

CIRCUIT PARTITIONS AND SIGNED INTERLACEMENT  
IN 4-REGULAR GRAPHS

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ABSTRACT. Let  $F$  be a 4-regular graph. Each circuit partition  $P$  of  $F$  has a corresponding touch-graph  $Tch(P)$ ; the circuits in  $P$  correspond to vertices of  $Tch(P)$ , and the vertices of  $F$  correspond to edges of  $Tch(P)$ . We discuss the connection between modified versions of the interlacement matrix of an Euler system of  $F$  and the cycle space of  $Tch(P)$ , over  $GF(2)$  and  $\mathbb{R}$ .

## 1. INTRODUCTION

This paper is concerned with the connection between two aspects of the structure of a 4-regular graph  $F$ : partitions of the edge set  $E(F)$  into circuits, and interlacement of vertices with respect to Euler systems of  $F$ . We begin our discussion by reviewing some relevant background and terminology.

The graphs we consider are unoriented multigraphs; loops and parallel edges are allowed. We think of every edge as consisting of two distinct half-edges, each half-edge incident on one vertex. The degree of a vertex is the number of incident half-edges, and a  $d$ -regular graph is one whose vertices all have degree  $d$ .

We use the term *circuit* for an undirected closed trail. A circuit cannot traverse an edge more than once, but it may traverse a vertex more than once. An Euler circuit is a circuit that includes every edge of a graph; a familiar argument shows that a 4-regular graph  $F$  has an Euler circuit if and only if  $F$  is connected. Every 4-regular graph  $F$  has an Euler system, i.e., a set that contains one Euler circuit for each connected component of  $F$ .

We use the term *circuit partition* for a partition of the edge set of a 4-regular graph into circuits (i.e., undirected closed trails). The idea of studying circuit partitions of 4-regular graphs was introduced by Kotzig [19], and developed further by Las Vergnas and Martin [20, 21, 22, 25]. Circuit

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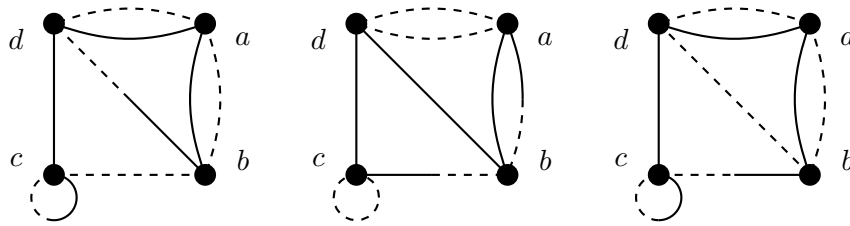


FIGURE 1. An Euler circuit  $C$ , a 3-element circuit partition  $P_1$  and a 2-element circuit partition  $P_2$ . To follow a circuit, maintain the same plain/dashed line status when traversing a vertex.

partitions in 4-regular graphs have found applications and generalizations in Kauffman's bracket description of the Jones polynomial [17], and in the interlace polynomials of Arratia, Bollobás and Sorkin [1, 2].

Some circuits in a 4-regular graph  $F$  are illustrated in Figure 1. On the left we see an Euler circuit,  $C$ . To trace  $C$ , start at any vertex and follow edges around  $F$ , making sure to maintain the plain/dashed line status when passing through a vertex. (The plain/dashed status may change in the middle of an edge.) For instance, if we follow  $C$  by starting at  $a$  and walking to  $b$  along the plain edge, we will encounter vertices in the order  $abdabccda$ . The same plain/dashed convention is used to indicate the circuits included in  $P_1$  and  $P_2$ . The circuits in  $P_1$  may be oriented to visit vertices in the orders  $abdcba$ ,  $ada$  and  $cc$ ; the circuits in  $P_2$  may be oriented to visit vertices in the orders  $abda$  and  $abccda$ .

*Note:* Recall that our circuits have neither preferred starting points nor preferred directions; for instance, the longer circuit of  $P_1$  might just as well be oriented to visit vertices in the order  $cdbabc$ .

Now, let  $F$  be an arbitrary 4-regular graph. A *transition* of  $F$  at a vertex  $v$  is a partition of the four half-edges incident at  $v$  into two pairs; for instance each part of Figure 1 indicates one transition at each vertex, with a pair of dashed half-edges and a pair of plain half-edges.  $F$  has three different transitions at each vertex. An Euler system  $C$  of  $F$  may be used to label the transitions of  $F$  in the following way. Temporarily choose an arbitrary orientation for each circuit included in  $C$ . Then for each vertex  $v \in V(F)$ , a person following the incident circuit of  $C$  makes two “entrances” to  $v$  and two “exits” from  $v$ ; say entrance 1 is followed by exit 1, and entrance 2 is followed by exit 2. The “entrances” and “exits” are the four half-edges of  $F$  incident at  $v$ . The transition that pairs entrance  $i$  with exit  $i$  for  $i \in \{1, 2\}$  is labeled  $\phi_C(v)$ ; the transition that pairs entrance  $i$  with exit  $j$  for  $i \neq j \in \{1, 2\}$  is labeled  $\chi_C(v)$ ; and the transition that pairs entrance 1 with entrance 2, and also pairs exit 1 with exit 2, is labeled  $\psi_C(v)$ . It is easy to see that each transition's label with respect to  $C$  remains the same if the orientation of a circuit of  $C$  is reversed.

If  $C$  and  $C'$  are different Euler systems of  $F$  then some transitions will have different  $\phi, \chi, \psi$  labels with respect to  $C$  and  $C'$ . For example, we leave it as an exercise for the reader to verify that in Figure 1 there is an Euler circuit  $C'$  of  $F$  with  $\phi_{C'}(a) = \psi_C(a)$ ,  $\phi_{C'}(b) = \chi_C(b)$ ,  $\phi_{C'}(c) = \psi_C(c)$  and  $\phi_{C'}(d) = \chi_C(d)$ . Moreover, only two of the twelve transitions in  $F$  have the same  $\phi, \chi, \psi$  labels with respect to  $C$  and  $C'$ .

It is easy to see that a circuit partition of  $F$  is completely determined by choosing one transition at each vertex. For example, in Figure 1  $P_1$  is determined by the transitions  $\psi_C(a), \phi_C(b), \chi_C(c)$  and  $\psi_C(d)$ , while  $P_2$  is determined by  $\phi_C(a), \chi_C(b), \phi_C(c)$  and  $\phi_C(d)$ .

The notion of interlacement with respect to Euler systems in 4-regular graphs has been studied by many authors; see for instance [6, 13, 29].

**Definition 1.1.** *If  $C$  is an Euler system of  $F$  then two vertices  $v \neq w \in V(F)$  are interlaced with respect to  $C$  if and only if there is a circuit of  $C$  on which  $v$  and  $w$  appear in the order  $vwvw$  or  $wvww$ . The interlacement matrix  $\mathcal{I}(C)$  is the  $V(F) \times V(F)$  matrix with entries in the 2-element field  $GF(2)$  given by: the  $vw$  entry is 1 if  $v$  and  $w$  are interlaced, and 0 otherwise.*

The fact that there is a connection between circuit partitions and interlacement has been discovered and rediscovered many times. Here is a statement that incorporates the versions of this connection that appear most often in the literature.

**Theorem 1.2.** *Suppose  $C$  is an Euler system of a 4-regular graph  $F$ , and  $P$  is a circuit partition of  $F$ . Let  $\mathcal{I}(C, P)$  be the symmetric  $GF(2)$ -matrix obtained from  $\mathcal{I}(C)$  by making these two kinds of changes.*

- (1) *If  $P$  involves the  $\phi_C(v)$  transition, remove the row and column corresponding to  $v$ .*
- (2) *If  $P$  involves the  $\psi_C(v)$  transition, change the  $vv$  entry to 1.*

*Then the  $GF(2)$ -nullity of  $\mathcal{I}(C, P)$  is  $|P| - c(F)$ , where  $|P|$  is the number of circuits in  $P$  and  $c(F)$  is the number of connected components in  $F$ .*

We refer to the formula  $|P| - c(F) = \text{nullity}(\mathcal{I}(C, P))$  as the *circuit-nullity formula*. It seems that the earliest discussion of some version of the formula appears in Brahana's 1921 study of curves on surfaces [10]. However the formula was not widely known until fifty years later, when a special case was discovered by Cohn and Lempel [13]. Both of these references state versions of the circuit-nullity formula which do not mention 4-regular graphs; Brahana refers to the connectivity of a surface and Cohn and Lempel refer to the number of orbits in a certain kind of permutation. Also, the version of Cohn and Lempel is restricted to oriented Euler circuits and circuit partitions; the  $\psi$  transitions are not relevant to the permutations they considered. Many other authors have rediscovered, refined or restated the circuit-nullity formula in various ways [3, 4, 9, 10, 13, 15, 16, 18, 23, 24, 26, 27, 30, 31, 32, 33, 34, 35, 37].

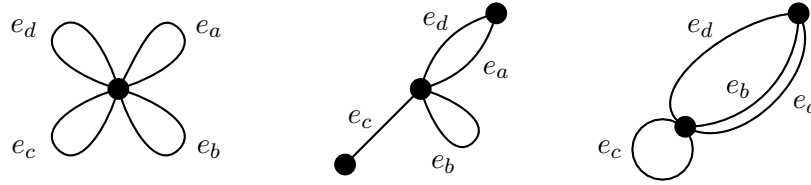


FIGURE 2. Touch-graphs from Figure 1.

We leave it as an exercise for the reader to confirm that the circuit-nullity formula holds in Figure 1, by calculating

$$\mathcal{I}(C, P_1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{I}(C, P_2) = (0).$$

Another important part of the theory of circuit partitions is the notion of a touch-graph. This notion appeared implicitly in the work of Jaeger [15], and explicitly in Bouchet's work on isotropic systems [7, 8].

**Definition 1.3.** *If  $P$  is a circuit partition in a 4-regular graph  $F$  then the touch-graph  $Tch(P)$  has a vertex  $v_\gamma$  for each circuit  $\gamma \in P$ , and an edge  $e_v$  for each vertex  $v \in V(F)$ ;  $e_v$  is incident on  $v_\gamma$  if and only if  $\gamma$  passes through  $v$ .*

The touch-graphs of the three circuit partitions of Figure 1 are pictured in Figure 2.

## 2. STATEMENT OF THE MAIN THEOREM

Two questions about the circuit-nullity formula should come to mind.

Question 1. Is there a version of the circuit-nullity formula that involves nullity over the reals instead of  $GF(2)$ ?

