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FEEDBACK VERTEX NUMBER OF SIERPIŃSKI-TYPE GRAPHS

LILI YUAN, BAOYINDURENG WU, AND BIAO ZHAO

Abstract. The feedback vertex number $\tau(G)$ of a graph G is the minimum number of vertices that can be deleted from G such that the resultant graph does not contain a cycle. We show that $\tau(S_p^n) = p^{n-1}(p-2)$ for the Sierpiński graph S_p^n with $p \geq 2$ and $n \geq 1$. The generalized Sierpiński triangle graph $\hat{S_p^n}$ is obtained by contracting all non-clique edges from the Sierpiński graph S_p^{n+1}. We prove that $\tau(\hat{S}_3^n) = (3^n+1)/2 = |V(\hat{S}_3^n)|/3$, and give an upper bound for $\tau(\hat{S}_p^n)$ for the case when $p \geq 4$.

1. Introduction

In this paper, we consider only simple, finite, undirected graphs and refer to [2] for undefined terminology and notation. For a graph $G = (V(G), E(G))$, the *order* and *size* are $|V(G)|$ and $|E(G)|$, respectively. We denote the order and the size of G by $|G|$ and $||G||$, respectively. For a vertex v, the *degree* of a vertex v, denoted by $d_G(v)$, is the number of edges which are incident with v in G, and the neighborhood of v, denoted by $N_G(v)$, is the set of the vertices adjacent to v in G.

For a set $X \subseteq V$, let $G - X$ be the graph obtained from G by removing vertices of X and all the edges that are incident to a vertex of X. The subgraph $G - (V(G) \setminus X)$ is said to be the *induced subgraph* of G induced by X, and is denoted by $G[X]$. We call X a *feedback vertex set* of G if $G-X$ is a forest. The *feedback vertex number* of G, denoted by $\tau(G)$, is the cardinality of a minimum feedback vertex set of G. The order of a maximum induced forest of G is denoted by $f(G)$. It is clear that $\tau(G) + f(G) = |G|$.

So, determining the feedback number of a graph G is equivalent to finding the maximum induced forest of G, first proposed by Erdős, Saks and Sós [5]. Some results on the maximum induced forest are obtained for several

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Corresponding author: Baoyindureng Wu.

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families of graphs, such as planar graphs with high girth [4, 17], outerplanar graphs [13, 28], triangle-free planar graphs [30]. The problem of determining the feedback vertex number has been proved to be NP-complete for general graphs [16]. However, the problem has been studied for some special graphs, such as hypercubes [6, 29], toroids [26], de Bruijn graphs [34], de Bruijn digraphs [33], 2-degenerate graphs [3]. A review of results and open problems on the feedback vertex number was given by Bau and Beineke [1].

The definition of Sierpiński graph was first introduced by Klavžar and Milutinović [19] in 1997. Graphs whose drawings can be viewed as approximations to the famous Sierpiński triangle have been studied extensively in the past 25 years, see a recent survey [8] for a collection of the results and the related works. We follow the notation there. For an integer k, let $\mathbb{N}_k$ be the set of all integers greater than or equal to k. In particular, $\mathbb{N}_1$ is denoted simply by $\mathbb{N}$. For an integer $p \in \mathbb{N}$, let $P = \{0, 1, \dots, p-1\}$. For convenience, let $[n] = \{1, \dots, n\}$ for an integer $n \in \mathbb{N}$. Let P^n be the set of all n-tuples on P.

Definition 1.1 ([8])**.** *For two integers $p \geq 1$ and $n \geq 0$, the Sierpiński graph S_p^n is given by*

$$V(S_p^n) = P^n, E(S_p^n) = \{\{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\} \mid i, j \in P, i \neq j; d \in [n]; \underline{s} \in P^{n-d}\}.$$

A vertex $s_1 s_2 \dots s_n$ of S_p^n is called an extreme vertex if $s_1 = \cdots = s_n$. There are precisely p extreme vertices in S_p^n. It is easy to see that $d_{S_p^n}(u) = p - 1$ for an extreme vertex u and $d_{S_p^n}(v) = p$ for all other vertices v. In the trivial cases $n = 0$ or $p = 1$, there is only one vertex and no edge, i.e. $S_p^0 \cong K_1 \cong S_1^n$; moreover, $S_p^1 \cong K_p$, where K_p denotes the complete graph of order p. The first interesting case is $p = 2$, where $S_2^n \cong P_{2^n}$. The drawings of the Sierpiński graphs S_3^3 and S_5^2 are shown in Fig. 1.

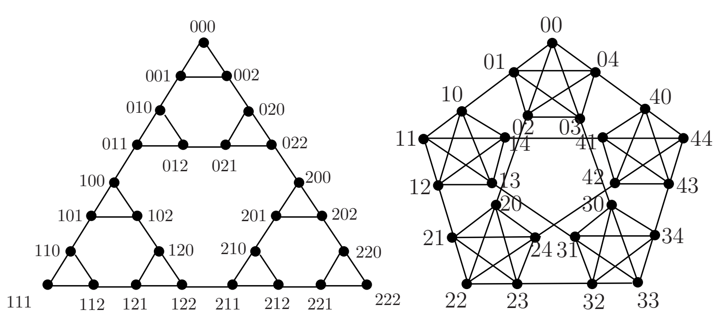

Figure 1. S_3^3 and S_5^2

Definition 1.2 ([8])**.** *For two integers $p \geq 1$ and $n \geq 0$, the generalized Sierpiński triangle graph $\hat{S}_p^n$ is obtained by contracting all nonclique edges (with the form $\{\underline{s}ij^l, \underline{s}ji^l\}$ for distinct $i, j \in P$ and $l \in [n]$, $\underline{s} \in P^{n-l}$) from the Sierpiński graph S_p^{n+1}.*

Sierpiński-type graphs have many interesting properties, see [9] for its planarity, [20] for its crossing number, [10, 15, 25, 18, 35, 36] for its various colorings, [23] for the global strong defensive alliances, [24] for the hub number, [21] for the Hamming dimension, [31, 32] for enumeration of matchings and spanning trees, [11, 12, 27, 22, 37] for distance and metric properties.

The main topic of the paper focuses on the feedback vertex number of Sierpiński-type graphs. In Section 2, we show that $\tau(S_p^n) = p^{n-1}(p-2)$. In Section 3, we show that $\tau(\hat{S}_3^n) = (3^n+1)/2 = |\hat{S}_3^n|/3$. In Section 4, we present an upper bound for the feedback vertex number of $\hat{S}_p^n$ for any $p \geq 4$. Some other relevant results are presented as well.

2. Sierpiński graphs

In this section, we determine the exact value of the feedback number of Sierpiński graph S_p^n. For two vertex subsets X, Y of a graph G, we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y. First we show the following.

Lemma 2.1. *For two integers $n \geq 1$ and $p \geq 2$, $\tau(S_p^n) \geq p^{n-1}(p-2)$.*

Proof. By induction on n. If $n = 1$, $S_p^n \cong K_p$. Since

$$\tau(S_p^n) = \tau(K_p) = p-2 = p^0(p-2) = p^{n-1}(p-2),$$

the result then follows. Next assume that $n \geq 2$ and X is a minimum feedback vertex set of S_p^n. Let $V_i = \{i\underline{k} |\ \underline{k} \in V(S_p^{n-1})\}$ for each $i \in P$. From the definition of S_p^n, we know that $S_p^n[V_i] \cong S_p^{n-1}$. Let $X_i = X \cap V_i$ for each $i \in P$. Since X is a minimum feedback vertex set of S_p^n, X_i is a feedback vertex set of $S_p^n[V_i]$. By the induction hypothesis, $|X_i| \geq p^{n-2}(p-2)$. Thus, $|X| \geq pp^{n-2}(p-2) = p^{n-1}(p-2)$. □

A 2-element set $Y \subseteq P^{m-1}$ for an integer $m \in \{2, \ldots, n\}$ is said to be *pairable* if $Y = \{\underline{s}a, \underline{s}b\}$, where $\underline{s} \in P^{m-2}$ and $\{a, b\} \in \binom{P}{2}$. In this case, let head(Y) = $\underline{\text{s}}$. Further, we define

$$C(Y) := C_1(Y) \cup C_2(Y),$$

where $C_1(Y) = \{\underline{s}aa, \underline{s}ab, \underline{s}ba, \underline{s}bb\}$ and $C_2(Y) = \{\underline{s}k(k-1), \underline{s}k(k+1) |\ k \in P \backslash \{a, b\}\}$, and $k-1, k+1$ are taken modulo p. In general, a set $Y \subseteq P^{m-1}$ is said to be *pairable* if Y can be partitioned into some 2-element sets $Y_1, \ldots, Y_q$ such that Y_i is pairable for each i, and $\text{head}(Y_i) \neq \text{head}(Y_j)$ for any i, j with $i \neq j$. Moreover, let us define

$$\text{head}(Y) := \bigcup_{i \in [q]} \text{head}(Y_i) \text{ and } C(Y) := \bigcup_{i \in [q]} C(Y_i).$$

Observe that if Y is pairable (since $\text{head}(Y_i) \neq \text{head}(Y_j)$ for any i, j with $i \neq j$), then it must be partitioned into some 2-element pairable sets in a unique way. So, we call $\{Y_1, \ldots, Y_q\}$ the pairable partition of Y, and call each Y_i a block of Y. Take $Y = \{0, 2\} \subseteq V(S_5^1)$ for an example. Then $C(Y) =$

$\{00, 02, 20, 22, 10, 12, 32, 34, 43, 40\} \subseteq V(S_5^2)$, see the subgraphs $S_5^1[Y]$ and $S_5^2[C(Y)]$ as depicted in red in Fig 2.

