t,$$

where $u_1 \equiv u_1(m) := \lfloor (\sqrt{24m+1}+1)/6 \rfloor$ and $u_2 \equiv u_2(m) := \lfloor (\sqrt{24m+1}-1)/6 \rfloor$. Then we have that

$$\begin{aligned}\phi(n) &= \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1) \mu(n/d) B_{\phi,m} \\ \mu(n) &= \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1) \mu(n/d) B_{\mu,m} \\ \lambda(n) &= \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1) \mu(n/d) B_{\lambda,m} \\ \Lambda(n) &= \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1) \mu(n/d) B_{\Lambda,m} \\ |\mu(n)| &= \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1) \mu(n/d) B_{|\mu|,m} \\ J_t(n) &= \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1) \mu(n/d) B_{J_t,m}.\end{aligned}$$

The corresponding formulas for the average orders, $\Sigma_{a,x}$, of these special arithmetic functions are obtained in an initial form by summing the right-hand-sides of the previous equations over all $n \leq x$.

We can also compare the results of the recurrence relations in the previous corollary to two other identical statements of these results. In particular, if we define the sequence $\{G_j\}_{j \geq 0} = \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \dots\}$ as in [3, §1] by the formula

$$G_j = \frac{1}{2} \left\lfloor \frac{j}{2} \right\rfloor \left\lfloor \frac{3j+1}{2} \right\rfloor,$$

then by performing a divisor sum over n in the previous equations, we see that the sequence pairs in the form of (1.1) satisfy

$$a_n = \sum_{k=1}^n \sum_{j=0}^{k-1} p(n-k) (-1)^{\lfloor j/2 \rfloor} b_{n-G_j}.$$

We immediately notice the similarity of the recurrence relation for b_n given in the last equation to the known result from [6, Thm. 1.4] which states that

$$b_n = \sum_{j=0}^n (-1)^{\lfloor j/2 \rfloor} b_{n-G_j},$$

and which was proved by a separate nonexperimental approach in the reference.

Remark (An Experimental Conjecture). Since we have a well-known recurrence relation for the partition function given by

$$p(n) = \sum_{k=1}^n (-1)^{k+1} (p(n - k(3k - 1)/2) + p(n - k(3k + 1)/2)),$$

we attempt to formulate an analogous formula for the $s_{i,j}^{(-1)}$ using (3.1), which leads us to the sums

$$\begin{aligned} a_n''' &:= \sum_{k=1}^n (-1)^{k+1} \left(s_{n,k(3k-1)/2}^{(-1)} + s_{n,k(3k+1)/2}^{(-1)} \right) \\ &\mapsto \{1, 1, 2, 3, 6, 7, 14, 17, 27, 34, 55, 63, \dots\}. \end{aligned}$$

A search in the integer sequences database suggests that this sequence denotes the number of partitions of n into relatively prime parts, or alternately, aperiodic partitions of n [7, A000837, A000837]. We notice the additional, and somewhat obvious and less interesting, identity which follows from the recurrence relation for $p(n)$ given above expanded in the form of

$$\sum_{k=0}^n (-1)^{\lceil k/2 \rceil} s_{n,G_k}^{(-1)} = 0.$$

3.2. Other properties related to the partition function.

Proposition 3.3 (Partition Function Subsequences). *Let n be a positive integer. For $\lceil n/2 \rceil < k \leq n$,*

$$s_{n,k}^{(-1)} = p(n - k).$$

The indices of the first few rows such that

$$s_{n,k}^{(-1)} = p(n - k)$$

is true for all $1 < k \leq n$ are $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\}$.

Proof. This result is immediate from the divisor sum in (3.1) where the only divisor of n in the range $\lceil n/2 \rceil < k \leq n$ is n itself. \square

Proposition 3.4 (Partition Function Subsequences for Prime n). *For n prime and $1 \leq k \leq n$,*

$$s_{n,k}^{(-1)} = p(n - k) - \delta_{1,k},$$

where $\delta_{i,j}$ is the Kronecker delta function.

Proof. This result is also immediate from the divisor sum in (3.1) where the only divisors of the prime n are 1 and n and $p(1 - k) = \delta_{k,1}$ by convention. In particular, we have that

$$s_{n,k}^{(-1)} = \mu(p)p(1 - k) + \mu(1)p(n - k),$$

for all $1 \leq k \leq n$. \square

The next two results which we initially obtained experimentally from tables of the matrix inverse entries follow along the same lines as the previous two propositions. Given the ease with which we proved the last formulas for prime n , we omit the one-line proofs of the next two results below. Note that by the formula in (3.1), we may also strengthen these results to prime powers of the form $n = p^k$ for $k \geq 1$ and any prime p .

Proposition 3.5. *For n prime,*

$$s_{n^2,k}^{(-1)} = \begin{cases} p(n^2 - k) - p(n - k), & \text{for } 1 \leq k \leq n, \\ p(n^2 - k), & \text{for } n < k \leq n^2. \end{cases}$$

Proposition 3.6. *For n prime,*

$$s_{2n,k}^{(-1)} = \begin{cases} p(2n - k) - p(n - k) - p(2 - k) + \delta_{1,k}, & \text{for } 1 \leq k \leq 2, \\ p(2n - k) - p(n - k), & \text{for } 2 < k \leq n, \\ p(2n - k), & \text{for } n < k \leq 2n. \end{cases}$$

A similar argument to the above can be used to show that if $q, r \in \mathbb{Z}^+$ are relatively prime positive integers, then we have that

$$s_{qr,k}^{(-1)} = \delta_{1,k} - p(q - k) - p(r - k) + p(qr - k),$$

which as we observe is another example of an additive formula we have obtained defining an inherently multiplicative structure in terms of additive functions. Notably, we can use this observation to show that if the arithmetic function a_n in (1.1) is multiplicative, then we have that $a_q \cdot a_r = b_{qr} - b_p - b_q + b_1$ for all positive integers p, q such that $(p, q) = 1$. We can then form subsequent generalizations for products of pairwise relatively prime integers, q_1, q_2, \dots, q_m , accordingly.