Answer 1. Yes, but the real version that has appeared in the literature is of limited generality. Brahana [10] discussed a skew-symmetric version of his matrix for systems of curves drawn on two-sided surfaces, suggesting a connection with topological orientability. Skew-symmetric versions of  $\mathcal{I}(C, P)$  have also been discussed by Bouchet [9], Jonsson [16], Lauri [23] and Macris and Pulé [24]. They all require that  $C$  and  $P$  be orientation-consistent, i.e.,  $P$  cannot involve any  $\psi_C$  transition.

Question 2. Does the equality  $\text{nullity}(\mathcal{I}(C, P)) = |P| - c(F)$  indicate a connection between  $P$  and the null space of  $\mathcal{I}(C, P)$ ?

Answer 2. Yes, but for full generality the connection involves a non-symmetric matrix in place of  $\mathcal{I}(C, P)$ . Building on earlier partial results [9, 15, 34], we introduced a modified form of  $\mathcal{I}(C, P)$  in [35], and showed that it is closely related to the touch-graph of  $P$ . This modified form of  $\mathcal{I}(C, P)$  is defined as follows.

**Definition 2.1.** ([35]) Let  $C$  be an Euler system of a 4-regular graph  $F$ , and  $P$  a circuit partition of  $F$ . Then the modified interlacement matrix  $M(C, P)$  is the  $V(F) \times V(F)$  matrix with entries in  $GF(2)$  obtained from  $\mathcal{I}(C)$  by making these two kinds of changes:

- (1) If  $P$  involves the  $\phi_C(v)$  transition, change the  $vv$  entry to 1, and change every other entry of the  $v$  column to 0.
- (2) If  $P$  involves the  $\psi_C(v)$  transition, change the  $vv$  entry to 1.

Observe that

$$M(C, P) = \begin{pmatrix} I & * \\ 0 & \mathcal{I}(C, P) \end{pmatrix},$$

where  $I$  is an identity matrix whose rows and columns correspond to the vertices of  $F$  where  $P$  involves the  $\phi_C$  transition. It follows that  $M(C, P)$  has the same nullity as  $\mathcal{I}(C, P)$ . The main theorem of [35] states that if we consider the rows of  $M(C, P)$  as elements of the vector space  $GF(2)^{E(Tch(P))}$  instead of  $GF(2)^{V(F)}$ , then the orthogonal complement of the row space of  $M(C, P)$  is the subspace spanned by the vertex cocycles of  $Tch(P)$ . (Recall that the cocycle of a vertex in a graph is the set of non-loop edges incident on that vertex.) To put it more simply: the row space of  $M(C, P)$  is the cycle space of  $Tch(P)$  over  $GF(2)$ .

As examples of this result from [35], consider that in Figure 1 we have

$$M(C, P_1) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M(C, P_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The row space of  $M(C, P_1)$  is generated by the first two rows, or equivalently, by  $e_a + e_d$  (the first row) and  $e_b$  (the difference between the first and second rows). The row space of  $M(C, P_2)$  is generated by the three nonzero rows, or equivalently, by  $e_a + e_b$ ,  $e_c$  and  $e_b + e_d$ . Consulting Figure 2, we see that these row spaces really do coincide with the cycle spaces of  $Tch(P_1)$  and  $Tch(P_2)$  over  $GF(2)$ .

Notice that the answers to Questions 1 and 2 are both of the form “Yes, but...” The second “but” is resolved over  $GF(2)$  by using the nonsymmetric matrix  $M(C, P)$  in place of the traditional (skew-)symmetric  $\mathcal{I}(C, P)$ . The purpose of the present paper is to observe that the first “but” is also resolved by using nonsymmetric matrices. In addition to determining the cycle space of  $Tch(P)$  rather than only the size of  $P$ , our result is more general than previously known versions of the circuit-nullity formula over  $\mathbb{R}$ ; there is no orientability requirement.

**Theorem 2.2.** Suppose  $C$  is an Euler system of a 4-regular graph  $F$ , and  $P$  is a circuit partition of  $F$ . Then there is a  $V(F) \times V(F)$  matrix  $M_{\mathbb{R}}(C, P)$  with integer entries, with these two properties.

- (1)  $M_{\mathbb{R}}(C, P)$  reduces to  $M(C, P)$  (modulo 2).
- (2) The row space of  $M_{\mathbb{R}}(C, P)$  is the cycle space of  $Tch(P)$  over  $\mathbb{R}$ .

If  $M_{\mathbb{R}}(C, P)$  satisfies Theorem 2.2, then  $M_{\mathbb{R}}(C, P)$  also satisfies the circuit-nullity formula over  $\mathbb{R}$ ; that is, the  $\mathbb{R}$ -nullity of  $M_{\mathbb{R}}(C, P)$  is  $|P| - c(F)$ . The reason is simple:  $M_{\mathbb{R}}(C, P)$  is a  $V(F) \times V(F)$  matrix whose rank is the dimension of the cycle space of  $Tch(P)$ ,

$$|E(Tch(P))| - |V(Tch(P))| + c(Tch(P)) = |V(F)| - |P| + c(Tch(P)).$$

Consequently the  $\mathbb{R}$ -nullity of  $M_{\mathbb{R}}(C, P)$  is  $|P| - c(Tch(P))$ . It is easy to prove that  $c(Tch(P)) = c(F)$ ; see Proposition 3.4 below.

Unless  $Tch(P)$  is a forest, there are infinitely many different matrices  $M_{\mathbb{R}}(C, P)$  which satisfy Theorem 2.2. For if  $M_{\mathbb{R}}(C, P)$  satisfies Theorem 2.2 and  $\rho$  is a nonzero row of  $M_{\mathbb{R}}(C, P)$ , then we may add  $\pm 2\rho$  to any row of  $M_{\mathbb{R}}(C, P)$  without disturbing either property specified in Theorem 2.2. Because of this nonuniqueness we will often refer to “an  $M_{\mathbb{R}}(C, P)$  matrix” rather than simply using the notation  $M_{\mathbb{R}}(C, P)$ .

Theorem 2.2 is proved in Section 3. In Section 4, we provide a standard form for  $M_{\mathbb{R}}(C, P)$ , denoted  $M_{\mathbb{R}}^0(C, P)$ . The standard form is defined using a signed version of  $C$ ; that is, for each  $v \in V(F)$ , one passage of a circuit of  $C$  through  $v$  is arbitrarily designated  $v^+$ , and the other is  $v^-$ . When  $C$  and  $P$  respect the same edge directions in  $F$ ,  $M_{\mathbb{R}}^0(C, P)$  is closely related to the skew-symmetric matrices used by Bouchet [9], Jonsson [16], Lauri [23] and Macris and Pulé [24]. Moreover, in this special case  $M_{\mathbb{R}}^0(C, P)$  has several attractive “naturalness” properties; for instance if  $C$  and  $C'$  are two Euler systems which respect the same edge directions then for each signed version of  $C$  there is a signed version of  $C'$  such that  $M_{\mathbb{R}}^0(C', C) = M_{\mathbb{R}}^0(C, C')^{-1}$ . The standard form does not have such nice properties in general. For instance, if  $C$  and  $C'$  are two Euler systems which do not respect the same edge directions, then  $M_{\mathbb{R}}^0(C, C')^{-1}$  may have fractional entries. An example of this type is presented in Section 5, along with a couple of other examples; one of them shows that in general we cannot require that  $M_{\mathbb{R}}^0(C, P)$  be skew-symmetric. In Section 6 we discuss the relationship between  $M_{\mathbb{R}}(C, P)$  and  $M_{\mathbb{R}}(C', P)$  matrices, where  $C$  and  $C'$  are Euler systems of  $F$ ; we also summarize the special features of the theory over  $GF(2)$ . In Sections 7 and 8 we discuss the special features of the orientation-consistent theory over  $\mathbb{R}$ , including the naturalness properties mentioned earlier in this paragraph. The paper ends with a brief account of the important result of Lauri [23] and Macris and Pulé [24], which gives a determinant formula for the number of Euler systems of  $F$  that respect the same edge directions as  $C$ .

Before proceeding to give details, we should mention that the present paper provides the foundation for algebraic characterizations of circle graphs using multimatroid properties analogous to matroid regularity [11, 12].

## 3. PROOF OF THE MAIN THEOREM

We begin with an elementary algebraic result. Let  $f : \mathbb{Z} \rightarrow GF(2)$  be the ring homomorphism with  $f(1) = 1$ . If  $G$  is a graph we obtain a homomorphism  $f : \mathbb{Z}^{E(G)} \rightarrow GF(2)^{E(G)}$  of abelian groups by applying  $f$  in each coordinate.

**Lemma 3.1.** *If  $S \subseteq \mathbb{Z}^{E(G)}$  then the rank of  $S$  in  $\mathbb{R}^{E(G)}$  is not less than the rank of  $f(S)$  in  $GF(2)^{E(G)}$ .*

*Proof.* As the rank is the cardinality of a maximal linearly independent subset, it is enough to show that if  $T \subseteq S$  and  $f(T)$  is linearly independent, then  $T$  is linearly independent too. Suppose instead that  $T$  is linearly dependent. Then there is a sum

$$\sum_{t \in T} q_t t = 0,$$

in which the coefficients  $q_t$  are real numbers, not all of which are 0. Eliminating irrational factors, we may presume the  $q_t$  are all rational; then multiplying by their denominators and dividing by the greatest common divisor, we may presume that the  $q_t$  are integers whose g.c.d. is 1. But then

$$\sum_{t \in T} f(q_t) f(t) = 0,$$

and the  $f(q_t)$  are not all 0. This contradicts the independence of  $f(T)$ .  $\square$

We take a moment to discuss our technical vocabulary. As mentioned in Section 1, we think of an edge in a graph as consisting of two distinct half-edges, each half-edge incident on one vertex. When we want to direct an edge, we designate one of its half-edges as initial, and the other as terminal. Notice that this convention provides every edge with two distinct directions, even if the edge is a loop.

A directed walk in a graph is a sequence  $W = v_1, h_1, h'_1, v_2, \dots, v_k, h_k, h'_k, v_{k+1}$  such that for each  $i$ ,  $h_{i+1}$  and  $h'_i$  are half-edges incident on  $v_{i+1}$ , and  $h_i$  and  $h'_i$  are the half-edges of an edge  $e_i$ . We consider the reversed sequence  $W' = v_{k+1}, h'_k, h_k, v_k, \dots, v_2, h'_1, h_1, v_1$  to define a different directed walk, even if  $k = 1$  and  $e_1$  is a loop. However,  $W$  and  $W'$  define the same undirected walk. When we say “ $W$  is a walk” without specifying that  $W$  is directed, we usually mean that  $W$  is undirected.