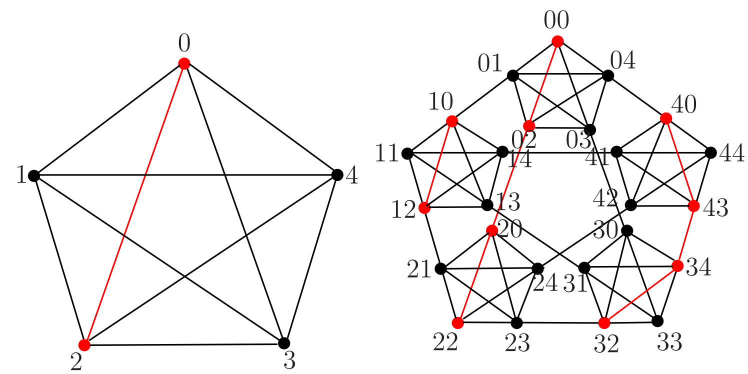

FIGURE 2. $S_5^1[Y]$ and $S_5^2[C(Y)]$, where $Y = \{0, 2\}$

Lemma 2.2. *If $Y \subseteq P^{m-1}$ is pairable for an integer $m \geq 2$, then $C(Y)$ is pairable, and* $\text{head}(C(Y)) = \{\underline{s}a|\ \underline{s} \in \text{head}(Y), a \in P\}$.

Proof. Let $\{Y_1, \dots, Y_q\}$ be the pairable partition of Y, where for each i, $Y_i = \{\underline{s_i}a_i, \underline{s_i}b_i\}$ for some $\underline{s_i} \in P^{m-2}$ and $\{a_i, b_i\} \in \binom{P}{2}$. By our definition,

$$C(Y) := \bigcup_{i \in [q]} C(Y_i),$$

where $C(Y_i) := C_1(Y_i) \cup C_2(Y_i)$, and $C_1(Y_i) = \{\underline{s_i}a_ia_i, \underline{s_i}a_ib_i, \underline{s_i}b_ia_i, \underline{s_i}b_ib_i\}$ and $C_2(Y_i) = \{\underline{s_i}k(k-1), \underline{s_i}k(k+1)|\ k \in P \setminus \{a_i, b_i\}$, where $k-1, k+1$ are taken modulo $p\}$.

Note that for an integer $i \in [q]$, $C_1(Y_i)$ can be partitioned into two 2-element pairable sets $\{\underline{s_i}a_ia_i, \underline{s_i}a_ib_i\}$ and $\{\underline{s_i}b_ia_i, \underline{s_i}b_ib_i\}$, and $C_2(Y_i)$ can be partitioned into $p-2$ two-element pairable sets $\{\underline{s_i}k(k-1), \underline{s_i}k(k+1)\}$, where $k \in P \setminus \{a_i, b_i\}$. It is clear that the set of the heads of these p sets is $\{\underline{s_i}a|\ a \in P\}$. Moreover, by our assumption, since $\underline{s_i} \neq \underline{s_j}$ for distinct integers i, j, $\underline{s_i}a \neq \underline{s_j}b$ for any $a, b \in P$ (even when $a = b$). This proves the lemma. □

In what follows, for an integer $m \geq 2$ and a pairable set $Y \subseteq P$, let $C^m(Y) = C(C^{m-1}(Y))$.

Lemma 2.3. *Let $Y_1 = \{1, 2\} \subseteq V(S_p^1)$, and $Y_m = C^{m-1}(Y_1)$ for $m \in [n] \setminus \{1\}$. Then Y_m is a pairable set with* $\text{head}(Y_m) = P^{m-1}$, $|Y_m| = 2p^{m-1}$ *and $S_p^m[Y_m]$ is a forest for each $m \in [n]$.*

Proof. First of all, since $Y_1 = \{1, 2\}$ is a pairable set, by Lemma 2.2, Y_2 is pairable. By definition,

$$Y_2 = C(Y_1) = \{11, 12, 21, 22\} \cup \{k(k-1), k(k+1) \mid k \in P \setminus \{1, 2\}\}.$$

Note that $\text{head}(Y_2) = \{0, 1, \ldots, p-1\} = P$ and $|Y_2| = 2p$. One can see that $S_p^2[Y_2]$ is a linear forest consisting of two paths, $11, 12, 21, 22$ and $32, 34, 43, 45, \cdots, (p-1)(p-2), (p-1)0, 0(p-1), 01$.

Assume that $m \geq 3$ and the lemma is true for lesser values of m. By the induction hypothesis, Y_{m-1} is a pairable set with $\text{head}(Y_{m-1}) = P^{m-2}$ and $|Y_{m-1}| = 2p^{m-2}$. Since $Y_m = C(Y_{m-1})$, by Lemma 2.2, Y^m is pairable, and

$$\text{head}(Y_m) = \{sa|\ a \in \text{head}(Y_{m-1}), a \in P\} = P^{m-1}.$$

Thus $|Y_m| = 2\,\text{head}(Y_m) = 2|P^{m-1}| = 2p^{m-1}$.

Next we show that $S_p^m[Y_m]$ is a forest of S_p^m. Let $Y_m = Y_{m1} \cup Y_{m2}$, where $Y_{m1} = C_1(Y_{m-1}), Y_{m2} = C_2(Y_{m-1})$.

Claim 2.4. $E_{S_p^m}[Y_{m1}, Y_{m2}] = \emptyset$.

Proof. Suppose not, and let $y_1 \in Y_{m1}$ and $y_2 \in Y_{m2}$ with $y_1y_2 \in E(S_p^m)$. By the definition of Y_{m1} and Y_{m2}, there exist blocks $\{\underline{s_1}a, \underline{s_1}b\}$ and $\{\underline{s_2}c, \underline{s_2}d\}$ of Y_{m-1} such that $y_1 \in C_1(\{\underline{s_1}a, \underline{s_1}b\})$ and $y_2 \in C_2(\{\underline{s_2}c, \underline{s_2}d\})$, where $\underline{s_1}, \underline{s_2} \in P^{m-2}$ and $a, b, c, d \in P$. Recall that $C_1(\{\underline{s_1}a, \underline{s_1}b\}) = \{\underline{s_1}aa, \underline{s_1}ab, \underline{s_1}ba, \underline{s_1}bb\}$ and $C_2(\{\underline{s_2}c, \underline{s_2}d\}) = \{\underline{s_2}k(k-1), \underline{s_2}k(k+1)|\ k \in P \setminus \{c, d\}\}$.
Case 1: $y_2 = \underline{s_2}k(k-1)$ for an element $k \in P \setminus \{c, d\}$.

Since $N_{S_p^m}(y_2) = \{\underline{s_2}(k-1)k\} \cup \{\underline{s_2}kc|\ c \in P \setminus \{k-1\}\}$, $y_1 \in \{\underline{s_1}aa, \underline{s_1}ab,$
$\underline{s_1}ba, \underline{s_1}bb\}$, and $y_1y_2 \in E(S_p^m)$, we have $(\{\underline{s_2}(k-1)k\} \cup \{\underline{s_2}kc|\ c \in P \setminus \{k-1\}\}) \cap \{\underline{s_1}aa, \underline{s_1}ab, \underline{s_1}ba, \underline{s_1}bb\} \neq \emptyset$. It follows that $\underline{s_1} = \underline{s_2}$. Moreover, since Y_{m-1} is pairable, the heads of different blocks are distinct. So, $\{\underline{s_1}a, \underline{s_1}b\} = \{\underline{s_2}c, \underline{s_2}d\}$ and thus $\{a, b\} = \{c, d\}$.

Now rewrite $N_{S_p^m}(y_2) = \{\underline{s_1}(k-1)k\} \cup \{\underline{s_1}kc|\ c \in P \setminus \{k-1\}\}$, where $k \in P \setminus \{a, b\}$. However, since $k \in P \setminus \{a, b\}$, it is clear that $N_{S_p^m}(y_2) \cap \{\underline{s_1}aa, \underline{s_1}ab, \underline{s_1}ba, \underline{s_1}bb\} = \emptyset$, a contradiction, implying $y_1y_2 \notin E(S_p^m)$.
Case 2: $y_2 = \underline{s_2}k(k+1)$ for an element $k \in P \setminus \{c, d\}$.

Proof of this case is similar to that of Case 1.

