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HAMILTONIAN PATHS IN $m \times n$ PROJECTIVE CHECKERBOARDS

DALLAN MCCARTHY AND DAVE WITTE MORRIS

Abstract. For any two squares ι and τ of an $m \times n$ checkerboard, we determine whether it is possible to move a checker through a route that starts at ι, ends at τ, and visits each square of the board exactly once. Each step of the route moves to an adjacent square, either to the east or to the north, and may step off the edge of the board in a manner corresponding to the usual construction of a projective plane by applying a twist when gluing opposite sides of a rectangle. This generalizes work of M. H. Forbush et al. for the special case where $m = n$.

1. Introduction

Place a checker in any square of an $m \times n$ checkerboard (or chessboard). We determine whether it is possible for the checker to move through the board, visiting each square exactly once. (In graph-theoretic terminology, we determine whether there is a hamiltonian path that starts at the given square.) Although other rules are also of interest (such as the well-known knight moves discussed in [6] and elsewhere), we require each step of the checker to move to an adjacent square that is either directly to the east or directly to the north, except that we allow the checker to step off the edge of the board.

Torus-shaped checkerboards are already understood (see, for example, [3]), so we allow the checker to step off the edge of the board in a manner that corresponds to the usual procedure for creating a projective plane, by applying a twist when gluing each edge of a rectangle to the opposite edge:

Definition 1.1 (cf. [2, Defn. 1.1])**.** The squares of an $m \times n$ checkerboard can be naturally identified with the set $\mathbb{Z}_m \times \mathbb{Z}_n$ of ordered pairs (p, q) of integers with $0 \le p \le m-1$ and $0 \le q \le n-1$. Define $E\colon \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ and $N\colon \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ by

$$(p,q)E = \begin{cases} (p+1, q) & \text{if } p < m-1 \\ (0, n-1-q) & \text{if } p = m-1 \end{cases}$$

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and

$$(p,q)N = \begin{cases} (p, q+1) & \text{if } q < n-1 \\ (m-1-p, 0) & \text{if } q = n-1. \end{cases}$$

The $m \times n$ *projective checkerboard* $\mathcal{B}_{m,n}$ is the digraph whose vertex set is $\mathbb{Z}_m \times \mathbb{Z}_n$, with a directed edge from σ to σE and from σ to σN, for each $\sigma \in \mathbb{Z}_m \times \mathbb{Z}_n$. We usually refer to the vertices of $\mathcal{B}_{m,n}$ as *squares*.

In a projective checkerboard $\mathcal{B}_{m,n}$ (with $m, n \geq 3$), only certain squares can be the initial square of a hamiltonian path, and only certain squares can be the terminal square. A precise determination of these squares was found by M. H. Forbush et al. [2] in the special case where $m = n$ (that is, when the checkerboard is square, rather than properly rectangular). In this paper, we find both the initial squares and the terminal squares in the general case. (Illustrative examples appear in Figures 1 to 4 on pages 3 to 6.)

Notation 1.2. For convenience, let

$$\cancel{m} = \lfloor m/2 \rfloor, \quad \cancel{m}^- = \lfloor (m-1)/2 \rfloor, \quad \cancel{m}^+ = \lfloor (m+1)/2 \rfloor = \lceil m/2 \rceil = \cancel{m}^- + 1,$$
$$\cancel{n} = \lfloor n/2 \rfloor, \quad \cancel{n}^- = \lfloor (n-1)/2 \rfloor, \quad \cancel{n}^+ = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil = \cancel{n}^- + 1.$$

Theorem 1.3. *Assume $m \geq n \geq 3$. There is a hamiltonian path in $\mathcal{B}_{m,n}$ whose initial square is (p,q) if and only if either:*

(1) $p = 0$ *and* $\cancel{n}^- \leq q \leq n-1$*, or*
(2) $\cancel{n}^- \leq p \leq m-1$ *and* $q = 0$*, or*
(3) $\cancel{m}^+ \leq p \leq m-1$ *and* $q = \cancel{n}$*, or*
(4) $0 \leq p \leq \cancel{n}$ *and* $q = \cancel{n}^-$*, or*
(5) $\cancel{m} \leq p \leq m - \cancel{n}^+$ *and* $\cancel{n} + 1 \leq q \leq n-1$*, or*
(6) $\cancel{n}^- \leq p \leq \cancel{m}^-$ *and* $0 \leq q \leq \cancel{n}$*.*

By rotating the checkerboard 180° (cf. Proposition 2.4), this theorem can be restated as follows:

Theorem 1.4. *Assume $m \geq n \geq 3$. There is a hamiltonian path in $\mathcal{B}_{m,n}$ whose terminal square is (x,y) if and only if either:*