We take a moment to explain a special case. Suppose  $W = v_1, h_1, h'_1, v_2, \dots, v_k, h_k, h'_k, v_{k+1}$  is a directed walk with  $k > 1$ , and there is an index  $i$  such that  $e_i$  is a loop. Then a new directed walk may be obtained from  $W$  by interchanging  $h_i$  and  $h'_i$ . This new directed walk is distinct from  $W$  because directed walks are sequences of half-edges. These two directed walks do not differ by simple reversal, so they define distinct undirected walks.

A trail is a walk without repeated edges, i.e.,  $e_i \neq e_j$  when  $i \neq j \in \{1, \dots, k\}$ . A path is a trail without repeated vertices except possibly at the beginning and end, i.e.,  $v_i \neq v_j$  when  $i \neq j$  and  $\{i, j\} \neq \{1, k+1\}$ .

A walk is closed if  $v_1 = v_{k+1}$ . We consider two closed directed walks to be the same if they differ only by a cyclic permutation. That is, if  $v_1 = v_{k+1}$  then  $v_1, h_1, h'_1, v_2, \dots, v_k, h_k, h'_k, v_{k+1}$  and  $v_i, h_i, h'_i, v_{i+1}, \dots, h'_k, v_{k+1} = v_1, h_1, \dots, h'_{i-1}, v_i$  determine the same closed directed walk. A closed trail is a circuit. (Some references agree with this usage, but others use “circuit” only for a closed path.)

For notation and terminology regarding cycles and cocycles in graphs, we follow Bollobás [5, Section II. 3] for the most part. We refer the reader there for proofs. Here is a summary.

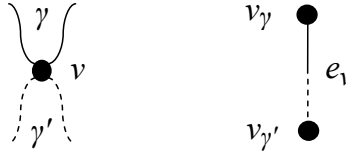
Suppose  $D$  is a directed version of a graph  $G$  and  $W$  is a directed walk in  $G$ . Let  $\mathbb{K}$  be a field, and  $\mathbb{K}^{E(G)}$  the vector space over  $\mathbb{K}$  with basis  $E(G)$ . There is a vector  $z_D(W) \in \mathbb{K}^{E(G)}$  determined by following  $W$  from beginning to end, and for each edge  $e \in E(G)$ , tallying  $+1$  in the  $e$  coordinate each time we pass through  $e$  in the  $D$  direction, and  $-1$  in the  $e$  coordinate each time we pass through  $e$  in the opposite direction. The *cycle space*  $Z_D(G)$  over  $\mathbb{K}$  is the subspace of  $\mathbb{K}^{E(G)}$  spanned by  $\{z_D(W) \mid W \text{ is a closed directed walk in } G\}$ . Also, if  $X \subseteq V(G)$  then there is an element  $u_D(X) \in \mathbb{K}^{E(G)}$  whose  $e$  coordinate, for each  $e \in E(G)$ , is  $+1$  if  $e$  is directed in  $D$  from a vertex in  $X$  to a vertex not in  $X$ ,  $-1$  if  $e$  is directed in  $D$  from a vertex not in  $X$  to a vertex in  $X$ , and  $0$  otherwise. The subspace of  $\mathbb{K}^{E(G)}$  spanned by  $\{u_D(X) \mid X \subseteq V(G)\}$  is the *cocycle space* of  $G$  over  $\mathbb{K}$ , denoted  $U_D(G)$ .

We recall seven properties of these spaces. (i) No special property is required of  $\mathbb{K}$ ; any field will do. (However we are primarily interested in  $\mathbb{K} = GF(2)$  or  $\mathbb{R}$ .) (ii) No special property is required of  $D$ ; any directed version of  $G$  yields spaces that correspond to all closed walks and all cocycles. (iii)  $Z_D(G)$  is spanned by the vectors  $z_D(W)$  such that  $W$  is a minimal directed circuit. (iv)  $U_D(G)$  is spanned by the vectors  $u_D(\{v\})$  such that  $v \in V(G)$ . (v) If  $G$  has  $c(G)$  connected components then the dimension of  $U_D(G)$  is  $|V(G)| - c(G)$ . (vi)  $U_D(G)$  and  $Z_D(G)$  are orthogonal complements. (We refer to this property as *cycle-cocycle duality*.) (vii) The orthogonality between  $U_D(G)$  and  $Z_D(G)$  rests on the simple observation that as we follow a closed directed walk, we must enter each subset  $X \subseteq V(G)$  the same number of times that we leave  $X$ . This simple observation goes back to the very beginning of graph theory, in Euler’s discussion of the seven bridges of Königsberg.

The machinery of cycle-cocycle duality may be summarized in matrix form, like this:

**Theorem 3.2.** *Given a spanning set  $S$  for  $Z_D(G)$ , let  $Z_S$  be the  $S \times E(G)$  matrix whose rows are the elements of  $S$ . Let  $U_{V(G)}$  be the  $E(G) \times V(G)$  matrix whose columns are the vectors  $u_D(\{v\})$ ,  $v \in V(G)$ . Then the rank of  $Z_S$  is  $|E(G)| - |V(G)| + c(G)$ , the rank of  $U_{V(G)}$  is  $|V(G)| - c(G)$ , and  $Z_S \cdot U_{V(G)} = 0$ .*

We may now restate Theorem 2.2 in the following equivalent form.

FIGURE 3.  $F$  is on the left, and  $Tch(P)$  is on the right.

**Theorem 3.3.** *Suppose  $C$  is an Euler system of a 4-regular graph  $F$ , and  $P$  is a circuit partition of  $F$ . Let  $D$  be a directed version of  $G = Tch(P)$ . Then there is a matrix  $M_{\mathbb{R}}(C, P)$  of integers, which has the following properties.*

- (1)  $M_{\mathbb{R}}(C, P)$  reduces (modulo 2) to  $M(C, P)$ .
- (2) In the notation of Theorem 3.2 with  $\mathbb{K} = \mathbb{R}$ ,  $Z_D(G)$  has a spanning set  $S$  such that  $Z_S = M_{\mathbb{R}}(C, P)$ .

Suppose now that  $F$  is a 4-regular graph. As mentioned in Section 1, if  $v \in V(F)$  then a transition at  $v$  is a partition of the four half-edges of  $F$  incident on  $v$  into two pairs. Each of the pairs is called a *single transition*. If  $P$  is a circuit partition of a 4-regular graph  $F$ , then  $P$  is determined by the choice of a transition  $P(v)$  at each vertex of  $F$ .

Recall that edges of  $Tch(P)$  correspond to vertices of  $F$  and vertices of  $Tch(P)$  correspond to circuits of  $F$ , as indicated in Figure 3. There is also a relationship between closed walks in  $F$  and closed walks in  $Tch(P)$ , which we proceed to describe.

As suggested in Figure 3, there is a natural 2-to-1 surjection

$$\pi_P : \{\text{half-edges of } F\} \rightarrow \{\text{half-edges of } Tch(P)\},$$

which we denote  $\pi_P(h) = \bar{h}$ . Suppose the four half-edges of  $F$  incident on  $v$  are  $h_v^1, h_v^2, h_v^3$  and  $h_v^4$ , and the two single transitions included in  $P(v)$  are  $\{h_v^1, h_v^2\}$  and  $\{h_v^3, h_v^4\}$ . Let  $\gamma \in P$  be the circuit that includes  $h_v^1$  and  $h_v^2$ , and let  $\gamma' \in P$  be the circuit that includes  $h_v^3$  and  $h_v^4$ . Then the two half-edges of  $e_v$  in  $Tch(P)$  are  $\bar{h}_v^1 = \bar{h}_v^2$  and  $\bar{h}_v^3 = \bar{h}_v^4$ . The half-edge  $\bar{h}_v^1 = \bar{h}_v^2$  is incident on the vertex  $v_\gamma \in V(Tch(P))$ , and the half-edge  $\bar{h}_v^3 = \bar{h}_v^4$  is incident on the vertex  $v_{\gamma'} \in V(Tch(P))$ .

This surjection  $\pi_P$  on half-edges induces a related surjection,

$$\pi_P : \{\text{closed directed walks in } F\} \rightarrow \{\text{closed directed walks in } Tch(P)\}.$$

Suppose  $W$  is the closed directed walk  $v_1, h_1, h'_1, v_2, \dots, v_k, h_k, h'_k, v_{k+1} = v_1$  in  $F$ . Then there are circuits  $\gamma_1, \dots, \gamma_k \in P$  such that  $\gamma_i$  includes the edge  $e_i \in E(F)$  whose half-edges are  $h_i$  and  $h'_i$ . Consider the list  $v_{\gamma_1}, \bar{h}_1', \bar{h}_2, v_{\gamma_2}, \dots, v_{\gamma_k}, \bar{h}_k', \bar{h}_1, v_{\gamma_1}$  of vertices and half-edges in  $Tch(P)$ . Each index  $i \in \{1, \dots, k\}$  is of one of the following three types. A type (a) index has  $\gamma_i \neq \gamma_{i+1}$ . In this case  $\bar{h}_i' \neq \bar{h}_{i+1}$  and  $e_{v_{i+1}} = \{\bar{h}_i', \bar{h}_{i+1}\}$  is a non-loop edge of  $Tch(P)$ . A type (b) index has  $\gamma_i = \gamma_{i+1}$ , and the single transition  $\{h'_i, h_{i+1}\}$  is excluded from  $P$ . In this case  $\bar{h}_i' \neq \bar{h}_{i+1}$  and  $e_{v_{i+1}} = \{\bar{h}_i', \bar{h}_{i+1}\}$  is a loop of

$Tch(P)$ . A type (c) index has  $\gamma_i = \gamma_{i+1}$ , and the single transition  $\{h'_i, h_{i+1}\}$  is included in  $P$ . In this case  $\overline{h'_i} = \overline{h_{i+1}}$  and the pair  $\{\overline{h'_i}, \overline{h_{i+1}}\}$  is not an edge of  $Tch(P)$ . We define  $\pi_P(W) = \overline{W}$  to be the closed directed walk in  $Tch(P)$  obtained from the list  $v_{\gamma_1}, \overline{h'_1}, \overline{h_2}, v_{\gamma_2}, \dots, v_{\gamma_k}, \overline{h'_k}, \overline{h_1}, v_{\gamma_1}$  by removing every subsequence  $\gamma_i, \overline{h_i}, \overline{h'_{i+1}}$  with  $i$  of type (c).