□

Claim 2.5. *$S_p^m[Y_{m2}]$ is a forest.*

Proof. Recall that $Y_{m2} = C_2(Y_{m-1})$ and $C_2(\{\underline{s_i}a_i, \underline{s_i}b_i\}) = \{\underline{s_i}k(k+1), \underline{s_i}k(k - 1)\ |\ k \in P \setminus \{a_i, b_i\}\}$ for a block $\{\underline{s_i}a_i, \underline{s_i}b_i\}$. One can easily check that $S_p^m[C_2(\{\underline{s_i}a_i, \underline{s_i}b_i\})]$ is a linear forest with at most two components. If $\{\underline{s_j}a_j, \underline{s_j}b_j\}$ is another block of Y_{m-1}, then by $\underline{s_i}, \underline{s_j} \in P^{m-2}$ with $\underline{s_i} \neq \underline{s_j}$, $E_{S_p^m}[\{\underline{s_i}k(k-1), \underline{s_i}k(k+1)\}, \{\underline{s_j}l(l-1), \underline{s_j}l(l+1)\}] = \emptyset$ for any $k \in P \setminus \{a, b\}$ and $l \in P \setminus \{c, d\}$. So, $E_{S_p^m}[C_2(s_ia_i, s_ib_i), C_2(\underline{s_j}a_j, \underline{s_j}b_j)] = \emptyset$. Thus $S_p^m[Y_{m2}]$ is a forest. □

Claim 2.6. *$S_p^m[Y_{m1}]$ is a forest.*

Proof. Suppose that C is an induced cycle of $S_p^m[Y_{m1}]$, and let $S = \{\underline{s}|\ \underline{s} \in P^{m-2}$ and there is a vertex $v \in V(C)$ such that $v \in C_1(\{\underline{s}a, \underline{s}b\})\}$. Since

$C_1(\{\underline{s}a, \underline{s}b\}) = \{\underline{s}aa, \underline{s}ab, \underline{s}ba, \underline{s}bb\}$ for a block $\{\underline{s}a, \underline{s}b\}$ of Y_{m-1}, it is clear that $S_p^m[C_1(\{\underline{s}a, \underline{s}b\})] \cong P_4$. It is not hard to see that $V(C) = \bigcup_{\underline{s}\in S} C_1(\{\underline{s}a, \underline{s}b\})$. Let $C' = S_p^{m-1}[\cup_{\underline{s}\in S}\{\underline{s}a, \underline{s}b\}]$. One can see that C' is a cycle of $S_p^{m-1}[Y_{m-1}]$, a contradiction. This proves that $S_p^m[Y_{m1}]$ contains no cycle. □

Summing up Claims 2.4–2.6, we conclude that $S_p^m[Y_m]$ is a forest. □

Theorem 2.7. *For integer $p \geq 2$, $n \geq 1$, $\tau(S_p^n) = p^{n-1}(p-2)$.*

Proof. By Lemma 2.3, $f(S_p^n) \geq 2p^{n-1}$, and thus $\tau(S_p^n) \leq p^n - f(S_p^n) \leq p^{n-1}(p-2)$. On the other hand, by Lemma 2.1, $\tau(S_p^n) \geq p^{n-1}(p-2)$. The result then follows. □

In [8], two kinds of regularization of Sierpiński graphs were introduced.

Definition 2.8 ([8])**.** *For $p \in N$ and $n \in N$, the graph $^+S_p^n$ is defined by*

$$V(^+S_p^n) = P^n \cup \{w\}, E(^+S_p^n) = E(S_p^n) \cup \{\{w, i^n\} |\ i \in P\}.$$

Lemma 2.9 ([8])**.** *If $p \in N$ and $n \in N$, then $|^+S_p^n| = p^n + 1$ and $\|^+S_p^n\| = p(p^n+1)/2$.*

Definition 2.10 ([8])**.** *For $p \in N$ and $n \in N$, the graph $^{++}S_p^n$ is defined by*

$$V(^{++}S_p^n) = P^n \cup \{p\bar{s} |\ \bar{s} \in P^{n-1}\},$$
$$E(^{++}S_p^n) = E(S_p^n) \cup \{\{p\bar{s}, p\bar{t}\} |\ \{\bar{s}, \bar{t}\} \in E(S_p^{n-1})\} \cup \{\{pi^{n-1}, i^n\} |\ i \in P\}.$$

Lemma 2.11 ([8])**.** *If $p \in N$ and $n \in N$, then $|^{++}S_p^n| = (p+1)p^{n-1}$ and $\|^{++}S_p^n\| = \frac{p+1}{2}p^n$.*

Drawings of $^+S_4^2$ and $^{++}S_4^2$ are shown in Fig. 3, where the left one is $^+S_4^2$ and the other one is $^{++}S_4^2$.

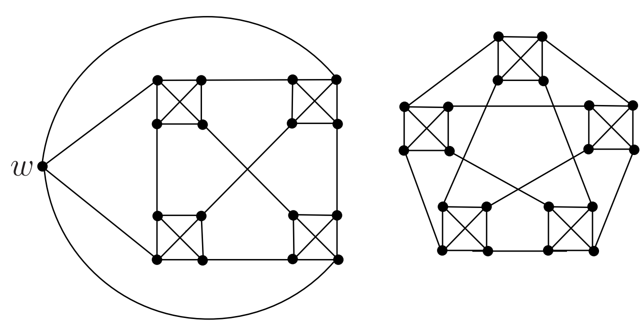

FIGURE 3. $^+S_4^2$ and $^{++}S_4^2$

Corollary 2.12. *For integers $p \geq 2$ and $n \geq 1$,*

$$\tau(^+S_p^n) = \begin{cases} p^{n-1}(p-2), & \text{if } p \geq 3 \\ 1, & \text{if } p = 2. \end{cases}$$

Proof. If $p = 2$, ${}^{+}S_p^n$ is a cycle, and thus $\tau({}^{+}S_p^n) = 1$. Now let us consider the case when $p \geq 3$. Since S_p^n is a subgraph of ${}^{+}S_p^n$, $\tau({}^{+}S_p^n) \geq \tau(S_p^n) = p^{n-1}(p-2)$. On the other hand, let Y_n be the set as in the proof of Lemma 2.3. Let $Y_n' = (Y_n \setminus \{1^n\}) \cup \{1^{n-1}0, w\}$. It can be checked that ${}^{+}S_p^n[Y_n']$ is a forest of ${}^{+}S_p^n$ with $|Y_n'| = 2p^{n-1} + 1$. Thus $\tau({}^{+}S_p^n) \leq p^{n-1}(p-2)$. The result follows. □

Corollary 2.13. *For integers $p \geq 2$ and $n \geq 1$, $\tau({}^{++}S_p^n) = \tau(S_p^n) + \tau(S_p^{n-1})$.*

Proof. By the definition of ${}^{++}S_p^n$, clearly, $\tau({}^{++}S_p^n) \geq \tau(S_p^n) + \tau(S_p^{n-1})$. Next, we prove $\tau({}^{++}S_p^n) \leq \tau(S_p^n) + \tau(S_p^{n-1})$. Let $Y_n = C^{n-1}(Y_1)$ be the set defined as in the proof of Lemma 2.3, where $Y_1 = \{1, 2\}$. On the other hand, we choose $Y_1^* = \{3, 4\}$, and $Y_{n-1}^* = C^{n-2}(Y_1^*)$ as defined in the proof of Lemma 2.3. By the proof of Lemma 2.3, ${}^{++}S_p^n[Y_n] \cong S_p^n[Y_n]$ is a forest, and ${}^{++}S_p^n[pY_{n-1}^*] \cong S_p^n[Y_{n-1}^*] \cong S_p^n[Y_{n-1}]$ is a forest. Moreover, $E_{{}^{++}S_p^n}[Y_n, pY_{n-1}^*] = \emptyset$. Thus ${}^{++}S_p^n[Y_n \cup pY_{n-1}^*]$ has no cycles, and $\tau({}^{++}S_p^n) \leq \tau(S_p^n) + \tau(S_p^{n-1})$. □

3. Sierpiński triangle graphs

A class of graphs that often has been mistaken for and also been called Sierpiński graphs can be obtained from the latter by simply contracting all nonclique edges. We will call them Sierpiński triangle graphs. Let $T := \{0, 1, 2\}$ and $\hat{T} := \{\hat{0}, \hat{1}, \hat{2}\}$. Denote the vertex obtained by contracting the edge $\{\underline{s}ij^{n-v+1}, \underline{s}ji^{n-v+1}\}$ by $\underline{s}\{i, j\}$. Then the vertex set of $\hat{S}_3^n$ can be written as

$$V(\hat{S}_3^n) = \hat{T} \cup \left\{ \underline{s}\{i, j\} |\ \underline{s} \in T^{v-1}, v \in [n], \{i, j\} \in \binom{T}{2} \right\}.$$

Further, we replace each vertex $\underline{s}\{i, j\}$ by $\underline{s}k$, where $k = 3 - i - j$. So, $E(\hat{S}_3^{n+1}) = \{\{\hat{k}, k^n j\} |\ k \in T, j \in T \setminus \{k\}\} \cup \{\{\underline{s}k, \underline{s}j\} |\ \underline{s} \in T^n, \{j, k\} \in \binom{T}{2}\}$ $\cup \{\{\underline{s}(3 - i - j)i^{n-v}k, \underline{s}j\} |\ \underline{s} \in T^{v-1}, v \in [n], i \in T, j, k \in T \setminus \{i\}\}$.