(1) $x = m-1$ *and* $0 \leq y \leq \cancel{n}$*, or*
(2) $0 \leq x \leq m - \cancel{n}^+$ *and* $y = n-1$*, or*
(3) $0 \leq x \leq \cancel{m} - 1$ *and* $y = \cancel{n}^-$*, or*
(4) $m - \cancel{n} - 1 \leq x \leq m-1$ *and* $y = \cancel{n}$*, or*
(5) $\cancel{n}^- \leq x \leq \cancel{m}^-$ *and* $0 \leq y \leq \cancel{n}^- - 1$*, or*
(6) $\cancel{m} \leq x \leq m - \cancel{n}^+$ *and* $\cancel{n}^- \leq y \leq n-1$*.*

Remark 1.5. By symmetry, there is no harm in assuming that $m \geq n$ when studying $\mathcal{B}_{m,n}$. Furthermore, if $\min(m,n) \leq 2$, then it is easy to see that $\mathcal{B}_{m,n}$ has a hamiltonian cycle. Therefore, every square is the initial square of a hamiltonian path (and the terminal square of some other hamiltonian path) in the cases not covered by Theorems 1.3 and 1.4.

For any square ι of $\mathcal{B}_{m,n}$, we determine not only whether there exists a hamiltonian path that starts at ι, but also the terminal square of each of these hamiltonian paths. This more detailed result is stated and proved in Section 5. It yields Theorems 1.3 and 1.4 as corollaries. As preparation for the proof, we recall some known results in Section 2, consider a very helpful special case in Section 3, and explain how to reduce the general problem to this special case in Section 4.

Remark 1.6. Suppose m and n are large. It follows from Theorem 1.3 that a square in $\mathcal{B}_{m,n}$ is much less likely to be the starting point of a hamiltonian path when the checkerboard is square than when it is very oblong:

- If $m = n$, then only a small fraction ($\approx 3/n$) of the squares are the initial square of a hamiltonian path.
- In contrast, if m is much larger than n, then about half of the squares are the initial square of a hamiltonian path.

The following question remains open (even when $m = n$).

Problem 1.7. *Which squares are the terminal square of a hamiltonian path that starts at* $(0,0)$ *in an* $m \times n$ *Klein-bottle checkerboard, where*

$$(m-1,q)E = (0,n-1-q) \quad \textit{and} \quad (p,n-1)N = (p,0).$$

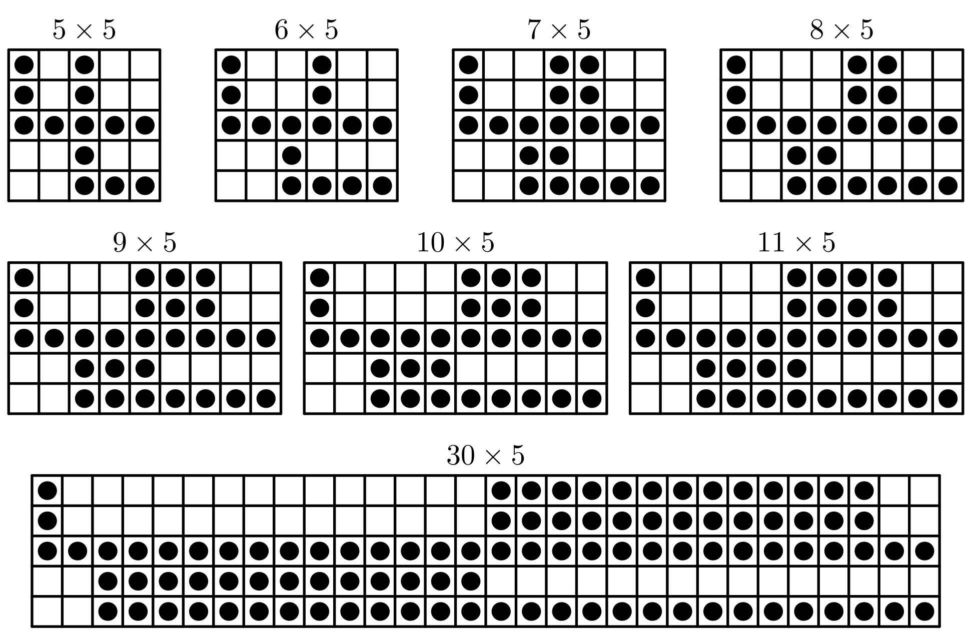

FIGURE 1. The initial squares (●) of hamiltonian paths in some $m \times 5$ projective checkerboards.

2. Preliminaries: definitions, notation, and previous results

We reproduce some of the elementary, foundational content of [2], slightly modified to eliminate that paper's standing assumption that $m = n$.