4. CONCLUSIONS

4.1. Summary. We have proved a unified form of the Lambert series factorization theorems from the references [3, 6] which allows us to exactly express matrix equations between the implicit arithmetic sequences, a_n and b_n , in (1.1) and in the classical special cases in (1.2). More precisely, we have noticed that the invertible matrices, A_n , from Schmidt’s article are expressed through the factorization theorem in (1.3) proved by Merca. We then proved new divisor sum formulas involving the partition function $p(n)$ for the corresponding inverse matrices which define the sequences, a_n , in terms of only these matrix entries and the secondary sequence of b_n as in the results from [6].

The primary application of our new matrix formula results is stated in Corollary 3.2. The corollary provides new exact finite (divisor) sum formulas for the special arithmetic functions, $\phi(n)$, $\mu(n)$, $\lambda(n)$, $\Lambda(n)$, $|\mu(n)|$, and $J_t(n)$, and the corresponding partial sums defining the average orders of these functions. One related result not explicitly stated in Schmidt’s article

provides a discrete (i.e., nondivisor-sum) convolution for the average order of the sum-of-divisors function, denoted by $\Sigma_{\sigma,x} := \sum_{n \leq x} \sigma(n)$, in the form of [6, §3.1]

$$\Sigma_{\sigma,x+1} = \sum_{s=\pm 1} \left(\sum_{0 \leq n \leq x} \sum_{k=1}^{\lfloor \frac{\sqrt{24n+25}-s}{6} \rfloor} (-1)^{k+1} \frac{k(3k+s)}{2} \cdot p(x-n) \right).$$

Other related divisor sum results that can be stated in terms of our new inverse matrix formulas implied by Theorem 3.1 are found, for example, in Merca's article [3, §5].

4.2. Generalizations. Merca showed another variant of the Lambert series factorization theorem stated in the form of [3, Cor. 6.1]

$$\sum_{n \geq 1} \frac{a_n q^{2n}}{1 - q^n} = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} \sum_{k=1}^{\lfloor n/2 \rfloor} (s_o(n-k, k) - s_e(n-k, k)) a_k \cdot q^n.$$

If we consider the generalized Lambert series formed by taking derivatives of (1.1) from [5] in the context of finding new relations between the generalized sum-of-divisors functions, $\sigma_{\alpha}(n)$, we can similarly formulate new, alternate forms of the factorization theorems unified by this article. For example, suppose that $k, m \geq 0$ are integers and consider the factorization theorem resulting from comparing the sums

$$\sum_{n \geq 1} \frac{a_n q^{(m+1)n}}{(1 - q^n)^{k+1}} = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} \sum_{i=1}^{\lfloor n/(m+1) \rfloor} s_{n-m,i} \frac{a_i}{(1 - q^i)^k} \cdot q^n.$$

Then we have another variation of the factorization theorem providing that the previous series in k and m are expanded by

$$\sum_{n \geq 1} \frac{a_n q^{(m+1)n}}{(1 - q^n)^{k+1}} = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} \sum_{i=1}^{\lfloor \frac{n}{m+1} \rfloor} \sum_{j=0}^{\lfloor \frac{n-m}{i} \rfloor} \binom{k-1+j}{k-1} s_{n-m-j,i} \cdot a_i \cdot q^n.$$

When $m \geq k$ the series coefficients of these modified Lambert series generating functions are given by

$$\sum_{\substack{d|n \\ d \leq \lfloor \frac{n}{m+1} \rfloor}} \binom{\frac{n}{d}-1-m+k}{k} a_d = \sum_{q=0}^n \sum_{i=1}^{\lfloor \frac{n-q}{m+1} \rfloor} \sum_{j=0}^{\lfloor \frac{n-q-m}{i} \rfloor} \binom{k-1+j}{k-1} s_{n-q-m-j,i} \cdot a_i \cdot p(q).$$

Thus, again, as in Merca's article, the applications and results in Corollary 3.2 can be repeated in the context of a slightly different motivation for considering these factorization theorems.

4.3. Topics for future research. Topics for future research based on the unified factorization theorem results we have proved within the article include investigating the properties of the generalizations defined in the last subsection, considering congruences for the partition function and the inverse matrix entries, $s_{n,k}^{(-1)}$, and finding useful new asymptotic formulas for the average orders of the special functions in Corollary 3.2.

The last topic is of particular interest since we have given an explicit formula for the Möbius function, $\mu(n)$, which holds for all $n \geq 0$. The problem of determining whether the average order, $M(x) := \sum_{n \leq x} \mu(n)$, of this particular special function is bounded by $M(x) = O(x^{1/2+\varepsilon})$ for all sufficiently small $\varepsilon > 0$ is equivalent to the Riemann hypothesis. In light of the significance of this problem, we must at least suggest our approach towards formulating new exact, nonapproximate properties of this average order sequence for all $x \geq 1$.

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