The Sierpiński triangle graph $\hat{S}_3^3$ is shown in Fig. 4.

Lemma 3.1 ([14])**.** *If $p \in N$ and $n \in N_0$, then $|\hat{S}_p^n| = p(p^n + 1)/2$ and $||\hat{S}_p^n|| = \frac{p-1}{2} p^{n+1}$.*

Theorem 3.2. *(i) For any $n \geq 0$, each minimum feedback vertex set of $\hat{S}_3^n$ contains at most one vertex in $\{\hat{0}, \hat{1}, \hat{2}\}$,*

(ii) for $n \geq 0$, $\hat{S}_3^n$ has a minimum feedback vertex set A_n such that $|A_n \cap \{\hat{0}, \hat{1}, \hat{2}\}| = 1$,

(iii) for $n \geq 1$, $\tau(\hat{S}_3^n) = 3\tau(\hat{S}_3^{n-1}) - 1$,

(iv) $\tau(\hat{S}_3^n) = (3^n + 1)/2 = |\hat{S}_3^n|/3$.

Proof. The proof proceeds by induction on n. For convenience, let $a_n = \tau(\hat{S}_3^n)$ and $V_n = V(\hat{S}_3^n)$ for any integer $n \geq 0$. If $n = 0$, $\hat{S}_3^n \cong K_3$. Trivially,

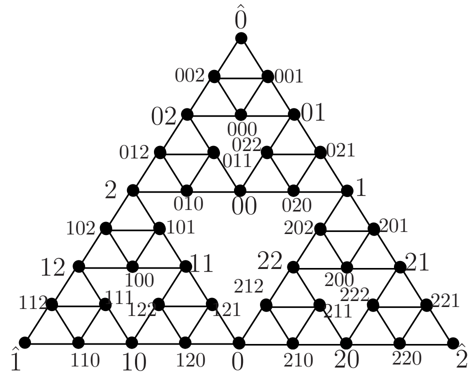

FIGURE 4. $\hat{S}_3^3$

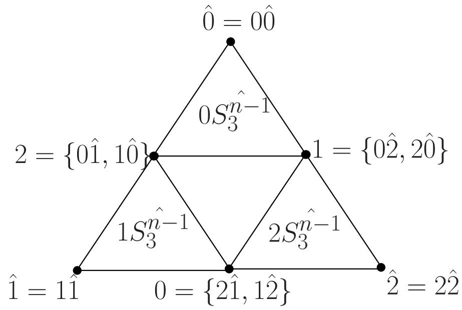

FIGURE 5. Relation between $\hat{S}_3^n$ and iS_3^{n-1} for each $i \in T$

$a_0 = 1$ and each minimum feedback vertex $\hat{S}_3^0$ consists of exactly one element of $\{\hat{0}, \hat{1}, \hat{2}\}$. Let $A_0 = \{\hat{0}\}$. Now assume that $n \geq 1$ and the results hold for $n - 1$.

First we prove (iii). By the induction hypothesis and the symmetry of $\hat{S}_3^{n-1}$, there exists a minimum feedback vertex set A_{n-1} of $\hat{S}_3^{n-1}$ containing the vertex $\hat{0}$. To prove $a_n \leq 3a_{n-1} - 1$, let $V_n^i = \{ia |\ a \in V_{n-1}\}$ for each $i \in T$, where $0\hat{0} = \hat{0}$, $1\hat{1} = \hat{1}$, and $2\hat{2} = \hat{2}$; $0\hat{1} = 2 = 1\hat{0}$, $0\hat{2} = 1 = 2\hat{0}$, and $2\hat{1} = 0 = 1\hat{2}$ (see Fig. 5 for an illustration).

One can see that $V_n = V_n^0 \cup V_n^1 \cup V_n^2$ and $\hat{S}_3^n[V_n^j] \cong \hat{S}_3^{n-1}$ for each $j \in T$. For each $j \in T$, we define an isomorphism f_j between $\hat{S}_3^{n-1}$ and $j\hat{S}_3^{n-1}$ as follows:

(1) $f_0(\hat{0}) = \hat{0}, f_0(\hat{1}) = 2, f_0(\hat{2}) = 1$;
(2) $f_1(\hat{0}) = 0, f_1(\hat{1}) = 2, f_1(\hat{2}) = \hat{1}$;
(3) $f_2(\hat{0}) = 0, f_2(\hat{1}) = \hat{2}, f_2(\hat{2}) = 1$.

Let

$$A_n = \bigcup_{j\in T} f_j(A_{n-1}). \tag{3.1}$$

It can be checked that A_n is a feedback vertex set of $\hat{S}_3^n$ with

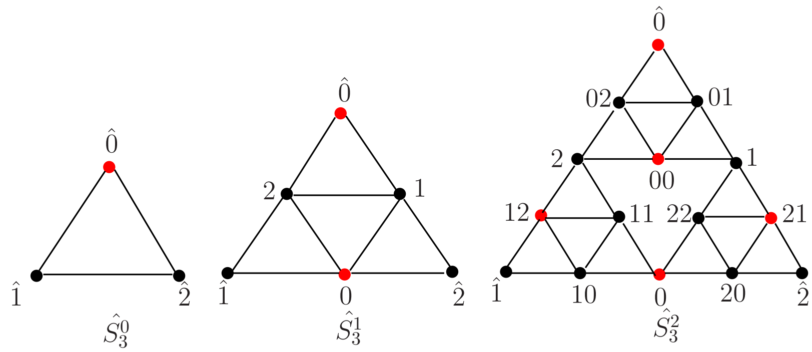

FIGURE 6. Minimum feedback vertex set A_n (red vertices) of $\hat{S}_3^n$ for $n \in \{0, 1, 2\}$ with property (ii)

$$|A_n| = 3a_{n-1} - 1. \tag{3.2}$$

So, $a_n \le 3a_{n-1} - 1$.

Next we prove $a_n \ge 3a_{n-1} - 1$. Let A_n be a minimum feedback vertex set of $\hat{S}_3^n$, and let $A_n^i = A_n \cap i\hat{S}_3^{n-1}$ for each $i \in T$. Since $i\hat{S}_3^{n-1} \cong \hat{S}_3^{n-1}$, A_n^i is a feedback vertex set of $i\hat{S}_3^{n-1}$. Hence, $|A_n^i| \ge a_{n-1}$. Note that $|A_n| = \sum_{i=0}^{2} |A_n^i| - |A_n \cap \{0, 1, 2\}|$. If $|A_n \cap \{0, 1, 2\}| \le 1$, then $a_n = |A_n| \ge 3a_{n-1} - 1$. Assume that $|\{0, 1, 2\} \cap A_n| = 2$, and without loss of generality, let $\{1, 2\} \subseteq A_n$. By the induction hypothesis (i), $|A_n^0| \ge a_{n-1} + 1$. Hence $a_n \ge a_{n-1} + 1 + a_{n-1} - 1 + a_{n-1} - 1 = 3a_{n-1} - 1$. If $|\{0, 1, 2\} \cap A_n| = 3$, by induction hypothesis (i), $|A_n^j| \ge a_{n-1} + 1$ for each $j \in \{0, 1, 2\}$. Hence $a_n \ge a_{n-1} + 1 + a_{n-1} + 1 + a_{n-1} + 1 - 3 = 3a_{n-1}$. This proves (iii).

Let A_n be the set constructed as (3.1) in the proof of (iii). By (3.2) and (iii), A_n is a minimum feedback vertex set with $\hat{0} \in A_n$ and $\hat{1} \notin A_n, \hat{2} \notin A_n$ (for an illustration, see Fig. 6). This proves (ii).

To prove (i), suppose that X_n is a minimum vertex set of $\hat{S}_3^n$ with $|X_n \cap \{\hat{0}, \hat{1}, \hat{2}\}| \ge 2$. Let $X_n^i = X_n \cap i\hat{S}_3^{n-1}$ for each $i \in T$. Since $i\hat{S}_3^{n-1} \cong \hat{S}_3^{n-1}$, X_n^i is a feedback vertex set of $i\hat{S}_3^{n-1}$. Hence, $|X_n^i| \ge a_{n-1}$. Note that $|X_n| = \sum_{i=0}^{2} |X_n^i| - |X_n \cap \{0, 1, 2\}|$.