Notation 2.1 ([2, Notation 3.1])**.** We use $[\sigma](X_1 X_2 \cdots X_k)$, where $X_i \in \{E, N\}$, to denote the walk in $\mathcal{B}_{m,n}$ that visits (in order) the squares

$$\sigma,\ \sigma X_1,\ \sigma X_1 X_2,\ \ldots,\ \sigma X_1 X_2 \ldots X_k.$$

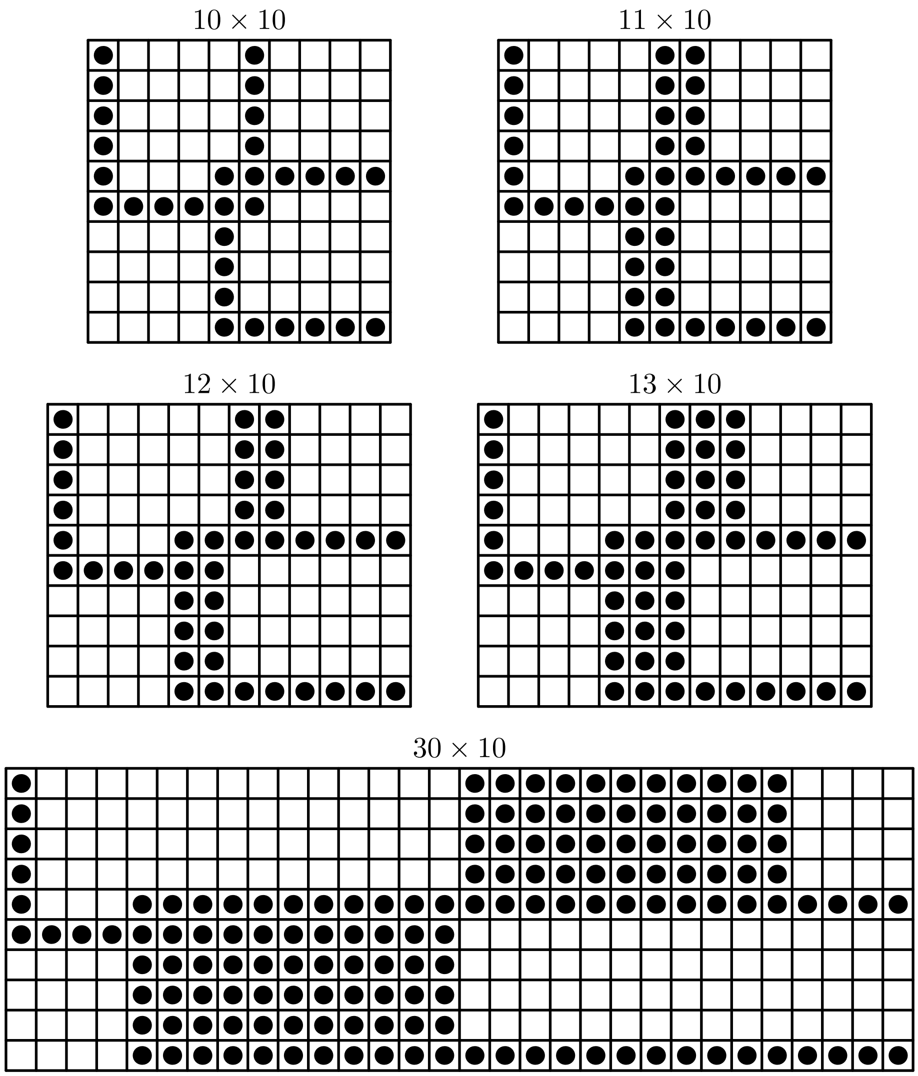

Figure 2. The initial squares (●) of hamiltonian paths in some $m \times 10$ projective checkerboards.

Definition 2.2 ([2, Defn. 2.14])**.** For $\sigma = (p, q) \in \mathcal{B}_{m,n}$, we define the *inverse* of σ to be $\widetilde{\sigma} = (m-1-p, n-1-q)$.

Remark 2.3. $\widetilde{\sigma}$ can be obtained from σ by rotating the checkerboard 180 degrees.

Proposition 2.4 ([2, Prop. 2.15])**.** *If there is a hamiltonian path from ι to τ in $\mathcal{B}_{m,n}$, then there is also a hamiltonian path from $\widetilde{\tau}$ to $\widetilde{\iota}$.*

More precisely, if $\mathcal{H} = [\iota](X_1 X_2 \cdots X_k)$ is a hamiltonian path from ι to τ, then the inverse *of $\mathcal{H}$ is the hamiltonian path $\widetilde{\mathcal{H}} = [\widetilde{\tau}](X_k X_{k-1} \cdots X_1)$ from $\widetilde{\tau}$ to $\widetilde{\iota}$.*

Definition 2.5 ([2, Defn. 2.14 and Prop. 2.15])**.** If $m = n$, then:

- The *transpose* σ^* of any square σ of $\mathcal{B}_{m,n}$ is defined by $(p,q)^* = (q,p)$.
- For a hamiltonian path $\mathcal{H} = [\iota](X_1 X_2 \cdots X_k)$ from ι to τ, the *transpose* of $\mathcal{H}$ is the hamiltonian path $\mathcal{H}^* = [\iota^*](X_1^* X_2^* \cdots X_k^*)$ from ι^* to τ^*, where $E^* = N$ and $N^* = E$.