**Proposition 3.4.** *There is a one-to-one correspondence between connected components of  $F$  and  $Tch(P)$ :  $\{v_1, \dots, v_k\}$  is the vertex set of a connected component of  $F$  if and only if  $\{e_{v_1}, \dots, e_{v_k}\}$  is the edge set of a connected component of  $Tch(P)$ .*

*Proof.* As  $F$  is 4-regular, every connected component of  $F$  has an Euler circuit. Two vertices of  $F$  belong to the same connected component if and only if they appear on the same one of these Euler circuits. The images of these Euler circuits under  $\pi_P$  are closed walks in  $Tch(P)$ , and two vertices of  $Tch(P)$  belong to the same connected component if and only if they appear on the same one of these closed walks.  $\square$

**Definition 3.5.** *Suppose  $C$  is an Euler system for a 4-regular graph  $F$ , and  $v \in V(F)$ . Then the induced circuits of  $C$  at  $v$  are the two closed trails obtained by following a circuit of  $C$  from  $v$  to  $v$ . We denote them  $C_1(C, v)$  and  $C_2(C, v)$ , with the indices arbitrary.*

That is,  $\{C_1(C, v), C_2(C, v)\}$  is the circuit partition defined by  $\chi_C(v)$  and the transitions  $\phi_C(w)$ ,  $w \neq v$ . The crucial property of the induced circuits is this:

**Theorem 3.6.** *Let  $C$  be an Euler system for a 4-regular graph  $F$ , and let  $\Gamma$  be a set of induced circuits, which includes one of  $C_1(C, v), C_2(C, v)$  for each  $v \in V(F)$ . Choose either of the two directions for each  $\gamma \in \Gamma$ . Then for every circuit partition  $P$  of  $F$  and every choice of a digraph  $D$  on  $Tch(P)$ , the set  $S = \{z_D(\overline{\gamma}) \mid \gamma \in \Gamma\}$  spans the subspace  $Z_D(Tch(P))$  of  $\mathbb{R}^{E(Tch(P))}$ .*

*Proof.* Every  $\gamma \in \Gamma$  is a directed closed walk in  $F$ , so  $\overline{\gamma}$  is a directed closed walk in  $Tch(P)$ . Consequently  $S \subseteq Z_D(Tch(P))$ . To prove that  $S$  spans  $Z_D(Tch(P))$ , it is enough to prove that the rank of  $S$  is at least

$$\dim Z_D(Tch(P)) = |E(Tch(P))| - |V(Tch(P))| + c(Tch(P)).$$

Let  $f : \mathbb{Z} \rightarrow GF(2)$  be the map of Lemma 3.1. Notice that  $M(C, P)$  is a  $GF(2)$ -matrix whose rows are the elements  $f(s)$  with  $s \in S$ , so the circuit-nullity formula over  $GF(2)$  tells us that the nullity of  $f(S)$  is  $|P| - c(F) = |V(Tch(P))| - c(Tch(P))$ . As  $|S| = |V(F)| = |E(Tch(P))|$ , the rank of  $f(S)$  is  $|f(S)| - \text{nullity}(f(S)) = |E(Tch(P))| - |V(Tch(P))| + c(Tch(P))$ . The proof is completed by Lemma 3.1, which tells us that the rank of  $S$  is not less than the rank of  $f(S)$ .  $\square$

Theorem 3.6 tells us that if  $\Gamma$  contains one directed induced circuit for each vertex of  $F$ , then Theorems 2.2 and 3.3 are satisfied by the  $V(F) \times V(F)$  matrix whose rows are the vectors  $z_D(\overline{\gamma})$ ,  $\gamma \in \Gamma$ .

4. A STANDARD FORM FOR  $M_{\mathbb{R}}(C, P)$ 

In this section we describe an  $M_{\mathbb{R}}(C, P)$  matrix obtained by using particular choices in the construction of Section 3. With these choices, all entries of the matrix lie in  $\{-1, 0, 1, 2\}$ . Moreover in the special case involving orientation-consistent circuits, the matrix contains the  $\mathcal{I}(C, P)$  matrix used by Bouchet [9], Jonsson [16], Lauri [23] and Macris and Pulé [24]. More details about this special case are given in Section 8.

Let  $C$  be an Euler system of  $F$ . Arbitrarily choose preferred orientations for the circuits of  $C$ . For each  $v \in V(F)$ , let the half-edges of  $F$  incident on  $v$  be denoted  $h_v^1, h_v^2, h_v^3$  and  $h_v^4$  in such a way that the circuit of  $C$  incident on  $v$  is  $\dots h_v^1, v, h_v^2, \dots, h_v^3, v, h_v^4, \dots$ . As the incident circuit of  $C$  does not have a preferred starting point, the distinction between the two passages of  $C$  through  $v$  is arbitrary; we use  $+$  and  $-$  to distinguish them notationally: one passage is  $h_v^1, v^+, h_v^2$  and the other is  $h_v^3, v^-, h_v^4$ . Let  $D$  be the directed version of  $Tch(P)$  in which the initial half-edge of the edge  $e_v$  is  $\overline{h_v^1}$ . Index the induced circuits  $C_1(C, v), C_2(C, v)$  so that  $C_1(C, v)$  includes  $h_v^v$ , and choose the preferred orientation of  $C_1(C, v)$  consistent with the preferred orientation of the incident circuit of  $C$ . Let  $M_{\mathbb{R}}^0(C, P)$  be the  $V(F) \times V(F)$  matrix whose  $v$  row is  $z_D(\overline{C_1(C, v)})$ , for each vertex  $v$ .

A compact way to encode this information is to write  $C$  as a set of double occurrence words, one for each connected component of  $F$ , and for each vertex  $v$ , to designate which appearance is  $v^+$  and which is  $v^-$ . Then for each  $v \in V(F)$ , the  $v$  row of  $M_{\mathbb{R}}^0(C, P)$  is obtained by tallying the contributions of passages through the vertices encountered as we follow the double occurrence word representing the incident circuit of  $C$ , from  $v^-$  to  $v^+$ . We proceed to calculate the resulting entries  $M_{\mathbb{R}}^0(C, P)_{vw}$ .

Suppose  $v \in V(F)$ . The circuit  $C_1(C, v)$  includes the passage  $h_v^1, v, h_v^4$  and no other passage through  $v$ . If  $\phi_C(v) = P(v)$  then the initial half-edge of  $e_v$  is  $\overline{h_v^1} = \overline{h_v^2}$ , and the terminal half-edge is  $\overline{h_v^3} = \overline{h_v^4}$ , so  $\overline{C_1(C, v)}$  traverses  $e_v$  in the positive direction. If  $\chi_C(v) = P(v)$  then the initial half-edge of  $e_v$  is  $\overline{h_v^1} = \overline{h_v^4}$ , and the terminal half-edge is  $\overline{h_v^2} = \overline{h_v^3}$ , so  $\overline{C_1(C, v)}$  does not traverse  $e_v$ . If  $\psi_C(v) = P(v)$  then the initial half-edge of  $e_v$  is  $\overline{h_v^1} = \overline{h_v^3}$ , and the terminal half-edge is  $\overline{h_v^2} = \overline{h_v^4}$ , so  $\overline{C_1(C, v)}$  traverses  $e_v$  in the positive direction. We have the following.

$$M_{\mathbb{R}}^0(C, P)_{vv} = \begin{cases} 1, & \text{if } P(v) \in \{\phi_C(v), \psi_C(v)\} \\ 0, & \text{if } P(v) = \chi_C(v) \end{cases}$$

Now, suppose  $v \neq w \in V(F)$ . If  $\phi_C(w) = P(w)$  then any passage of  $C_1(C, v)$  through  $w$  contributes 0 to  $M_{\mathbb{R}}^0(C, P)_{vw}$ . If  $\chi_C(w) = P(w)$  then the initial half-edge of  $e_w$  is  $\overline{h_w^1} = \overline{h_w^4}$ , and the terminal half-edge is  $\overline{h_w^2} = \overline{h_w^3}$ . Consequently if  $\overline{C_1(C, v)}$  includes the passage  $\overline{h_w^1}, \overline{h_w^2}$  then this passage contributes 1 to  $M_{\mathbb{R}}^0(C, P)_{vw}$ ; and if  $\overline{C_1(C, v)}$  includes the passage  $\overline{h_w^3}, \overline{h_w^4}$  then this passage contributes  $-1$  to  $M_{\mathbb{R}}^0(C, P)_{vw}$ . If  $\psi_C(w) = P(w)$  then

the initial half-edge of  $e_w$  is  $\overline{h_w^1} = \overline{h_w^3}$ , and the terminal half-edge is  $\overline{h_w^2} = \overline{h_w^4}$ . Consequently if  $\overline{C_1(C, v)}$  includes the passage  $\overline{h_w^1}, \overline{h_w^2}$  then this passage contributes 1 to  $M_{\mathbb{R}}^0(C, P)_{vw}$ ; and if  $\overline{C_1(C, v)}$  includes the passage  $\overline{h_w^3}, \overline{h_w^4}$  then this passage also contributes 1 to  $M_{\mathbb{R}}^0(C, P)_{vw}$ . In sum, for  $v \neq w \in V(F)$  we have the following.

$$M_{\mathbb{R}}^0(C, P)_{vw} = \begin{cases} 0, & \text{if } v \text{ and } w \text{ lie in different connected components of } F \\ 0, & \text{if } \phi_C(w) = P(w) \\ 0, & \text{if } \chi_C(w) = P(w) \text{ and } v \text{ and } w \text{ are not interlaced with respect to } C \\ 1, & \text{if } \chi_C(w) = P(w) \text{ and a circuit of } C \text{ is } v^- \dots w^+ \dots v^+ \dots w^- \dots \\ -1, & \text{if } \chi_C(w) = P(w) \text{ and a circuit of } C \text{ is } v^- \dots w^- \dots v^+ \dots w^+ \dots \\ 1, & \text{if } \psi_C(w) = P(w) \text{ and } v \text{ and } w \text{ are interlaced with respect to } C \\ 0, & \text{if } \psi_C(w) = P(w) \text{ and a circuit of } C \text{ is } v^- \dots v^+ \dots w \dots w \dots \\ 2, & \text{if } \psi_C(w) = P(w) \text{ and a circuit of } C \text{ is } v^- \dots w \dots w \dots v^+ \dots \end{cases}$$