If $|\{0, 1, 2\} \cap X_n| = 0$, $|X_n^j| \ge a_{n-1}$ for each $j \in \{0, 1, 2\}$. Hence $a_n \ge a_{n-1} + a_{n-1} + a_{n-1} = 3a_{n-1}$, contradicting (iii). If $|\{0, 1, 2\} \cap X_n| = 1$, by induction hypothesis (i), $|X_n^j| \geq a_{n-1}+1$ for some $j \in \{0,1,2\}$. Thus

$$a_n = |X_n| = \sum_{j=0}^{2} |X_n^j| - 1 \geq (a_{n-1}+a_{n-1}+a_{n-1}+1) - 1 = 3a_{n-1},$$

contradicting (iii). If $|\{0,1,2\} \cap X_n| = 2$, then at least two of $|X_n^1|, |X_n^2|, |X_n^3|$ are greater than or equal to $a_{n-1}+1$. Thus $a_n = |X_n| = \sum_{j=0}^{2} |X_n^j| - 2 \geq (a_{n-1}+a_{n-1}+1+a_{n-1}+1)-2 = 3a_{n-1}$, contradicting (iii). If $|\{0,1,2\} \cap X_n| = 3$, by the induction hypothesis (i), $|X_n^j| \geq a_{n-1}+1$ for each $j \in \{0,1,2\}$. Hence $a_n \geq a_{n-1}+1+a_{n-1}+1+a_{n-1}+1-3 = 3a_{n-1}$, contradicting (iii). This proves (i).

Finally we prove (iv). Since $a_0 = 1$ and by (iii), we have $\tau(\hat{S}_3^n) = (3^n + 1)/2$. Moreover, by Lemma 3.1, $\tau(\hat{S}_3^n) = |\hat{S}_3^n|/3$. □

4. Generalized Sierpiński triangle graphs $\hat{S}_p^n$ for $p \geq 4$

By Definition 1.2, for $p \in N$ and $n \in N_0$, the generalized Sierpiński triangle graph $\hat{S}_p^n$ is obtained by contracting all nonclique edges from the Sierpiński graph S_p^{n+1}. We denote the vertex obtained from contracting the edge $\{\underline{s}ij^{n-v+1}, \underline{s}ji^{n-v+1}\} \in E(S_p^{n+1})$ by $\underline{s}\{i,j\}$. Then

$$V(\hat{S}_p^n) = \hat{P} \cup \left\{ \underline{s}\{i,j\} |\ \underline{s} \in P^{v-1}, v \in [n], \{i,j\} \in \binom{P}{2} \right\},$$

where $\hat{P} = \{\hat{k} \mid k \in P\}$, and

$$\begin{aligned} E(\hat{S}_p^{n+1}) &= \{\{\hat{k}, k^n\{j,k\}\}|\ k \in P, j \in P \setminus \{k\}\} \\ &\cup \left\{ \{\underline{s}\{i,j\}, \underline{s}\{i,k\}\} |\ \underline{s} \in P^n, i \in P, \{j,k\} \in \binom{P \setminus \{i\}}{2} \right\} \\ &\cup \{\{\underline{s}ki^{n-v}\{i,j\}, \underline{s}\{i,k\}\}|\ \underline{s} \in P^{v-1}, v \in [n], i \in P, j,k \in P \setminus \{i\}\}. \end{aligned}$$

The Sierpiński triangle graphs $\hat{S}_6^2$ and $\hat{S}_5^2$ are shown in Fig. 7 and in Fig. 8, respectively. Trivially, $\hat{S}_p^0 \cong K_p$. Recall that $\tau(G) + f(G) = |G|$ for any graph G. For convenience, we consider $f(\hat{S}_p^n)$ instead of $\tau(\hat{S}_p^n)$. It is not hard to see that for $n \leq 2$, if p is even,

$$f(\hat{S}_p^n) = \begin{cases} 2, & \text{if } n = 0 \\ \frac{3p}{2}, & \text{if } n = 1 \\ p^2 + \frac{p}{2}, & \text{if } n = 2. \end{cases}$$

If p is odd,

$$f(\hat{S}_p^n) = \begin{cases} 2, & \text{if } n = 0 \\ \frac{3p-1}{2}, & \text{if } n = 1 \\ p^2 + \frac{p}{2} - \frac{1}{2}, & \text{if } n = 2. \end{cases}$$

In view of Section 3, it remains to consider the case when $p \geq 4$.

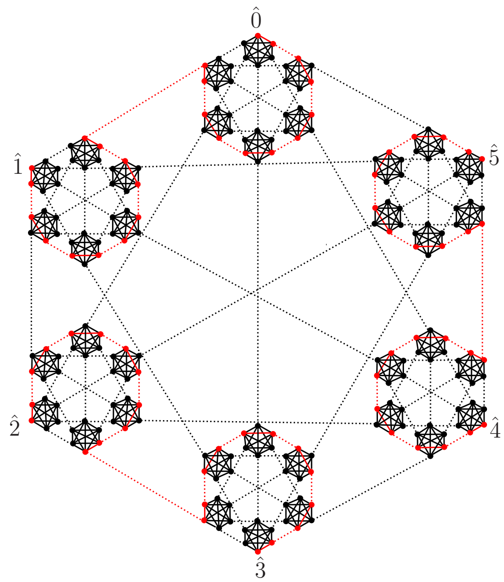

FIGURE 7. $\hat{S}_6^2$: a contraction of a dotted edge of S_6^3 represents a vertex of $\hat{S}_6^2$, $\hat{S}_6^2[B_2^*]$ is the subgraph colored red

Theorem 4.1. *For integers $n \geq 3$ and $p \geq 4$,*

$$f(\hat{S_p^n}) \geq \begin{cases} p^n - \frac{p^{n-1}}{8} + \frac{p^{n-2}+\cdots+p}{8} + \frac{5p}{8}, & \text{if } p \text{ is even} \\ p^n - \frac{p^{n-1}+p^{n-2}-5p+3}{8}, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Let $V_n = V(\hat{S}_p^n)$ and $iB_n^* = \{ib|\ b \in B_n^*\}$ for each $i \in P$, where $i\hat{i} = \hat{i}$ and $i\hat{j} = j\hat{i} = \{i, j\}$, $B_n^* \subseteq V_n$ will be defined recursively in terms of the parity of p as follows.

Case 1: p is even. Let $S = \{0, 2, \ldots, p-2\}$ and $A = \{\{s_1, s_2\}|\ s_1, s_2 \in S, s_1 \neq s_2\}$. Note that $A \subseteq V_n$.

For any $s \in S$, let $A_s^2 = \{\hat{s}, s\hat{+}1, \{s, s+1\}, s\{i, i+1\}, (s+1)\{i, i+1\}|\ i \in P \setminus \{s\}\} \subseteq V_2$. It is easy to see that $\hat{S}_p^2[A_s^2]$ is a path of length $2p$ connecting $\hat{s}$ and $s\hat{+}1$. For any $k \geq 3$, some subsets of V_k are defined by $A_s^k = sA_s^{k-1} \cup (s+1)A_s^{k-1}$, and further

$$B'_k = \bigcup_{s_1, s_2 \in S, s_1 \neq s_2} (s_1 A_{s_2}^{k-1} \cup (s_1+1)A_{s_2}^{k-1} \cup s_2 A_{s_1}^{k-1} \cup (s_2+1)A_{s_1}^{k-1}),$$

$$B_k = \bigcup_{s_1, s_2 \in S, s_1 \neq s_2} (s_1 A_{s_2}^{k-1} \cup (s_1+1)A_{s_2}^{k-1} \cup s_2 A_{s_1}^{k-1} \cup (s_2+1)A_{s_1}^{k-1}) \setminus \{s_1, s_2\}),$$

$$B_k^0 = B_k, B_k^{n-k} = \{\underline{j}b|\ \underline{j} \in P^{n-k}, b \in B_k\}.$$

By the definition above, one can see that $B'_k = B_k \setminus A$, $\bigcup_{j\in P} jB_k^{n-1-k} = B_k^{n-k}$, and $A \cap (\bigcup_{s\in S} A_s^n) = \emptyset$, $A \cap B_k^{n-k} = \emptyset$. Let $B_2^* = \bigcup_{s\in S} A_s^2$. It can be seen that $\hat{S}_p^2[B_2^*]$ is a forest (see $\hat{S}_6^2[B_2^*]$ depicted in the color red in Fig. 7).

Claim 4.2. *For any $k \geq 3$, let $B_k^* = (\bigcup_{j\in P} jB_{k-1}^*) \setminus A$. Then*

(i) $B_n^* = (\bigcup_{s\in S} A_s^n) \cup (\bigcup_{3\leq k\leq n} B_k^{n-k})$,

(ii) for each $s \in S$, $\hat{S}_p^n[A_s^n]$ is a path of order $2^{n-1}p+1$ joining $\hat{s}$ and $\widehat{s+1}$,

(iii) for any $3 \leq k \leq n$, $\hat{S}_p^n[B_k^{n-k}]$ is a forest consisting of $p^{n-k}\binom{p/2}{2}$ paths of order $2^k p - 1$,

(iv) $\hat{S}_p^n[B_n^*]$ is a forest.

