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A LOWER BOUND ON THE HYPERGRAPH RAMSEY NUMBER $R(4,5;3)$

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Abstract. The finite version of Ramsey's theorem says that for positive integers r, k, $a_1,\dots,a_r$, there exists a least number $n = R(a_1,\dots,a_r;k)$ so that if X is an n-element set and all k-subsets of X are r-coloured, then there exists an i and an a_i-set A so that all k-subsets of A are coloured with the ith colour.

In this paper, the bound $R(4,5;3) \geq 35$ is shown by using a SAT solver to construct a red–blue colouring of the triples chosen from a 34-element set.

1. Introduction

In 1930, Ramsey proved the following theorem:

Theorem 1.1 ([11])**.** *Let $r, k, a_1,\dots,a_r$ be given positive integers. Then there is an integer n with the following property. If all k-subsets of an n-set are coloured with r colours, then for some i, $1 \leq i \leq r$, there exists an a_i-set entirely coloured in colour i (all of its k-subsets have colour i).*

The smallest n for which Ramsey's theorem holds, we call a Ramsey number and is denoted by $R(a_1,\dots,a_r;k)$. This notation is used by the survey by Radziszowski [10]. Note that there are at least two other notations for these numbers in the literature, namely: $R_k(a_1,\dots,a_r)$, used for example in [5], or $R^{(k)}(a_1,\dots,a_r)$, used in [2]. Since colouring of all k-subsets can be viewed as colouring of the edges of complete k-uniform hypergraphs, numbers $R(a_1,\dots,a_r;k)$, for $k \geq 3$, are also called hypergraph Ramsey numbers.

For $k = 2$, only ten exact values for nontrivial Ramsey numbers are known (see [10] for details). For $k = 3$, only one exact nontrivial value is known, namely $R(4,4;3) = 13$, where $R(4,4;3) \geq 13$ was proved in 1969 by Isbell [7] and equality was shown by McKay and Radziszowski [9] in 1991.

In this paper we deal with the number $R(4,5;3)$. In 1983 Isbell [8] proved that $R(4,5;3) \geq 24$, and in 1998 Exoo [4] presented a colouring which gives the bound $R(4,5;3) \geq 33$. Up to the author's knowledge

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the best upper bound for $R(4,5;3)$ can be obtained by using one step of the Erdős–Szekeres recursion [3] and known bounds for Ramsey numbers with smaller parameters, see [10]. These give the estimation $R(4,5;3) \leq R(R(3,5;3), R(4,4;3); 2) + 1 = R(5,13;2) + 1 \leq 1139$. In this paper we show that $R(4,5;3) \geq 35$. This bound is shown by producing a 2-colouring of the triples of set with 34-elements. The colouring is formed by dividing all triples into classes and then by looking through 2-colouring of the classes. Similar approaches were used to establish lower bound for graph Ramsey numbers ($k = 2$). For example, Harborth and Krause [6] searched through colourings such that the colouring matrix is partitioned into cyclic orbits.

2. Colouring

Theorem 2.1. *Let $V = \{0, \dots, 33\}$. There exists a colouring $c : \binom{V}{3} \to \{red, blue\}$, such that no 4-subset of V is entirely coloured in red, and no 5-subset of V is entirely coloured in blue.*

Proof. This colouring is achieved by first dividing all such triples into 176 classes and then by giving a 2-colouring of the classes. For each two integers a, b satisfying $1 \leq a \leq 11$ and $a + 1 \leq b \leq 22$, define the class:

$$C_{ab} = \Big\{\{0 + d, a + d, b + d\} : 0 \leq d \leq 33\Big\},$$

where addition is modulo 34. It is easy to see that there are 176 classes, each contains 34 3-sets and the classes are pairwise disjoint. Hence, we have a partition of $\binom{V}{3}$ into 176 disjoint classes. To find a proper colouring of the classes we use a SAT solver. We construct a formula with 176 variables—one for every class C_{ab}. For every $E \in \binom{V}{3}$, let $f(E)$ be the variable for the class that contains E. We assume that E is blue if the variable $f(E)$ is true, and E is red if $f(E)$ is false. We use the following formula:

$$\Big[\bigwedge_{S \in \binom{V}{4}} \bigvee_{E \in \binom{S}{3}} f(E)\Big] \wedge \Big[\bigwedge_{S \in \binom{V}{5}} \bigvee_{E \in \binom{S}{3}} \neg f(E)\Big].$$

The first part of the formula says that in every 4-set at least one of 3-subsets is coloured in blue (variable is true). Similarly, the second part says that in every 5-sets at least one of 3-subsets is coloured in red. It may happen that for some $S \in \binom{V}{4}$ we have two triples E and E' such that $f(E) = f(E')$ and we have two repeated literals in the clause $\bigvee_{E \in \binom{S}{3}} f(E)$. We simplify the clauses by removing such repetitions. Similarly it may happen that for two sets S and $S' \in \binom{V}{4}$ the clauses $\bigvee_{E \in \binom{S}{3}} f(E)$ and $\bigvee_{E \in \binom{S'}{3}} f(E)$ are equivalent. We simplify the formula by removing such repetitions. Similarly we simplify the second part of the formula.

Finally, a formula with 176 variables and 9552 clauses is found. We use the SparrowToRiss [1] SAT solver and find out that the formula is satisfied. The SAT solver finishes in less than five minutes on a personal computer[1]

[1]Computer with processor Intel® Core™ i7-4790, 3.60GHz.

a\b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0	0	1	0	1	0	1	1	0	1	1	1	1	0	0	0	0	0	1	0	0
2		0	0	0	0	0	1	1	0	1	0	1	1	1	1	0	1	1	0	0	1
3			1	0	1	0	1	1	0	0	0	1	0	1	0	0	0	0	1	1	1
4				1	0	0	1	0	0	1	0	1	0	0	1	0	0	0	0	1	1
5					1	1	0	1	1	1	0	0	0	1	0	0	0	1	1	1	0
6						0	0	0	0	0	0	1	0	1	0	1	1	0	0	1	0
7							1	1	0	0	0	0	1	0	0	0	0	0	0	0	1
8								0	0	1	0	0	1	0	1	1	0	0	1	0	0
9									0	0	0	0	1	0	0	0	1	0	0	0	0
10										0	1	0	1	0	1	1	1	0	0	1	1
11											0	0	0	1	0	0	1	1	0	1	0

FIGURE 1. The colouring of the classes C_{ab} without red 4-subset and blue 5-subset.

and returns the assignment that gives the proper colouring of the classes presented in Figure 1.

One can easily find, using a computer, that the colouring is proper, i.e. each clique K_4 contains at least one 3-set coloured in blue and each clique K_5 contains at least one 3-set coloured in red. On the website, `https://www.inf.ug.edu.pl/ramsey`, we posted a simplified formula, satisfying assignment, the corresponding colouring of all triples, and a C++ program that verifies that the colouring is proper. $\square$

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