The reader will have no trouble verifying the following properties of  $M_{\mathbb{R}}^0(C, P)$ . Suppose we let  $V(F) = V_\phi \cup V_\chi \cup V_\psi$ , in such a way that  $v \in V_\alpha$  if and only if  $\alpha_C(v) = P(v)$ . Then  $M_{\mathbb{R}}^0(C, P)$  is

$$\begin{matrix} & V_\phi & V_\chi & V_\psi \\ \begin{matrix} V_\phi \\ V_\chi \\ V_\psi \end{matrix} & \begin{pmatrix} I & M_1 & M_2 \\ 0 & M_3 & M_4 \\ 0 & M_5 & M_6 \end{pmatrix}, \end{matrix}$$

where the indicated submatrices have the following properties.  $I$  is an identity matrix, the entries of  $M_1$  all lie in  $\{-1, 0, 1\}$ , and the entries of  $M_2$  all lie in  $\{0, 1, 2\}$ .  $M_3$  is a skew-symmetric matrix with entries in  $\{-1, 0, 1\}$ . (In the special case  $V_\psi = \emptyset$ ,  $M_3$  is the matrix  $\mathcal{I}(C, P)$  used by Bouchet [9] (when  $V_\phi$  is empty), Jonsson [16], Lauri [23] and Macris and Pulé [24].)  $M_4$  has entries from  $\{0, 1, 2\}$  and  $M_5$  has entries from  $\{-1, 0, 1\}$ . There is a limited symmetry connecting  $M_4$  and  $M_5$ : if the  $vw$  entry of  $M_4$  is 0 or 2 then the  $wv$  entry of  $M_5$  is 0; and if the  $vw$  entry of  $M_4$  is 1 then the  $wv$  entry of  $M_5$  is 1 or  $-1$ .  $M_6$  has diagonal entries equal to 1 and all other entries from  $\{0, 1, 2\}$ ; it reduces (mod 2) to a symmetric matrix. Interchanging the appearances of  $v^-$  and  $v^+$  on  $C$  produces three changes in  $M_{\mathbb{R}}^0(C, P)$ : if  $P(v) = \chi_C(v)$  then the  $v$  column of  $M_{\mathbb{R}}^0(C, P)$  is multiplied by  $-1$ ; if  $P(w) = \chi_C(w)$  then  $M_{\mathbb{R}}^0(C, P)_{vw}$  is multiplied by  $-1$ ; and if  $P(w) = \psi_C(w)$  then  $M_{\mathbb{R}}^0(C, P)_{vw}$  is changed by the replacement  $0 \leftrightarrow 2$ . Notice that all three changes have no effect modulo 2, reflecting the fact that  $M(C, P)$  is a uniquely defined matrix over  $GF(2)$ . Notice also that if  $P$  does not involve any  $\psi_C$  transition then the third kind of change does not occur, so the effect of interchanging  $v^-$  and  $v^+$  on  $C$  can be described using elementary row and column operations; this special case is detailed in Section 8.

## 5. FOUR EXAMPLES

Our first example illustrates the fact that if  $C$  and  $P$  do not respect the same edge directions, it may be that there is no skew-symmetric matrix that reduces to  $\mathcal{I}(C, P) \pmod{2}$  and has nullity  $|P| - c(F)$ .

Let  $F$  be the 4-regular graph with  $V(F) = \{a, b, c\}$  that is obtained from  $K_3$  by doubling edges. Then  $F$  has an Euler circuit described by the double occurrence word  $abcabc$ . We will use the standard form  $M_{\mathbb{R}}^0(C, P)$  corresponding to  $a^+b^-c^+a^-b^+c^-$ , and the natural notation for edges of  $F$ , e.g., the two edges connecting  $a$  to  $b$  are  $a^+b^- = b^-a^+$  and  $a^-b^+ = b^+a^-$ . Let  $P$  be the circuit partition that includes  $\gamma_1 = \{a^+b^-, a^-b^+\}$ ,  $\gamma_2 = \{a^+c^-, a^-c^+\}$  and  $\gamma_3 = \{b^+c^-, b^-c^+\}$ . Then  $Tch(P) \cong K_3$ . Let  $D$  be the oriented version of  $Tch(P)$  used in Section 4:  $e_a$  is directed from  $v_{\gamma_2}$  to  $v_{\gamma_1}$ ,  $e_b$  is directed from  $v_{\gamma_1}$  to  $v_{\gamma_3}$  and  $e_c$  is directed from  $v_{\gamma_3}$  to  $v_{\gamma_2}$ . Then  $Z_D(Tch(P))$  is spanned by the vector  $(1, 1, 1)$ .

$P$  involves the  $\psi_C$  transition at every vertex, so

$$\mathcal{I}(C, P) = M(C, P) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The  $GF(2)$ -nullity of  $M(C, P)$  is 2, as predicted by the circuit-nullity formula, and the rows of  $M(C, P)$  span the cycle space  $Z_D(Tch(P))$  over  $GF(2)$ .

It is a simple matter to check that every skew-symmetric version of  $M(C, P)$  is of nullity 0 or 1 over  $\mathbb{R}$ , so the circuit-nullity formula over  $\mathbb{R}$  is not satisfied by any skew-symmetric version of  $M(C, P)$ . However the definition of Section 4 yields

$$M_{\mathbb{R}}^0(C, P) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The nullity of  $M_{\mathbb{R}}^0(C, P)$  is 2, and the row space of  $M_{\mathbb{R}}^0(C, P)$  is  $Z_D(Tch(P))$ .

Our second example illustrates Theorem 2.2 for the standard form of Section 4. Let  $F$  be the simple 4-regular graph with  $V(F) = \{a, b, c, d, e, f, g, h\}$  and Euler circuit  $C$  given by the signed double occurrence word

$$e^-a^-b^-f^-e^+h^-g^-f^+a^+d^-h^+c^-b^+g^+c^+d^+.$$

Consider the circuit partition  $P$  that involves the  $\phi_C(a)$ ,  $\chi_C(e)$  and  $\chi_C(g)$  transitions, along with the  $\psi_C$  transition at every other vertex. Then  $P$  includes four circuits:  $\gamma_1 = \{ab, bc, cd, da, af, fe, ea\}$ ,  $\gamma_2 = \{bf, fg, gb\}$ ,  $\gamma_3 = \{ch, hg, gc\}$  and  $\gamma_4 = \{de, eh, hd\}$ . The construction of Section 4

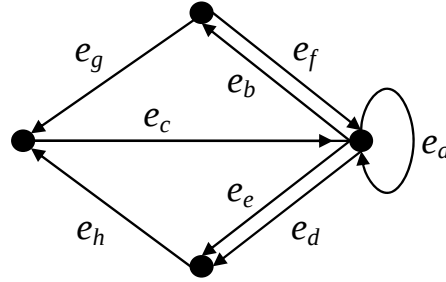


FIGURE 4. The directed touch-graph from the second example.

yields the directed version of  $Tch(P)$  illustrated in Figure 4, and the matrix

$$M_{\mathbb{R}}^0(C, P) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

It is not hard to see that the rows of  $M_{\mathbb{R}}^0(C, P)$  span the cycle space of  $Tch(P)$  over  $\mathbb{R}$ . Some rows represent individual circuits, like  $e_b + e_f$  (the fifth row) or  $e_e + e_f - e_g + e_h$  (the sixth row); other rows represent combinations of circuits, like  $e_a + (e_b + e_f) + (e_e + e_f - e_g + e_h)$  (the first row). Also,  $M_{\mathbb{R}}^0(C, P)$  reduces to  $M(C, P) \pmod{2}$ , and the product  $M_{\mathbb{R}}^0(C, P) \cdot U_{V(Tch(P))}$  is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = 0.$$

Notice that if we add  $-2$  times the third row of  $M_{\mathbb{R}}^0(C, P)$  to the fourth row, and add  $-2$  times the sixth row to each of the first two rows, then the resulting matrix has the same reduction  $\pmod{2}$  and the same row space as  $M_{\mathbb{R}}^0(C, P)$ , and its entries are all in  $\{-1, 0, 1\}$ . We do not know whether it is always possible to eliminate entries outside  $\{-1, 0, 1\}$  in this way.

Our third example involves two Euler circuits of  $K_5$ :  $C$  is given by the double occurrence word  $abdcaecbed$  and  $C'$  is given by the double occurrence word  $abcdecadbe$ . On the left below is the  $M_{\mathbb{R}}^0(C, C')$  matrix for the signed version  $a^-b^-d^-c^-a^+e^-c^+b^+e^+d^+$  of  $C$ ; its inverse appears on the right. (We

abuse notation slightly by writing  $M_{\mathbb{R}}^0(C, C')$  rather than  $M_{\mathbb{R}}^0(C, \{C'\})$ .)

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & -2 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

The inverse matrix is an  $M_{\mathbb{R}}(C', C)$  matrix, though it is not in standard form. The  $M_{\mathbb{R}}^0(C, C')$  matrix for the signed version  $a^-b^+d^+c^-a^+e^-c^+b^-e^+d^-$  of  $C$  is on the left below; its inverse appears on the right.

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}^{-1} = \left(\frac{1}{3}\right) \begin{pmatrix} -1 & -1 & 2 & 3 & -1 \\ 1 & 1 & -2 & 0 & 1 \\ -2 & -2 & 1 & 3 & 1 \\ 1 & -2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -3 & 1 \end{pmatrix}$$

In this case  $M_{\mathbb{R}}^0(C, C')^{-1}$  is not a matrix of integers, so it is certainly not an  $M_{\mathbb{R}}(C', C)$  matrix; but  $3 \cdot M_{\mathbb{R}}^0(C, C')^{-1}$  is an  $M_{\mathbb{R}}(C', C)$  matrix. Also,  $\det M_{\mathbb{R}}^0(C, C') = 3$  tells us that the rows of  $M_{\mathbb{R}}^0(C, C')$  generate a proper subgroup of  $\mathbb{Z}^{E(Tch(C'))}$ . Every edge of  $Tch(C')$  is a loop, though, so the cycle space of  $Tch(C')$  includes all of  $\mathbb{Z}^{E(Tch(C'))}$ .

Our fourth example includes  $C'$  and another Euler circuit  $C''$  of  $K_5$ , given by the double occurrence word  $abecdbcade$ . Using the signed form  $a^-b^+c^-d^+e^-c^+a^+d^-b^-e^+$  of  $C'$ , we obtain

$$M_{\mathbb{R}}^0(C', C'') = \begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

Using the signed form  $a^+b^-e^-c^-d^-b^+c^+a^-d^+e^+$  of  $C''$ , we obtain

$$M_{\mathbb{R}}^0(C'', C') = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} = M_{\mathbb{R}}^0(C', C'')^{-1}.$$

## 6. THE EFFECT OF A $\kappa$ -TRANSFORMATION

The fundamental operation of the theory of Euler systems of 4-regular graphs was introduced by Kotzig [19].