Proof. The proof proceeds by induction on n. First we prove the statements for $n = 3$. Since $B_2^* = \bigcup_{s\in S} A_s^2$,

$$\begin{aligned} B_3^* &= (\bigcup_{j\in P} jB_2^*) \setminus A = (\bigcup_{j\in P} j \bigcup_{s\in S} A_s^2) \setminus A \\ &= (\bigcup_{s\in S} (sA_s^2 \cup (s+1)A_s^2 \cup (\bigcup_{j\in P\setminus\{s,s+1\}} jA_s^2))) \setminus A \\ &= (\bigcup_{s\in S} (sA_s^2 \cup (s+1)A_s^2) \cup (\bigcup_{s\in S} \bigcup_{j\in P\setminus\{s,s+1\}} jA_s^2)) \setminus A \\ &= (\bigcup_{s\in S} A_s^3) \cup B'_3 \setminus A \\ &= \bigcup_{s\in S} A_s^3 \cup B_3, \end{aligned}$$

proving (i).

Now we prove (ii). Since $\hat{S}_p^3[A_s^3] = \hat{S}_p^3[sA_s^2 \cup (s+1)A_s^2]$, $\hat{S}_p^3[sA_s^2] \cong \hat{S}_p^3[(s+1)A_s^2] \cong \hat{S}_p^2[A_s^2]$. Recall that for a fixed $s \in S$, $\hat{S}_p^2[A_s^2]$ is path of order $2p+1$ connecting $\hat{s}$ and $\widehat{s+1}$. It follows that $\hat{S}_p^3[sA_s^2]$ and $\hat{S}_p^3[(s+1)A_s^2]$ are paths of order $2p+1$ joining $s\hat{s}$ and $s(\widehat{s+1})$, $(s+1)\hat{s}$ and $(s+1)(\widehat{s+1})$, respectively. Moreover, since $s\hat{s} = \hat{s}$, $(s+1)(\widehat{s+1}) = (\widehat{s+1})$ and $s(\widehat{s+1}) = (s+1)\hat{s} = \{s, s+1\}$, $\hat{S}_p^3[A_s^3]$ is a path connecting $\hat{s}$ and $\widehat{s+1}$ of order $2(2p+1) - 1 = 4p+1$.

To prove (iii), let us consider a pair of elements $s_1, s_2 \in S$ with $s_1 \neq s_2$. Since

$$\hat{S}_p^3[s_1 A_{s_2}^2] \cong \hat{S}_p^3[(s_1+1)A_{s_2}^2] \cong \hat{S}_p^3[s_2 A_{s_1}^2] \cong \hat{S}_p^3[(s_2+1)A_{s_1}^2] \cong \hat{S}_p^2[A_s^2],$$

$\hat{S}_p^3[s_1 A_{s_2}^2]$, $\hat{S}_p^3[(s_1+1)A_{s_2}^2]$, $\hat{S}_p^3[s_2 A_{s_1}^2]$, $\hat{S}_p^3[(s_2+1)A_{s_1}^2]$ are paths of order $2p+1$ joining $s_1\hat{s_2}$ and $s_1(\widehat{s_2+1})$, $(s_1+1)\hat{s_2}$ and $(s_1+1)(\widehat{s_2+1})$, $s_2\hat{s_1}$ and $s_2(\widehat{s_1+1})$, $(s_2+1)\hat{s_1}$ and $(s_2+1)(\widehat{s_1+1})$, respectively. Moreover, $s_1\hat{s_2} = s_2\hat{s_1} = \{s_1, s_2\}$, $s_1(\widehat{s_2+1}) = (s_2+1)\hat{s_1} = \{s_1, s_2+1\}$, $s_2(\widehat{s_1+1}) = (s_1 +$

$1)\hat{s_2} = \{s_1+1, s_2\}$, $(s_2+1)(\widehat{s_1+1}) = (s_1+1)(\widehat{s_2+1}) = \{s_1+1, s_2+1\}$. Hence $\hat{S}_p^3[s_1A_{s_2}^3 \cup (s_1+1)A_{s_2}^3 \cup s_2A_{s_1}^3 \cup (s_2+1)A_{s_1}^3]$ is a cycle of order $4(2p+1)-4 = 8p$, and thus $\hat{S}_p^3[s_1A_{s_2}^3 \cup (s_1+1)A_{s_2}^3 \cup s_2A_{s_1}^3 \cup (s_2+1)A_{s_1}^3] - \{s_1, s_2\}$ is a path of order $8p-1$.

By the different choices for $s_1, s_2 \in S$, we obtain the total number of $\binom{p/2}{2}$ such paths in $\hat{S}_p^3[B_3]$. This proves (iii).

By (i), (ii), and (iii), $\hat{S}_p^3[B_3^*]$ is a forest, proving (iv).

Now assume that $n \geq 4$ and the statements (i)–(iv) are true for smaller values of n. Since

$$B_{n-1}^* = (\cup_{s\in S} A_s^{n-1}) \cup (\cup_{3\leq k\leq n-1} B_k^{n-1-k})$$

and $B_n^* = (\cup_{j\in P} jB_{n-1}^*) \setminus A$, we have

$$\begin{aligned}
B_n^* &= (\bigcup_{j\in P}\bigcup_{s\in S} jA_s^{n-1}) \cup (\bigcup_{j\in P}\bigcup_{3\leq k\leq n-1} jB_k^{n-1-k}) \setminus A \\
&= \bigcup_{s\in S}(sA_s^{n-1} \cup (s+1)A_s^{n-1}) \cup (\bigcup_{j\in P\setminus\{s,s+1\}} jA_s^{n-1})) \\
&\quad \cup (\bigcup_{3\leq k\leq n-1} B_k^{n-k}) \setminus A \\
&= \bigcup_{s\in S}(sA_s^{n-1} \cup (s+1)A_s^{n-1}) \cup (\bigcup_{s\in S}\bigcup_{j\in P\setminus\{s,s+1\}} jA_s^{n-1}) \\
&\quad \cup (\bigcup_{3\leq k\leq n-1} B_k^{n-k}) \setminus A \\
&= \bigcup_{s\in S} A_s^n \cup (\bigcup_{3\leq k\leq n-1} B_k^{n-k}) \cup B_n' \setminus A \\
&= \bigcup_{s\in S} A_s^n \cup (\bigcup_{3\leq k\leq n-1} B_k^{n-k}) \cup B_n \\
&= \bigcup_{s\in S} A_s^n \cup (\bigcup_{3\leq k\leq n} B_k^{n-k}).
\end{aligned}$$

This proves (i).

Now we prove (ii). By the induction hypothesis (ii), $\hat{S}_p^{n-1}[A_s^{n-1}]$ is path connecting $\hat{s}$ and $\widehat{s+1}$ of order $2^{n-2}p+1$. Since $\hat{S}_p^n[sA_s^{n-1}] \cong \hat{S}_p^n[(s+1)A_s^{n-1}] \cong \hat{S}_p^{n-1}[A_s^{n-1}]$, $\hat{S}_p^n[A_s^n]$ is path connecting $\hat{s}$ and $\widehat{s+1}$ of order $2(2^{n-2}p+1)-1 = 2^{n-1}p+1$ (by a similar reason to the case when $n=3$). This proves (ii).

To prove (iii), let $3 \leq k \leq n-1$. Since

$$B_k^{n-k} = \{jb|\ j \in P^{n-k}, b \in B_k\} = \bigcup_{j\in p} jB_k^{n-1-k},$$

we have

$$\hat{S}_p^n[B_k^{n-k}] = \hat{S}_p^n[\bigcup_{j\in p} jB_k^{n-1-k}].$$

For any $j \in P$, $\hat{S}_p^n[jB_k^{n-1-k}] \cong \hat{S}_p^{n-1}[B_k^{n-1-k}]$. By the induction hypothesis (iii), there are $p^{n-1-k}\binom{\frac{p}{2}}{2}$ paths of order $2^k p-1$ in $\hat{S}_p^n[jB_k^{n-1-k}]$. Moreover, since $E_{\hat{S}_p^n}[iB_k^{n-1-k}, jB_k^{n-1-k}] = \emptyset$ for any $i, j \in P$ with $i \neq j$, there exists $pp^{n-1-k}\binom{\frac{p}{2}}{2}$ paths of order $2^k p - 1$ in $\hat{S}_p^n[B_k^{n-k}]$.

If $k = n$, $B_n^{n-n} = B_n = B'_n \setminus A$. Since $\hat{S}_p^n[s_1A_{s_2}^{n-1} \cup (s_1+1)A_{s_2}^{n-1} \cup s_2A_{s_1}^{n-1} \cup (s_2+1)A_{s_1}^{n-1}]$ is a cycle of order $4(2^{n-2}p+1)-4$ and $\hat{S}_p^n[s_1A_{s_2}^{n-1} \cup (s_1+1)A_{s_2}^{n-1} \cup s_2A_{s_1}^{n-1} \cup (s_2+1)A_{s_1}^{n-1} \setminus \{s_1, s_2\}]$ is a path of order $2^n p - 1$ in $\hat{S}_p^n[B_n]$. By the different choice for $s_1, s_2 \in S$, we obtain total number of $\binom{p/2}{2}$ such paths in $\hat{S}_p^n[B_n]$. This proves (iii).