2A. **Direction-forcing diagonals.**

Definition 2.6 ([2, Defn. 2.1])**.** Define a symmetric, reflexive relation $\sim$ on the set of squares of $\mathcal{B}_{m,n}$ by $\sigma \sim \tau$ if

$$\{\sigma E, \sigma N\} \cap \{\tau E, \tau N\} \neq \emptyset.$$

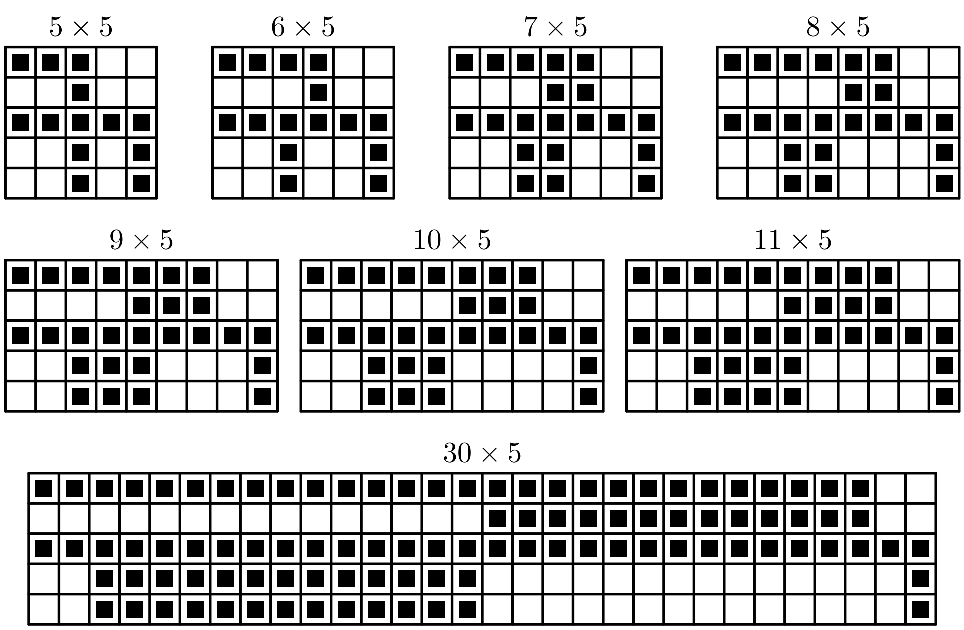

FIGURE 3. The terminal squares (■) of hamiltonian paths in some $m \times 5$ projective checkerboards.

The equivalence classes of the transitive closure of $\sim$ are *direction-forcing diagonals*. For short, we refer to them simply as *diagonals*. Thus, the diagonal containing σ is

$$\{\sigma, \sigma NE^{-1}, \sigma(NE^{-1})^2, \ldots, \sigma EN^{-1}\}.$$

Notation 2.7 ([2, Notn. 2.3])**.** For $0 \leq i \leq m+n-2$, let

$$S_i = \{\, (p,q) \in \mathcal{B}_{m,n} \mid p+q = i \,\}.$$

We call S_i a *subdiagonal*.