**Definition 6.1.** *If  $C$  is an Euler system of a 4-regular graph  $F$  and  $v \in V(F)$  then a  $\kappa$ -transform  $C * v$  is an Euler system obtained from  $C$  by reversing one of the induced circuits  $C_i(C, v)$  within a circuit of  $C$ .*

If  $C$  is given with preferred orientations for its circuits, then Definition 6.1 provides two choices for the preferred orientation of the circuit of  $C * v$  incident at  $v$ . For instance, if  $C$  is the directed Euler circuit of  $K_5$  given by the double occurrence word  $abdcaecbed$  then  $C * a$  is given by  $acdbaecbed$  or  $abdcadebce$ .

Kotzig [19] proved that if  $C$  and  $C'$  are any two Euler systems of  $F$  then there is a sequence  $v_1, \dots, v_k$  of vertices of  $F$  such that  $C' = C * v_1 * \dots * v_k$ . We refer to this fundamental result as *Kotzig's theorem*.

It is not hard to see that the effect of a  $\kappa$ -transformation on transition labels is given by the following.

**Proposition 6.2.** *Transition labels with respect to  $C$  and  $C * v$  differ only in these two ways.*

- $\phi_C(v) = \psi_{C*v}(v)$  and  $\psi_C(v) = \phi_{C*v}(v)$ .
- If  $w$  is interlaced with  $v$  then  $\chi_C(w) = \psi_{C*v}(w)$  and  $\psi_C(w) = \chi_{C*v}(w)$ .

Recall that if we are given  $C$  and  $P$ ,  $M(C, P)$  is the matrix over  $GF(2)$  specified in Definition 2.1. Proposition 6.2 implies the following three properties, which we describe collectively as “naturality” of  $M(C, P)$  with respect to  $\kappa$ -transformations. See [35] for a detailed discussion. (Special cases of the third property appear also in earlier work of Bouchet [9] and Jaeger [15].)

**Corollary 6.3.** ([35]) *If  $P$  is a circuit partition of  $F$  and  $C, C'$  are Euler systems of  $F$  then the following properties hold.*

- (1) *If  $v \in V(F)$  then  $M(C * v, P)$  is obtained from  $M(C, P)$  by adding the  $v$  row to the  $w$  row whenever  $w \neq v$  and  $w$  is interlaced with  $v$  on  $C$ .*
- (2)  $M(C', P) = M(C', C) \cdot M(C, P)$ .
- (3)  $M(C, C') = M(C', C)^{-1}$ .

*Proof.* The first assertion follows from Proposition 6.2.

For the second property recall that by Kotzig's theorem, there is a sequence  $v_1, \dots, v_k$  of vertices of  $F$  such that  $C' = C * v_1 * \dots * v_k$ . The first property tells us that this sequence of  $\kappa$ -transformations induces a corresponding sequence of elementary row operations, which transforms the double matrix

$$(I = M(C, C) \quad M(C, P))$$

into the double matrix

$$(M(C', C) \quad M(C', P)).$$

It follows that if  $E$  is the product of elementary matrices corresponding to the induced elementary row operations, then  $E \cdot I = M(C', C)$  and  $E \cdot M(C, P) = M(C', P)$ .

For the third property, notice that the second property tells us that  $I = M(C', C') = M(C', C) \cdot M(C, C')$ .  $\square$

Over  $\mathbb{R}$ , in contrast, we do not have a uniquely defined  $M_{\mathbb{R}}(C, P)$  matrix. Consequently the naturality properties of  $M_{\mathbb{R}}(C, P)$  over  $\mathbb{R}$  are less precise than the properties of Corollary 6.3.

**Corollary 6.4.** *If  $C$  and  $C'$  are Euler systems of  $F$  then the following properties hold.*

- (1) *Every  $M_{\mathbb{R}}(C, C')$  matrix is nonsingular, and has the property that*

$$(\det M_{\mathbb{R}}(C, C')) \cdot M_{\mathbb{R}}(C, C')^{-1}$$

*is an  $M_{\mathbb{R}}(C', C)$  matrix.*

- (2) *Let  $P$  be a circuit partition of  $F$ . Given an  $M_{\mathbb{R}}(C', C)$  matrix and an  $M_{\mathbb{R}}(C, P)$  matrix, the product*

$$M_{\mathbb{R}}(C', C) \cdot M_{\mathbb{R}}(C, P)$$

*is an  $M_{\mathbb{R}}(C', P)$  matrix.*

*Proof.* As  $M_{\mathbb{R}}(C, C')$  satisfies Theorem 2.2, it is a nonsingular matrix of integers that reduces to  $M(C, C') \pmod{2}$ ; it follows that  $\det M_{\mathbb{R}}(C, C')$  reduces to  $\det M(C, C') \pmod{2}$ . The circuit-nullity formula tells us that  $M(C, C')$  is a nonsingular  $GF(2)$ -matrix, so  $\det M_{\mathbb{R}}(C, C')$  is an odd integer. It follows that  $(\det M_{\mathbb{R}}(C, C')) \cdot M_{\mathbb{R}}(C, C')^{-1}$  is a nonsingular matrix of integers that reduces  $\pmod{2}$  to  $M(C, C')^{-1}$ . Corollary 6.3 tells us that  $M(C, C')^{-1} = M(C', C)$ , so  $(\det M_{\mathbb{R}}(C, C')) \cdot M_{\mathbb{R}}(C, C')^{-1}$  is an  $M_{\mathbb{R}}(C', C)$  matrix.

For the second property, notice that the nonsingularity of  $M_{\mathbb{R}}(C', C)$  implies that the row space of  $M_{\mathbb{R}}(C', C) \cdot M_{\mathbb{R}}(C, P)$  is the same as the row space of  $M_{\mathbb{R}}(C, P)$ . Corollary 6.3 tells us that  $M_{\mathbb{R}}(C', C) \cdot M_{\mathbb{R}}(C, P)$  reduces to  $M(C', P) \pmod{2}$ , so  $M_{\mathbb{R}}(C', C) \cdot M_{\mathbb{R}}(C, P)$  is an  $M_{\mathbb{R}}(C', P)$  matrix.  $\square$

Multiplying by  $\det M_{\mathbb{R}}(C, C')$  is necessary in part 1 because as we saw in Section 5, if  $|\det M_{\mathbb{R}}(C, C')| > 1$  then  $M_{\mathbb{R}}(C, C')^{-1}$  may have entries that are not integers.

## 7. THE EFFECT OF A TRANSPOSITION

In addition to  $\kappa$ -transformations, Kotzig [19] also defined “ $\varrho$ -transformations” on Euler systems. We follow Arratia, Bollobás and Sorkin [1, 2] and use a different name for this operation.

**Definition 7.1.** *If  $C$  is an Euler system of a 4-regular graph  $F$  and  $v, w \in V(F)$  are interlaced with respect to  $C$ , then the transposition  $C * (vw)$  is an Euler system obtained from  $C$  by interchanging the  $v$ -to- $w$  trails within a circuit of  $C$ .*

Several properties of transpositions are readily apparent. One property is that if the circuits of  $C$  are given with preferred orientations, then the circuits of  $C * (vw)$  inherit preferred orientations in a natural way. Also, the transpositions  $C * (vw)$  and  $C * (wv)$  are the same. Moreover,

a transposition can be effected by performing three  $\kappa$ -transformations: if  $C = vT_1wT_2vT_3wT_4$  then

$$\begin{aligned} ((C * v) * w) * v &= ((v\overleftarrow{T_2}w\overleftarrow{T_1}vT_3wT_4) * w) * v \\ &= (v\overleftarrow{T_2}w\overleftarrow{T_3}vT_1wT_4) * v = vT_3wT_2vT_1wT_4 = C * (vw), \end{aligned}$$

where  $\overleftarrow{T_i}$  indicates reversal of the trail  $T_i$ . Another property is that  $C$  and  $C * (vw)$  respect the same edge directions. In fact, Kotzig [19], Pevzner [28] and Ukkonen [36] proved that if  $C$  and  $C'$  are two Euler systems of  $F$ , then  $C$  and  $C'$  respect the same edge directions if and only if it is possible to obtain  $C'$  from  $C$  using transpositions.

It is not hard to see that the effect of a transposition on transition labels is given by the following.

**Proposition 7.2.** *If  $v$  and  $w$  are interlaced with respect to  $C$  then transition labels with respect to  $C$  and  $C * (vw)$  differ only in these ways:  $\chi_{C*(vw)}(v) = \phi_C(v)$ ,  $\phi_{C*(vw)}(v) = \chi_C(v)$ ,  $\chi_{C*(vw)}(w) = \phi_C(w)$  and  $\phi_{C*(vw)}(w) = \chi_C(w)$ .*

Despite the fact that a transposition's effect on transition labels is less complicated than the effect of a  $\kappa$ -transformation, Euler systems related through transpositions may give rise to  $M_{\mathbb{R}}^0$  matrices that are related in complicated ways. For example, the following Euler circuits of  $K_5$  yield the matrices below.

$$\begin{aligned} C &: a^-e^-c^+b^+d^+c^-a^+b^-e^+d^- \\ C * (cd) &: a^-e^-c^+a^+b^-e^+d^-c^-b^+d^+ \\ C' &: abcdecadbce \end{aligned}$$

$$\begin{aligned} M_{\mathbb{R}}^0(C, C') &= \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 \end{pmatrix} \\ M_{\mathbb{R}}^0(C * (cd), C') &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Notice that  $\det M_{\mathbb{R}}^0(C, C') = -3$  and  $\det M_{\mathbb{R}}^0(C * (cd), C') = -1$ , so although  $M_{\mathbb{R}}^0(C, C')$  and  $M_{\mathbb{R}}^0(C * (cd), C')$  are row equivalent over  $\mathbb{R}$ , they are not row equivalent over  $\mathbb{Z}$ .

## 8. THE ORIENTED CASE

In this section we show that in case  $C$  and  $P$  respect the same edge directions, the standard form  $M_{\mathbb{R}}^0(C, P)$  described in Section 4 has naturality properties over  $\mathbb{Z}$  that are very similar to the naturality properties of  $M(C, P)$  over  $GF(2)$ , stated in Corollary 6.3. Moreover,  $M_{\mathbb{R}}^0(C, P)$  includes the skew-symmetric signed interlacement matrices of Brahana [10], Bouchet [9], Jonsson [16], Lauri [23] and Macris and Pulé [24].