By (i), (ii), and (iii), we conclude that $\hat{S}_p^n[B_n^*]$ is a linear forest, proving (iv). □

Claim 4.3. *For any $n \geq 3$,*

$$|B_n^*| = p \times |B_{n-1}^*| - \frac{p(p-1)}{2} - \binom{p/2}{2}. \tag{4.1}$$

Proof. Since $B_n^* = \bigcup_{j\in P} jB_{n-1}^* \setminus A$ and $s_1\hat{s_2} = s_2\hat{s_1} = \{s_1, s_2\}$, we have

$$|\bigcup_{j\in P} jB_{n-1}^*| = p \times |B_{n-1}^*| - \frac{p(p-1)}{2}.$$

Moreover, since $|A| = \binom{\frac{p}{2}}{2}$, $|B_n^*| = p \times |B_{n-1}^*| - \frac{p(p-1)}{2} - \binom{p/2}{2}$. □

By Claim 4.2 and Claim 4.3, if $n \geq 3$, $\hat{S}_p^n$ has an induced forest of order

$$p^n - \frac{p^{n-1}}{8} + \frac{p^{n-2} + \cdots + p}{8} + \frac{5p}{8}.$$

Case 2: p is odd. Let $S = \{0, 2, \cdots, p-3\}$ and $A = \{\{s_1, s_2\} | \ s_1, s_2 \in S, s_1 \neq s_2\}$. For any $s \in S$, let

$$A_s^2 = \{\hat{s}, \widehat{s+1}, \{s, s+1\}, s\{i, i+1\}, (s+1)\{i, i+1\} | \ i \in P \setminus \{s\}\} \subseteq V_2.$$

It is easy to see that $\hat{S}_p^2[A_s^2]$ is a path of length $2p$ connecting $\hat{s}$ and $s \hat{+} 1$. For any $k \geq 3$, we define some subsets of V_k as: $A_s^k = sA_s^{k-1} \cup (s+1)A_s^{k-1}$, and further

$$B'_k = \bigcup_{s_1,s_2\in S, s_1\neq s_2} (s_1A_{s_2}^{k-1} \cup (s_1+1)A_{s_2}^{k-1} \cup s_2A_{s_1}^{k-1} \cup (s_2+1)A_{s_1}^{k-1}),$$

$$B_k = \bigcup_{s_1,s_2\in S, s_1\neq s_2} (s_1A_{s_2}^{k-1} \cup (s_1+1)A_{s_2}^{k-1} \cup s_2A_{s_1}^{k-1} \cup (s_2+1)A_{s_1}^{k-1}) \setminus \{s_1, s_2\}),$$

$$B_k^0 = B_k, B_k^{n-k} = \{\underline{j}b \mid \underline{j} \in P^{n-k}, b \in B_k\}.$$

With the slight difference to the case when p is even, we define some more subsets of V_k as follows.

$$T' = \{\widehat{p-1}, (p-1)\{i, i+1\} \mid i \in P-1\},$$

$$T_k = \bigcup_{s\in S}((p-1)A_s^{k-1} \cup s(p-1)^{k-3}T' \cup (s+1)(p-1)^{k-3}T'),$$

$$T_k^0 = T_k, T_k^{n-k} = \{\underline{j}t \mid \underline{j} \in P^{n-k}, t \in T_k\}.$$

By the definition above, one can see that $B'_k = B_k \setminus A$, $\bigcup_{j\in P} jB_k^{n-1-k} = B_k^{n-k}$, $\bigcup_{j\in P} jT_k^{n-1-k} = T_k^{n-k}$, $A \cap (\bigcup_{s\in S} A_s^n) = \emptyset$, $A \cap B_k^{n-k} = \emptyset$, $A \cap T_k^{n-k} = \emptyset$, and $A \cap (p-1)^{n-2}T' = \emptyset$.

It is easy to see that $\hat{S}_p^2[T']$ is path of order p connecting $(p-1)\{0,1\}$ and $\widehat{p-1}$. Let $B_2^* = (\cup_{s\in S} A_s^2) \cup T'$. Trivially, $\hat{S}_p^2[B_2^*]$ is a linear forest (see Fig. 8 for $\hat{S}_5^2[B_2^*]$, depicted in the color red).

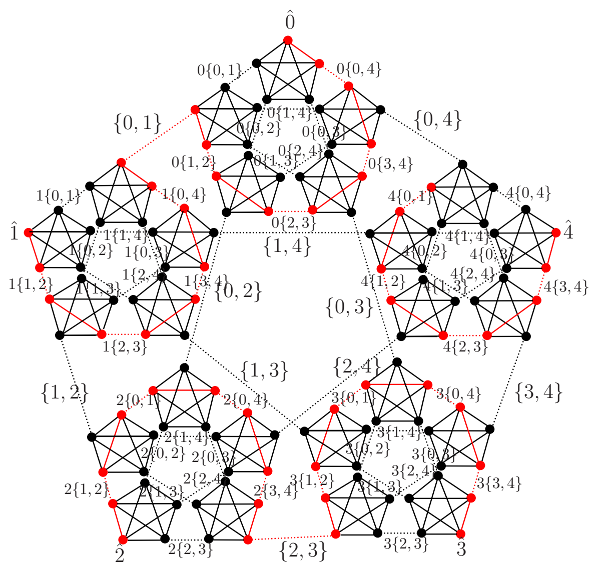

FIGURE 8. $\hat{S}_5^2$: contraction of a dotted edge of S_5^3 represents a vertex of $\hat{S}_5^2$, $\hat{S}_5^2[B_2^*]$ is the subgraph colored red

Claim 4.4. *For any* $k \geq 3$, *let* $B_k^* = \bigcup_{j \in P} jB_{k-1}^* \setminus A$. *Then*

(i) $B_n^* = (\bigcup_{s \in S} A_s^n) \cup (\bigcup_{3 \leq k \leq n} B_k^{n-k}) \cup (\bigcup_{3 \leq k \leq n} T_k^{n-k}) \cup (p-1)^{n-2}T'$,

(ii) for each $s \in S$, $\hat{S}_p^n[A_s^n]$ *is a path of order* $2^{n-1}p+1$ *joining* $\hat{s}$ *and* $\widehat{s+1}$,

(iii) for any $3 \leq k \leq n$, $\hat{S}_p^n[B_k^{n-k}]$ *is a forest consisting of* $p^{n-k}\binom{(p-1)/2}{2}$ *paths of order* $2^k p - 1$,

(iv) for any $3 \leq k \leq n$, $\hat{S}_p^n[T_k^{n-k}]$ *is a forest consisting of* $p^{n-k}(p-1)/2$ *paths of order* $2^{k-2}p + 2p - 1$,

(v) $\hat{S}_p^n[(p-1)^{n-2}T']$ *is a path of order* p,

(vi) $\hat{S}_p^n[B_n^*]$ *is a forest.*

Proof. The proof proceeds by induction on n. First we prove statements (i)–(vi) for $n = 3$. Since $B_2^* = (\bigcup_{s \in S} A_s^2) \cup T'$, we have

$$\begin{aligned} B_3^* &= (\bigcup_{j \in P} jB_2^*) \setminus A \\ &= (\bigcup_{j \in P} j(\bigcup_{s \in S} A_s^2 \cup T')) \setminus A \\ &= ((\bigcup_{j \in P \setminus \{p-1\}} j \bigcup_{s \in S} A_s^2) \setminus A) \cup (\bigcup_{j \in P} jT') \cup (\bigcup_{s \in S} (p-1)A_s^2) \\ &= (\bigcup_{s \in S} A_s^3) \cup B_3 \cup T_3 \cup (p-1)T'. \end{aligned}$$

The proof of (ii) and (iii) is similar to that of (ii) and (iii) for the case when p is even. Next we prove (iv). By the definition,

$$T_3 = \bigcup_{s \in S} ((p-1)A_s^2 \cup sT' \cup (s+1)T').$$

It can be seen that for a fixed $s \in S$,

- $\hat{S}_p^3[(p-1)A_s^2]$ is a path of order $2p+1$ joining $(p-1)\hat{s}$ and $(p-1)(\widehat{s+1})$,
- $\hat{S}_p^3[sT']$ is a path of order p joining $s(p-1)\{0,1\}$ and $s(\widehat{p-1})$,
- $\hat{S}_p^3[(s+1)T']$ is a path of order p joining $(s+1)(p-1)\{0,1\}$ and $(s+1)(\widehat{p-1})$.

Moreover, since $(p-1)\hat{s} = s(\widehat{p-1}) = \{s, p-1\}$, $(s+1)(\widehat{p-1}) = (p-1)(\widehat{s+1}) = \{s+1, p-1\}$, $\hat{S}_p^3[(p-1)A_s^2 \cup sT' \cup (s+1)T']$ is a path of order $2p+1+2p-2 = 4p-1$. So, by the different choices for $s \in S$, we obtain total number of $(p-1)/2$ such paths in $\hat{S}_p^3[T_3]$.