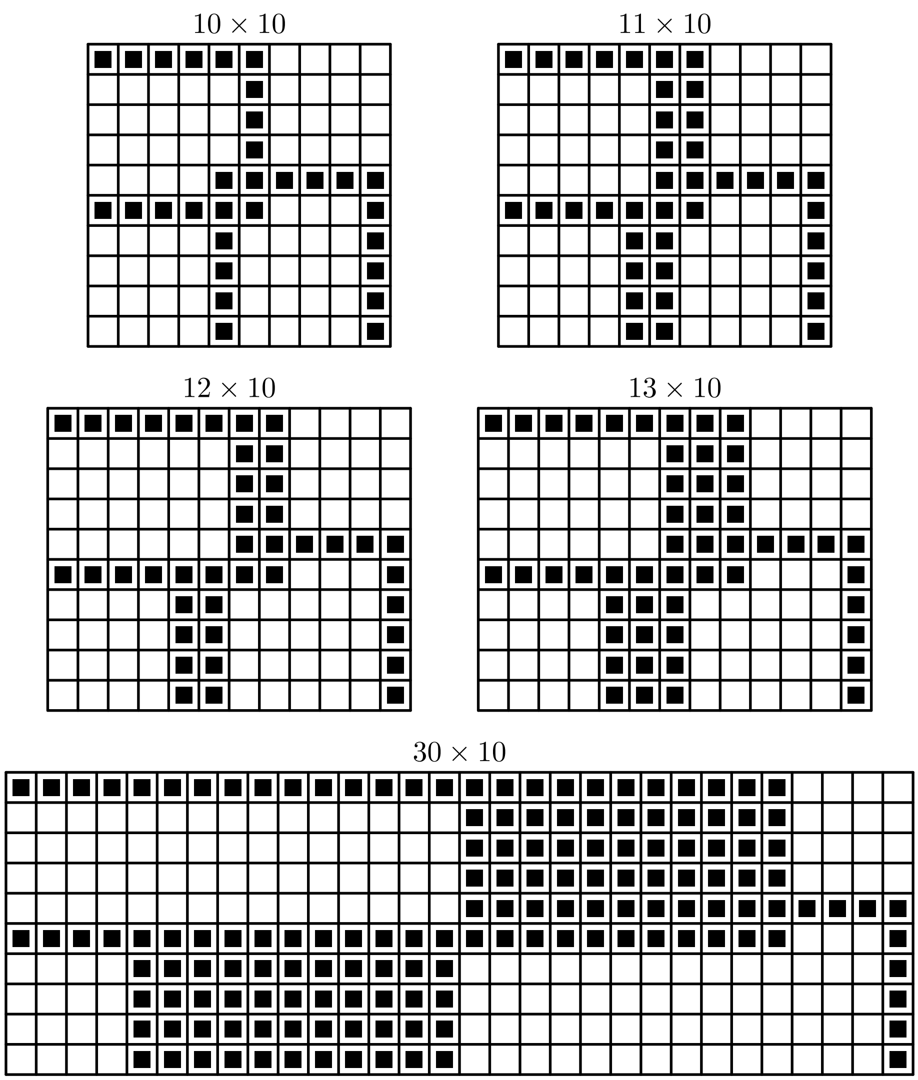

FIGURE 4. The terminal squares (■) of hamiltonian paths in some $m \times 10$ projective checkerboards.

Proposition 2.8 ([2, Prop. 2.4])**.** *For each i with $0 \le i \le m+n-3$, the set $D_i = S_i \cup S_{m+n-3-i}$ is a diagonal. The only other diagonal D_{m+n-2} consists of the single square $(m-1, n-1)$.*

Corollary 2.9 ([2, Notn. 2.5])**.** *Let D be a diagonal, other than D_{m+n-2}. Then we may write $D = S_a \cup S_b$ with $a \le b$ and $a + b = m+n-3$.*

Definition 2.10 ([2, Defn. 2.7])**.** If $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_{m,n}$, then the diagonal containing the terminal square τ is called the *terminal* diagonal of $\mathcal{H}$. All other diagonals are *nonterminal* diagonals.

Definition 2.11 ([2, Defn. 2.8])**.** Let $\mathcal{H}$ be a hamiltonian path in $\mathcal{B}_{m,n}$. A square σ *travels east* (in $\mathcal{H}$) if the edge from σ to σE is in $\mathcal{H}$. Similarly, σ *travels north* (in $\mathcal{H}$) if the edge from σ to σN is in $\mathcal{H}$.

The following important observation is essentially due to R. A. Rankin [5, proof of Thm. 2].

Proposition 2.12 ([2, Prop. 2.9], cf. [4, Prop. on p. 82], [1, Lem. 6.4c])**.** *If $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_{m,n}$, then, for each nonterminal diagonal D, either every square in D travels north, or every square in D travels east. For short, we say that either D travels north or D travels east.* □

Proposition 2.13 ([2, Prop. 2.10], cf. [1, Lem. 6.4b])**.** *Let D be the terminal diagonal of a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_{m,n}$, with initial square ι and terminal square τ, and let $\sigma \in D$.*

- *if $\tau N \neq \iota$, then $\tau N E^{-1}$ travels east;*
- *if $\tau E \neq \iota$, then $\tau E N^{-1}$ travels north;*
- *if σ travels east and $\sigma N \neq \iota$, then $\sigma N E^{-1}$ travels east;*
- *if σ travels east, then $\sigma E N^{-1}$ does not travel north;*
- *if σ travels north and $\sigma E \neq \iota$, then $\sigma E N^{-1}$ travels north; and*
- *if σ travels north, then $\sigma N E^{-1}$ does not travel east.* □

Corollary 2.14 ([2, Cor. 2.11], cf. [1, Lem. 6.4a])**.** *If $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_{m,n}$, then the diagonal that contains ιE^{-1} and ιN^{-1} is the terminal diagonal.*

The following corollary follows from Proposition 2.13 by induction.