Suppose  $C$  is a directed Euler system of  $F$ , and the edges of  $F$  are directed consistently with the given directions for the circuits of  $C$ . These edge directions will remain fixed. If  $P$  is a circuit partition of  $F$ , then the circuits of  $P$  can be oriented consistently with the given edge directions if and only if  $P(v) \neq \psi_C(v) \forall v \in V(F)$ . Recall the notational scheme of Section 4: for each  $v \in V$ , one passage of  $C$  through  $v$  is  $h_v^1, v^+, h_v^2$  and the other is  $h_v^3, v^-, h_v^4$ . Let  $\mathcal{I}_{\mathbb{R}}(C)$  be the  $V(F) \times V(F)$  matrix whose diagonal entries all equal 0, and whose  $vw$  entry is given by:  $\mathcal{I}_{\mathbb{R}}(C)_{vw} = 1$  if  $v$  and  $w$  occur on  $C$  in the order  $v^+w^-v^-w^+$ ,  $\mathcal{I}_{\mathbb{R}}(C)_{vw} = -1$  if  $v$  and  $w$  occur on  $C$  in the order  $v^+w^+v^-w^-$ , and  $\mathcal{I}_{\mathbb{R}}(C)_{vw} = 0$  if  $v$  and  $w$  are not interlaced on  $C$ . Then

$$M_{\mathbb{R}}^0(C, P) = \begin{pmatrix} I & \mathcal{J}_{\mathbb{R}}(C, P) \\ 0 & \mathcal{I}_{\mathbb{R}}(C, P) \end{pmatrix},$$

where  $I$  is an identity matrix whose rows and columns correspond to vertices  $v \in V(F)$  with  $\phi_C(v) = P(v)$ ,  $\mathcal{I}_{\mathbb{R}}(C, P)$  is the submatrix of  $\mathcal{I}_{\mathbb{R}}(C)$  whose rows and columns correspond to vertices  $v \in V(F)$  with  $\chi_C(v) = P(v)$ , and  $\mathcal{J}_{\mathbb{R}}(C, P)$  is the submatrix of  $\mathcal{I}_{\mathbb{R}}(C)$  whose rows (resp. columns) correspond to vertices  $v \in V(F)$  with  $\phi_C(v) = P(v)$  (resp.  $\chi_C(v) = P(v)$ ).

Two properties of these matrices are apparent.

- Both  $\mathcal{I}_{\mathbb{R}}(C)$  and  $\mathcal{I}_{\mathbb{R}}(C, P)$  are skew-symmetric.
- If we interchange  $v^+$  and  $v^-$  on  $C$ , the effect on both  $\mathcal{I}_{\mathbb{R}}(C)$  and  $M_{\mathbb{R}}^0(C, P)$  is to multiply the  $v$  row and the  $v$  column by  $-1$ .

Some new notation will be useful. Suppose  $T$  is a sub-trail of a circuit of  $C$ . Let  $\phi_C(T) \in \mathbb{Z}^{V(F)}$  be the vector whose  $x$  coordinate, for each  $x \in V(F)$  with  $P(x) = \phi_C(x)$ , is obtained by tallying passages of  $T$  through  $x$ , with  $x^+$  contributing 1 and  $x^-$  contributing  $-1$ . If  $P(x) = \chi_C(x)$  then the  $x$  coordinate of  $\phi_C(T)$  is 0. Let  $\chi_C(T) \in \mathbb{Z}^{V(F)}$  be the vector obtained in the same way, but tallying contributions only with respect to those  $x$  with  $P(x) = \chi_C(x)$ . Also, for each vertex  $x \in V(F)$  let  $\rho_x(M_{\mathbb{R}}^0(C, P))$  denote the  $x$  row of  $M_{\mathbb{R}}^0(C, P)$ . The definition of  $M_{\mathbb{R}}^0(C, P)$  may now be rephrased as follows: if a circuit of  $C$  is  $x^-C_1(C, x)x^+C_2(C, x)$  then

$$\rho_x(M_{\mathbb{R}}^0(C, P)) = \chi_C(C_1(C, x)) + \phi_C(x^+).$$

Kotzig [19], Pevzner [28] and Ukkonen [36] proved that if  $C$  and  $C'$  are two Euler systems of  $F$ , then  $C$  and  $C'$  respect the same edge directions if and only if it is possible to obtain  $C'$  from  $C$  using transpositions. Consequently,

in order to describe the relationship between  $M_{\mathbb{R}}^0(C, P)$  and  $M_{\mathbb{R}}^0(C', P)$  it suffices to understand the relationship between  $M_{\mathbb{R}}^0(C, P)$  and  $M_{\mathbb{R}}^0(C * (vw), P)$ .

**Proposition 8.1.** *Suppose the edges of  $F$  are directed consistently with the circuits of  $C$ , and  $C$  includes a circuit  $v^+T_1w^+T_2v^-T_3w^-T_4$ . Consider the signed version of  $C * (vw)$  obtained from  $C$  by using  $v^+T_3w^-T_2v^-T_1w^+T_4$ . Let  $P$  be a circuit partition such that  $\psi_C(x) \neq P(x) \forall x \in V(F)$ . Then  $M_{\mathbb{R}}^0(C, P)$  and  $M_{\mathbb{R}}^0(C * (vw), P)$  are related through elementary row operations, as follows:*

- (1)  $\rho_v(M_{\mathbb{R}}^0(C * (vw), P)) = \rho_w(M_{\mathbb{R}}^0(C, P))$ .
- (2)  $\rho_w(M_{\mathbb{R}}^0(C * (vw), P)) = -\rho_v(M_{\mathbb{R}}^0(C, P))$ .
- (3) If  $x \in V(F) - \{v, w\}$  then

$$\begin{aligned} & \rho_x(M_{\mathbb{R}}^0(C * (vw), P)) \\ &= \rho_x(M_{\mathbb{R}}^0(C, P)) + \mathcal{I}_{\mathbb{R}}(C)_{xw}\rho_v(M_{\mathbb{R}}^0(C, P)) - \mathcal{I}_{\mathbb{R}}(C)_{xv}\rho_w(M_{\mathbb{R}}^0(C, P)). \end{aligned}$$

*Proof.* Property 1 follows from Proposition 7.2 and the rephrased definition of  $M_{\mathbb{R}}^0(C, P)$  given above:

$$\begin{aligned} \rho_v(M_{\mathbb{R}}^0(C * (vw), P)) &= \chi_{C*(vw)}(T_1w^+T_4) + \phi_{C*(vw)}(v^+) \\ &= \chi_C(T_1) + \chi_C(T_4) + \phi_C(w^+) + \chi_C(v^+) \\ &= \chi_C(T_4v^+T_1) + \phi_C(w^+) = \rho_w(M_{\mathbb{R}}^0(C, P)). \end{aligned}$$

The proof of Property 2 uses the fact that  $\sum_{i=1}^4 \chi_C(T_i) = 0$ :

$$\begin{aligned} \rho_w(M_{\mathbb{R}}^0(C * (vw), P)) &= \chi_{C*(vw)}(T_2v^-T_1) + \phi_{C*(vw)}(w^+) \\ &= \chi_C(T_2) + \chi_C(T_1) + \phi_C(v^-) + \chi_C(w^+) \\ &= -\chi_C(T_3) - \chi_C(T_4) - \phi_C(v^+) - \chi_C(w^-) \\ &= -\chi_C(T_3w^-T_4) - \phi_C(v^+) = -\rho_v(M_{\mathbb{R}}^0(C, P)). \end{aligned}$$

Property 3 has many cases, with  $x^-$  and  $x^+$  in various positions. We detail three cases, and leave the rest to the reader.

If  $x$  is not interlaced with either  $v$  or  $w$ , then  $C_1(C, x)$  and  $C_1(C * (vw), x)$  may not be the same trail, but they will involve the same passages through vertices, so  $\rho_x(M_{\mathbb{R}}^0(C, P)) = \rho_x(M_{\mathbb{R}}^0(C * (vw), P))$ .

Suppose  $x^-$  appears in  $T_1$  and  $x^+$  appears in  $T_2$ ; say  $T_1 = T_{11}x^-T_{12}$  and  $T_2 = T_{21}x^+T_{22}$ . Then  $\mathcal{I}_{\mathbb{R}}(C)_{xv} = 0$ ,  $\mathcal{I}_{\mathbb{R}}(C)_{xw} = 1$  and

$$\begin{aligned} & \rho_x(M_{\mathbb{R}}^0(C * (vw), P)) \\ &= \chi_{C*(vw)}(T_{12}w^+T_4v^+T_3w^-T_{21}) + \phi_{C*(vw)}(x^+) \\ &= \chi_C(T_{12}) + \chi_C(T_4) + \phi_C(v^+) + \chi_C(T_3) + \chi_C(T_{21}) + \phi_C(x^+) \\ &= \chi_C(T_{12}) + \chi_C(w^+T_{21}) - \chi_C(w^+) + \phi_C(x^+) + \chi_C(T_4) + \phi_C(v^+) + \chi_C(T_3) \\ &= \chi_C(T_{12}w^+T_{21}) + \phi_C(x^+) + \chi_C(T_3w^-T_4) + \phi_C(v^+) \\ &= \rho_x(M_{\mathbb{R}}^0(C, P)) + \rho_v(M_{\mathbb{R}}^0(C, P)). \end{aligned}$$

Suppose  $x^-$  appears in  $T_4$  and  $x^+$  appears in  $T_2$ ; say  $T_2 = T_{21}x^+T_{22}$  and  $T_4 = T_{41}x^-T_{42}$ . Then  $\mathcal{I}_{\mathbb{R}}(C)_{xv} = 1 = \mathcal{I}_{\mathbb{R}}(C)_{xw}$  and

$$\begin{aligned}
& \rho_x(M_{\mathbb{R}}^0(C * (vw), P)) \\
&= \chi_{C*(vw)}(T_{42}v^+T_3w^-T_{21}) + \phi_{C*(vw)}(x^+) \\
&= \chi_C(T_{42}) + \phi_C(v^+) + \chi_C(T_3) + \phi_C(w^-) + \chi_C(T_{21}) + \phi_C(x^+) \\
&= \chi_C(T_{42}v^+T_1w^+T_{21}) - \chi_C(v^+T_1w^+) + \phi_C(x^+) - \phi_C(w^+) + \phi_C(v^+) \\
&\quad + \chi_C(T_3) \\
&= \phi_C(x^+) + \chi_C(T_{42}v^+T_1w^+T_{21}) + \chi_C(w^-) - \chi_C(T_4v^+T_1) + \chi_C(T_4) \\
&\quad - \phi_C(w^+) + \phi_C(v^+) + \chi_C(T_3) \\
&= \rho_x(M_{\mathbb{R}}^0(C, P)) - \chi_C(T_4v^+T_1) - \phi_C(w^+) + \chi_C(T_3w^-T_4) + \phi_C(v^+) \\
&= \rho_x(M_{\mathbb{R}}^0(C, P)) - \rho_w(M_{\mathbb{R}}^0(C, P)) + \rho_v(M_{\mathbb{R}}^0(C, P)).
\end{aligned}$$

□

Proposition 8.1 uses the same set of elementary row operations to obtain  $M_{\mathbb{R}}^0(C * (vw), P)$  from  $M_{\mathbb{R}}^0(C, P)$ , for every circuit partition  $P$  with  $\psi_C(x) \neq P(x) \forall x \in V(F)$ . This lack of dependence on  $P$  leads to strong naturality properties, just as it does in the proof of Corollary 6.3. We believe these properties have not appeared in the literature, except for the special case of  $M_{\mathbb{R}}^0(C', C) = M_{\mathbb{R}}^0(C, C')^{-1}$  involving Euler circuits with  $\phi_C(v) \neq \phi_{C'}(v) \forall v \in V(F)$ , which is due to Bouchet [9].