Now we show (v). Since $\hat{S}_p^2[T']$ is path of order p connecting $(p-1)\{0,1\}$ and $\widehat{p-1}$ and $(p-1)\widehat{p-1} = \widehat{p-1}$, $\hat{S}_p^3[(p-1)^{3-2}T']$ is path of order p connecting $(p-1)^2\{0,1\}$ and $\widehat{p-1}$.

By (i)–(vi), we conclude that $\hat{S}_p^3[B_3^*]$ is a linear forest.

Next assume that $n \geq 4$ and statements (i)–(v) are true for smaller values of n. It is clear that

$$\bigcup_{j\in P}\bigcup_{3\leq k\leq n-1} jB_k^{n-1-k} = \bigcup_{3\leq k\leq n-1} B_k^{n-k},$$

$$\bigcup_{j\in P}\bigcup_{3\leq k\leq n-1} jT_k^{n-1-k} = \bigcup_{3\leq k\leq n-1} T_k^{n-k}.$$

Since p is odd, $p-1$ is even, hence we have

$$(\bigcup_{j\in P\setminus\{p-1\}}\bigcup_{s\in S} jA_s^{n-2}) \setminus A = (\bigcup_{s\in S} A_s^n) \cup B_n.$$

Since

$$B_{n-1}^* = (\bigcup_{s\in S} A_s^{n-1}) \cup (\bigcup_{3\leq k\leq n-1} B_k^{n-1-k}) \cup (\bigcup_{3\leq k\leq n-1} T_k^{n-1-k}) \cup (p-1)^{n-3}T'$$

and $B_n^* = \bigcup_{j\in P} jB_{n-1}^* \setminus A$, we have

$$\begin{aligned}
B_n^* &= \bigcup_{j\in P}\bigcup_{s\in S} jA_s^{n-1} \cup \bigcup_{j\in P}\bigcup_{3\leq k\leq n-1} jB_k^{n-1-k} \cup \bigcup_{j\in P}\bigcup_{3\leq k\leq n-1} jT_k^{n-1-k} \\
&\quad \cup (\bigcup_{j\in P} j(p-1)^{n-3}T') \setminus A \\
&= (\bigcup_{j\in P\setminus\{p-1\}}\bigcup_{s\in S} jA_s^{n-1}) \setminus A \cup \bigcup_{3\leq k\leq n-1} B_k^{n-k} \cup \bigcup_{3\leq k\leq n-1} T_k^{n-k} \\
&\quad \cup \bigcup_{s\in S}(p-1)A_s^{n-1} \cup \bigcup_{j\in P} j(p-1)^{n-3}T' \\
&= \bigcup_{s\in S} A_s^n \cup B_n \cup \bigcup_{3\leq k\leq n-1} B_k^{n-k} \cup \bigcup_{3\leq k\leq n-1} T_k^{n-k} \cup \bigcup_{s\in S}(p-1)A_s^{n-1} \\
&\quad \cup \bigcup_{j\in P\setminus\{p-1\}} j(p-1)^{n-3}T' \cup (p-1)^{n-2}T' \\
&= \bigcup_{s\in S} A_s^n \cup \bigcup_{3\leq k\leq n} B_k^{n-k} \cup \bigcup_{3\leq k\leq n-1} T_k^{n-k} \cup T_k \cup (p-1)^{n-2}T' \\
&= \bigcup_{s\in S} A_s^n \cup \bigcup_{3\leq k\leq n} B_k^{n-k} \cup \bigcup_{3\leq k\leq n} T_k^{n-k} \cup (p-1)^{n-2}T'.
\end{aligned}$$

proving (i).

The proof of (ii) and (iii) is similar to that of (ii) and (iii) for the case when p is even.

To prove (iv), we first consider the case when $3 \leq k \leq n-1$. Since

$$T_k^{n-k} = \{\underline{j}b |\ \underline{j} \in P^{n-k}, b \in T_k\} = \bigcup_{j\in p} jT_k^{n-1-k},$$

we have

$$\hat{S_p^n}[T_k^{n-k}] = \hat{S_p^n}[\bigcup_{j\in p} jT_k^{n-1-k}].$$

For any $j \in P$, $\hat{S}_p^n[jT_k^{n-1-k}] \cong \hat{S}_p^n[T_k^{n-1-k}]$. By the induction hypothesis (iv), there are $p^{n-1-k}(p-1)/2$ paths of order $2^{k-2}p+2p-1$ in $\hat{S}_p^{n-1}[jT_k^{n-1-k}]$. Moreover, since $E_{\hat{S}_p^n}[iT_k^{n-1-k}, jT_k^{n-1-k}] = \emptyset$ for any $i, j \in P$ with $i \neq j$, there exists $p^{n-k}(p-1)/2$ paths of order $2^{k-2}p + 2p - 1$ in $\hat{S}_p^n[T_k^{n-k}]$.

If $k = n$, $T_n^{n-n} = T_n = \bigcup_{s\in S}((p-1)A_s^{n-1} \cup s(p-1)^{n-3}T' \cup (s+1)(p-1)^{n-3}T')$. Note that for a fixed $s \in S$,

- $\hat{S}_p^n[(p-1)A_s^{n-1}]$ is a path of order $2^{n-2}p+1$ joining $(p-1)\hat{s}$ and $(p-1)\widehat{(s+1)}$,
- $\hat{S}_p^n[s(p-1)^{n-3}T']$ is a path of order p joining $s(p-1)^{n-2}\{0,1\}$ and $s(p-1)^{n-3}\widehat{(p-1)} = s\widehat{(p-1)}$,
- $\hat{S}_p^n[(s+1)(p-1)^{n-3}T']$ is a path of order p joining $(s+1)(p-1)^{n-2}\{0,1\}$ and $(s+1)(p-1)^{n-3}\widehat{(p-1)} = (s+1)\widehat{(p-1)}$.

Furthermore, since $(p-1)\hat{s} = s\widehat{(p-1)} = \{s, p-1\}$, $(s+1)\widehat{(p-1)} = (p-1)\widehat{(s+1)} = \{s+1, p-1\}$, $\hat{S}_p^n[(p-1)A_s^{n-1} \cup s(p-1)^{n-3}T' \cup (s+1)(p-1)^{n-3}T']$ is a path of order $2^{n-2}p + 2p - 1$. By the different choices for $s \in S$, we obtain the total number of $(p-1)/2$ such paths in $\hat{S}_p^n[T_n]$. This proves (iv).

Since $\hat{S}_p^2[T']$ is a path of order p connecting $(p-1)\{0,1\}$ and $p\hat{-}1$. Moreover, since $(p-1)^{n-2}p\hat{-}1 = p\hat{-}1$, $\hat{S}_p^n[(p-1)^{n-2}T']$ is path of order p connecting $(p-1)^{n-1}\{0,1\}$ and $p\hat{-}1$, proving (v).

By (i)–(v), $\hat{S}_p^n[B_n^*]$ is a forest. This proves (vi). □

Claim 4.5. *For any $n \geq 3$,*

$$|B_n^*| = p \times |B_{n-1}^*| - \frac{p(p-1)}{2} - \binom{(p-1)/2}{2}, \tag{4.2}$$

Proof. Since $B_n^* = \bigcup_{j\in P} jB_{n-1}^* \setminus A$ and $s_1\hat{s_2} = s_2\hat{s_1} = \{s_1, s_2\}$, we have

$$|\bigcup_{j\in P} jB_{n-1}^*| = p \times |B_{n-1}^*| - \frac{p(p-1)}{2}.$$

Since $|A| = \binom{(p-1)/2}{2}$, $|B_n^*| = p \times |B_{n-1}^*| - p(p-1)/2 - \binom{(p-1)/2}{2}$. □

By Claim 4.4 and Claim 4.5, if $n \geq 3$, $\hat{S}_p^n$ has an induced forest of order

$$p^n - \frac{p^{n-1} + p^{n-2} - 5p + 3}{8}.$$

□

Since $\tau(G) + f(G) = |V(G)|$ for a graph G, Theorem 4.1 provides an upper bound for $\tau(\hat{S}_p^n)$. We suspect the upper bound is the exact value of the feedback vertex number of the generalized Sierpiński triangle graph $\hat{S}_p^n$ for the case when $p \geq 4$.

Conjecture. *For integers $n \geq 3$ and $p \geq 4$,*

$$f(\hat{S}_p^n) = \begin{cases} p^n - \frac{p^{n-1}}{8} + \frac{p^{n-2}+\cdots+p}{8} + \frac{5p}{8}, & \text{if } p \text{ is even} \\ p^n - \frac{p^{n-1}+p^{n-2}-5p+3}{8}, & \text{if } p \text{ is odd.} \end{cases}$$

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College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, Xinjiang, China
E-mail address: yuanli0920@163.com

College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, Xinjiang, China
E-mail address: baoywu@163.com

College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, Xinjiang, China
E-mail address: zhb_xj@163.com