Corollary 2.15 ([2, Cor. 2.12])**.** *Let D be the terminal diagonal of a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_{m,n}$, with initial square ι and terminal square τ, and let $|D|$ denote the cardinality of D.*

(1) *For each $\sigma \in D$, there is a unique integer $u(\sigma) \in \{1, 2, \ldots, |D|\}$ with $\sigma = \tau (N E^{-1})^{u(\sigma)}$; the square σ travels east iff $u(\sigma) < u(\iota E^{-1})$.*

(2) *Similarly, there is a unique integer $v(\sigma) \in \{1, 2, \ldots, |D|\}$ with $\sigma = \iota E^{-1} (E N^{-1})^{v(\sigma)}$; the square σ travels east iff $v(\sigma) < v(\tau)$.*

Corollary 2.16 ([2, Cor. 2.13]). *A hamiltonian path is uniquely determined by specifying*

(1) *its initial square;*
(2) *its terminal square; and*
(3) *which of its nonterminal diagonals travel east.*

2B. **Further restrictions on hamiltonian paths.**

Proposition 2.17 ([2, Thm. 3.2]). *If* $m, n \geq 3$*, then* $(0,0)$ *is not the initial square of any hamiltonian path in* $\mathcal{B}_{m,n}$*. Therefore,* D_{m+n-2} *is not the terminal diagonal of any hamiltonian path (and* $\mathcal{B}_{m,n}$ *does not have a hamiltonian cycle).*

Lemma 2.18 ([2, Lem. 3.4]). *Suppose* $S_a \cup S_b$ *is the terminal diagonal of a hamiltonian path* $\mathcal{H}$ *in* $\mathcal{B}_{m,n}$*, with* $m, n \geq 3$ *and* $a \leq b$*. Choose* $(p, q) \in S_{b+1}$*, and let* P *be the unique path in* $\mathcal{H}$ *that starts at* (p, q) *and ends in* S_a*, without passing through* S_a*. Then the terminal square of* P *is the inverse of* (p, q)*.*

Definition 2.19. Let $S_a \cup S_b$ be the terminal diagonal of a hamiltonian path, with $a \leq b$, and let $S_i \cup S_j$ be some other diagonal of $\mathcal{B}_{m,n}$, with $i \leq j$ and $i + j < m + n - 2$. We say that:

(1) $S_i \cup S_j$ is an *outer diagonal* if $i < a$ (or, equivalently, $j > b$).
(2) $S_i \cup S_j$ is an *inner diagonal* if $i > a$ (or, equivalently, $j < b$).

Lemma 2.18 has the following important consequence.

Corollary 2.20 (cf. [2, Thm. 3.5]). *Assume that* $\mathcal{H}$ *is a hamiltonian path from* ι *to* τ *in* $\mathcal{B}_{m,n}$*, with* $m \geq n \geq 3$*. Define* $\mathcal{H}_E$ *and* $\mathcal{H}_N$ *to be the subdigraphs of* $\mathcal{B}_{m,n}$*, such that*

- ι *has invalence* 0*, but the invalence of all other squares is* 1 *in both* $\mathcal{H}_E$ *and* $\mathcal{H}_N$*,*
- τ *has outvalence* 0*, but the outvalence of all other squares is* 1 *in both* $\mathcal{H}_E$ *and* $\mathcal{H}_N$*,*
- *each inner diagonal travels exactly the same way in* $\mathcal{H}_E$ *and* $\mathcal{H}_N$ *as it does in* $\mathcal{H}$*, and*
- *each outer diagonal travels east in* $\mathcal{H}_E$*, but travels north in* $\mathcal{H}_N$*.*

Then:

(1) $\mathcal{H}_E$ *is a hamiltonian path from* ι *to* τ*.*
(2) $\mathcal{H}_N$ *is a hamiltonian path from* ι *to* τ *if and only if the diagonal* $S_{n-1} \cup S_{m-2}$ *is not outer.*

3. Hamiltonian paths in which all nonterminal diagonals travel north

We eventually need to understand all of the hamiltonian paths in $\mathcal{B}_{m,n}$, but this section considers only the much simpler special case in which every nonterminal diagonal is required to travel north. Although this may seem to be a very restrictive assumption, Proposition 4.6 below will allow us to obtain the general case from this one.