**Corollary 8.2.** *Suppose  $C$  and  $C'$  are Euler systems of  $F$ , whose circuits are oriented consistently with the same edge directions. Then for each signed version of  $C$  there is a corresponding signed version of  $C'$  such that  $M_{\mathbb{R}}^0(C', C) = M_{\mathbb{R}}^0(C, C')^{-1}$ . Moreover if  $P$  is a circuit partition that respects the same edge directions, then these signed versions of  $C$  and  $C'$  have  $M_{\mathbb{R}}^0(C', P) = M_{\mathbb{R}}^0(C', C) \cdot M_{\mathbb{R}}^0(C, P)$ .*

*Proof.* According to the theorem of Kotzig [19], Pevzner [28] and Ukkonen [36] mentioned above, there is a sequence of transpositions that transforms a signed version of  $C$  into a signed version of  $C'$  using the sign convention of Proposition 8.1. Proposition 8.1 also gives us an induced sequence of elementary row operations, which transforms the double matrix

$$(I = M_{\mathbb{R}}^0(C, C) \quad M_{\mathbb{R}}^0(C, P))$$

into the double matrix

$$(M_{\mathbb{R}}^0(C', C) \quad M_{\mathbb{R}}^0(C', P)).$$

It follows that if  $E$  is the product of elementary matrices corresponding to the induced elementary row operations, then  $E \cdot I = M_{\mathbb{R}}^0(C', C)$  and  $E \cdot M_{\mathbb{R}}^0(C, P) = M_{\mathbb{R}}^0(C', P)$ . In particular, if  $P = C'$  we deduce that  $E \cdot I = M_{\mathbb{R}}^0(C', C)$  and  $E \cdot M_{\mathbb{R}}^0(C, C') = M_{\mathbb{R}}^0(C', C') = I$ . □

**Corollary 8.3.** *Let  $C$  and  $C'$  be Euler systems of  $F$ , whose circuits are oriented consistently with the same edge directions. Consider arbitrary signed versions of  $C$  and  $C'$ . Then there is a matrix  $\Delta$  with the following properties.*

- (1) *Every diagonal entry of  $\Delta$  is  $\pm 1$ , and every other entry of  $\Delta$  is 0.*
- (2)  $M_{\mathbb{R}}^0(C', C) = \Delta \cdot M_{\mathbb{R}}^0(C, C')^{-1} \cdot \Delta$ .
- (3) *If  $P$  is any circuit partition that respects the same edge directions, then  $M_{\mathbb{R}}^0(C', P) = M_{\mathbb{R}}^0(C', C) \cdot \Delta \cdot M_{\mathbb{R}}^0(C, P) \cdot \Delta$ .*

*Proof.* Let  $C''$  denote the signed version of  $C'$  that corresponds to the given signed version of  $C$ , as in Corollary 8.2. For any circuit partition  $P$  with  $P(x) \neq \psi_C(x) \forall x \in V(F)$ ,  $M_{\mathbb{R}}^0(C', P)$  is the matrix obtained from  $M_{\mathbb{R}}^0(C'', P)$  by multiplying by  $-1$  the row and column of  $M_{\mathbb{R}}^0(C'', P)$  corresponding to each  $x \in V(F)$  such that the positions of  $x^-$  and  $x^+$  in  $C'$  and  $C''$  are different. Consequently if  $\Delta$  is the diagonal matrix whose  $xx$  entry is 1 (resp.  $-1$ ) when the positions of  $x^-$  and  $x^+$  in  $C'$  and  $C''$  are the same (resp. different), then  $M_{\mathbb{R}}^0(C', P) = \Delta \cdot M_{\mathbb{R}}^0(C'', P) \cdot \Delta$ . Assertions (2) and (3) now follow from Corollary 8.2:

$$M_{\mathbb{R}}^0(C', C) = \Delta \cdot M_{\mathbb{R}}^0(C'', C) \cdot \Delta = \Delta \cdot M_{\mathbb{R}}^0(C, C'')^{-1} \cdot \Delta = \Delta \cdot M_{\mathbb{R}}^0(C, C')^{-1} \cdot \Delta$$

$$\begin{aligned} M_{\mathbb{R}}^0(C', P) &= \Delta \cdot M_{\mathbb{R}}^0(C'', P) \cdot \Delta \\ &= \Delta \cdot M_{\mathbb{R}}^0(C'', C) \cdot M_{\mathbb{R}}^0(C, P) \cdot \Delta \\ &= M_{\mathbb{R}}^0(C', C) \cdot \Delta \cdot M_{\mathbb{R}}^0(C, P) \cdot \Delta \end{aligned}$$

□

Lauri [23] and Macris and Pulé [24] gave a formula for the number of Euler systems of  $F$  that respect the same edge directions. We close with a quick explanation of this important result.

**Lemma 8.4.** *Suppose  $C$  and  $C'$  are Euler systems of  $F$ , whose circuits are oriented consistently with the same edge directions. Then for any signed versions of  $C$  and  $C'$ ,*

$$\det M_{\mathbb{R}}^0(C, C') = 1.$$

*Proof.* Suppose first that  $C' = C * (vw)$  and the signed versions of  $C$  and  $C'$  are related as in Proposition 8.1. Then Proposition 8.1 tells us how to obtain  $M_{\mathbb{R}}^0(C', C') = I$  from  $M_{\mathbb{R}}^0(C, C')$ . The determinant is not affected by the row operations of part 3 of Proposition 8.1, and the row operations of parts 1 and 2 – interchanging the  $v$  and  $w$  rows, and multiplying one of these rows by  $-1$  – both have the effect of multiplying the determinant by  $-1$ . We conclude that in this case  $\det M_{\mathbb{R}}^0(C, C') = \det I = 1$ .

If some other signed versions of  $C$  and  $C'$  are used, then the effect is to replace  $M_{\mathbb{R}}^0(C, C')$  with  $\Delta \cdot M_{\mathbb{R}}^0(C, C') \cdot \Delta$ , as in the proof of Corollary 8.3. As  $\det \Delta = \pm 1$ , this replacement does not change the determinant.

The general case follows from part 3 of Corollary 8.3 by induction, because  $C'$  can be obtained from  $C$  using a sequence of transpositions. □

**Corollary 8.5.** *Let  $C$  be an Euler system of  $F$ , and  $P$  a circuit partition with  $\psi_C(v) \neq P(v) \forall v \in V(F)$ . Then the following conditions are equivalent.*

- (1)  $P$  is an Euler system.
- (2)  $\det M_{\mathbb{R}}^0(C, P) = 1$ .
- (3)  $\det M_{\mathbb{R}}^0(C, P) \neq 0$ .
- (4)  $\det \mathcal{I}_{\mathbb{R}}(C, P) = 1$ .
- (5)  $\det \mathcal{I}_{\mathbb{R}}(C, P) \neq 0$ .

*Proof.* Lemma 8.4 gives us the implication  $1 \Rightarrow 2$ . The equality

$$M_{\mathbb{R}}^0(C, P) = \begin{pmatrix} I & \mathcal{J}_{\mathbb{R}}(C, P) \\ 0 & \mathcal{I}_{\mathbb{R}}(C, P) \end{pmatrix}$$

tells us that  $\det M_{\mathbb{R}}^0(C, P) = \det \mathcal{I}_{\mathbb{R}}(C, P)$ , so we have  $2 \Leftrightarrow 4$  and  $3 \Leftrightarrow 5$ . The implication  $2 \Rightarrow 3$  is obvious. According to Theorem 2.2, condition 3 implies that every edge of  $Tch(P)$  is a loop; this in turn implies that  $P$  is an Euler system.  $\square$

**Theorem 8.6.** *(Lauri [23] and Macris and Pulé [24]) Let  $C$  be any signed version of any Euler system of  $F$ . Then the number of Euler systems of  $F$  that respect the edge directions given by  $C$  is  $\det(I + \mathcal{I}_{\mathbb{R}}(C))$ .*

*Proof.* Let  $v_1, \dots, v_n$  be the vertices of  $F$ , and let  $x_1, \dots, x_n$  be independent indeterminates. For each subset  $S \subseteq \{1, \dots, n\}$ , let  $P_S$  be the circuit partition of  $F$  that involves  $\phi_C(v_i)$  whenever  $i \in S$ , and  $\chi_C(v_i)$  whenever  $i \notin S$ . Let

$$\mathcal{E} = \{S \subseteq \{1, \dots, n\} \mid P_S \text{ is an Euler system of } F\}.$$

Let  $X$  be the matrix with entries  $x_1, \dots, x_n$  on the diagonal, and other entries 0. Then Corollary 8.5 tells us that

$$\det(X + \mathcal{I}_{\mathbb{R}}(C)) = \sum_{S \subseteq \{1, \dots, n\}} \left( \prod_{i \in S} x_i \right) \det M_{\mathbb{R}}^0(C, P_S) = \sum_{S \in \mathcal{E}} \left( \prod_{i \in S} x_i \right).$$

That is,  $\det(X + \mathcal{I}_{\mathbb{R}}(C))$  is a version of the indicator function of the set  $\mathcal{E}$ . The theorem follows by setting  $x_1, \dots, x_n$  equal to 1.  $\square$

Theorem 8.6 implies that in polynomial time, one can calculate the number of Euler systems of  $F$  that respect the edge directions defined by  $C$ . Ge and Štefankovič [14] proved that in contrast, the problem of counting all the Euler systems of  $F$  is  $\#P$ -complete.

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