Proposition 3.1. *Assume $S_a \cup S_b$ is the terminal diagonal of a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_{m,n}$, with $m \geq n \geq 3$ and $a \leq b$. Let τ_+ be the southeasternmost square in S_b. If all nonterminal diagonals travel north in $\mathcal{H}$, then $a \leq m-2 \leq b$, and $\tau_+ E$ is the initial square of either $\mathcal{H}$ or the inverse of $\mathcal{H}$ (or the transpose or transpose-inverse of $\mathcal{H}$, if $m = n = a+2 = b+1$), unless $a+1 = b = n = m-2$, in which case the initial square (of either $\mathcal{H}$ or $\widetilde{\mathcal{H}}$) might also be $\tau_+ = (n, 0)$.*

Proof. From Corollary 2.20(2), we know that $a \leq n-1$ and $m-2 \leq b$. Since $n \leq m$, this immediately implies $a \leq m-2 \leq b$, unless $a = n-1 = m-1$. But then $b = m+n-3-a = m-2 < a$, which contradicts the fact that $a \leq b$.

For convenience, write $\tau_+ = (x, y)$. Assume, for the purposes of contradiction, that the initial square is not as described. We consider two cases.

Case 1: Assume $x = m-1$. Note that $\tau_+ E = (0, n-1-y)$ is the inverse of τ_+, so τ_+ cannot be the terminal square of $\mathcal{H}$ (since $\tau_+ E$ is not the initial square of the inverse of $\mathcal{H}$, because of our assumption that the initial square is not as described).

Assume, for the moment, that $\tau_+ E$ is not in the terminal diagonal. Then, by assumption, $\tau_+ E$ travels north. So τ_+ cannot travel east. (Otherwise, the hamiltonian path $\mathcal{H}$ would contain the cycle $[\tau_+](EN^{2y+1})$ because $b > n-1$.) Therefore, since τ_+ is not the terminal square of $\mathcal{H}$, we conclude that τ_+ travels north. Since $\tau_+ E$ is not the initial square (and must therefore be entered from either τ_+ or $\tau_+ E N^{-1}$, we conclude that $\tau_+ E N^{-1}$ travels north. So $\mathcal{H}$ contains the cycle $[\tau_+](N^{2n})$. This is a contradiction.

We may now assume that $\tau_+ E$ is in the terminal diagonal. However, $\tau_+ E N^{-1}$ is also in the terminal diagonal (since it is obviously in the same diagonal as τ_+, which is in the terminal diagonal). It follows that $\tau_+ E N^{-1} \in S_a$ and $\tau_+ E \in S_b$, with $b = a+1$. Since

$$b = x + y = (m-1) + y \geq m-1,$$

this implies

$$2m-3 \geq m+n-3 = a+b = 2b-1 \geq 2(m-1)-1 = 2m-3.$$

Therefore, we must have equality throughout both strings of inequalities, so

$$y = 0, \quad m = n, \quad b = m-1, \quad \text{and} \quad a = m-2.$$

(Since $m = n$, the desired contradiction can be obtained from [2, Thm. 3.12], but, for completeness, we provide a direct proof.) Since $(m-1, 0) = (m-1, y) = \tau_+$ is not the terminal square, it must travel either north or east. We consider these two possibilities individually.

Assume, for the moment, that $(m-1, 0)$ travels east (to $(0, m-1)$, because $m = n$). Clearly, $(0, m-1)$ does not travel north (because $\mathcal{H}$ does not contain the cycle $[(m-1,0)](EN))$. Also, $(0,m-1) = \tau_+ E$ is not the terminal square (because it is not the initial square of the transpose-inverse of $\mathcal{H}$, by our assumption that the initial square is not as described). So $(0,m-1)$ must travel east. Since $(0,m-1)N = (m-1,0)$ is not the initial square (because our assumption that the initial square is not as described implies $(0,m-1) = \tau_+ E$ is not the initial square of the transpose of $\mathcal{H}$), we conclude from Proposition 2.13 that $(m-2,0) = (0,m-1)NE^{-1}$ also travels east. And $(1,m-1)$ travels north, because it is not in the terminal diagonal. So $\mathcal{H}$ contains the cycle

$$(0,m-1) \xrightarrow{E} (1,m-1) \xrightarrow{N} (m-2,0) \xrightarrow{E} (m-1,0) \xrightarrow{E} (0,m-1).$$

This is a contradiction.

We may now assume that $(m-1,0)$ travels north. Since $(m-1,0)E = \tau_+ E$ is not the initial square, we conclude from Proposition 2.13 that $(0,m-2) = (m-1,0)EN^{-1}$ also travels north. By applying the same argument to the transpose of $\mathcal{H}$, we see that $(0,m-1)$ and $(m-2,0)$ must travel east. Also, $(1,m-1)$ travels north, because it is not in the terminal diagonal. So $\mathcal{H}$ contains the cycle

$$(m-1,0) \xrightarrow{N^{2m-2}} (0,m-2) \xrightarrow{N} (0,m-1) \xrightarrow{E} (1,m-1) \xrightarrow{N} (m-2,0) \xrightarrow{E} (m-1,0).$$

This contradiction completes the proof of Case 1.

Case 2: Assume $x < m-1$. Since $(x,y) = \tau_+$ is the southeasternmost square in S_b, we must have $y = 0$ (otherwise, $(x+1,y-1)$ is a square in S_b is farther southeast), so $b = x < m-1$. Since we know from the first sentence of the proof that $m-2 \le b$, we conclude that $b = m-2$ (and $a = m+n-3-b = n-1$). Therefore

$$\tau_+ = (m-2,0), \text{ so } \tau_+ E = (m-1,0).$$

Note that $(0,n-1)$ cannot travel north (otherwise, $\mathcal{H}$ contains the cycle $[(m-1,0)](N^{2n})$, since $(0,n-1)$ is the only square of this cycle that is in the terminal diagonal). Also, $(0,n-1)$ is not the terminal square (since $(m-1,0) = \tau_+ E$ is not the initial square of the inverse of $\mathcal{H}$, by our assumption that the initial square is not as described). Therefore $(0,n-1)$ must travel east. Then, since $(0,n-1)N = (m-1,0) = \tau_+ E$ is not the initial square, we know that $(m-2,0) = \tau_+$ also travels east.

Since $\mathcal{H}$ cannot contain the cycle

$$(m-1,0) \xrightarrow{N^{2n-1}} (0,n-1) \xrightarrow{E} (1,n-1) \xrightarrow{N} (m-2,0) \xrightarrow{E} (m-1,0),$$

we know that $(1,n-1)$ does not travel north. Therefore this square is in the terminal diagonal, which means $b = 1 + (n-1) = n$, so we have $a + 1 = n = b = m-2$. Hence, we are in the exceptional case at the end of the statement of the proposition. Therefore, $(1,n-1)$ is not the terminal square of $\mathcal{H}$ (since $(m-2,0) = \tau_+$ is not the initial square of the inverse of $\mathcal{H}$). Since we have already seen that it does not travel north, we conclude that $(1,n-1)$ travels east.

Applying the argument of the preceding paragraph to the inverse of $\mathcal{H}$ tells us that $(1, n-1)$ travels east in $\widetilde{\mathcal{H}}$. Taking the inverse, this means $(m-2, 0)$ travels east in $\mathcal{H}$. Also, we know that $(2, n-1)$ travels north (because it is not in the terminal diagonal), and we know that $(m-3, 0)$ travels east (because $(m-3,0)EN^{-1} = (1, n-1)$ travels east and $(m-3,0)E = \tau_+$ is not the initial square). Therefore, $\mathcal{H}$ contains the cycle

$$(m-1, 0) \xrightarrow{N^{2n-1}} (0, n-1) \xrightarrow{E^2} (2, n-1) \xrightarrow{N} (m-3, 0) \xrightarrow{E^2} (m-1, 0).$$

This is a contradiction. □

The above proposition usually allows us to assume that the initial square of a hamiltonian cycle is $\tau_+ E$ (if all nonterminal diagonals travel north). The following result finds the possible terminal squares in this case.

The proof of this proposition (and also Lemma 3.3) constructs hamiltonian paths in projective checkerboards of various sizes, with particular initial squares and terminal squares. These paths are specified by stating the initial square and a list of arcs to traverse, using the format of Notation 2.1. It would be tedious to formally prove that the specified walk visits every square exactly once (and terminates at the desired square), but, in each case, it should not be difficult for the reader to verify that this is true. Illustrations of the hamiltonian paths are provided in order to facilitate this.

Proposition 3.2. *Let*

- $m \geq n \geq 3$,
- $S_a \cup S_b$ *be a diagonal in* $\mathcal{B}_{m,n}$,
- τ_+ *be the southeasternmost square in* S_b, *and*
- τ *be any square in* $S_a \cup S_b$.

There is a hamiltonian path $\mathcal{H}$ *from* $\tau_+ E$ *to* τ *in which all nonterminal diagonals travel north if and only if* τ *is either* $(\not{m}^-, a - \not{m}^-)$ *or* $(\not{m}, b - \not{m})$ *(and* $\tau = (\not{m}^-, a - \not{m}^-)$ *if* $a = b$*), and* $a \leq m-2 \leq b$.

Proof. Let $\sigma_a = (\not{m}^-, a - \not{m}^-)$ and $\sigma_b = (\not{m}, b - \not{m})$.

($\Rightarrow$) Corollary 2.20(2) tells us that $a \leq m-2 \leq b$. (See the first paragraph of the proof of Proposition 3.1.) This establishes one conclusion of the proposition.

We now wish to show that τ is either σ_a or σ_b, and that $\tau = \sigma_a$ if $a = b$. Assume the contrary.

Note that τ_+ must travel north in $\mathcal{H}$, since $\tau_+ E$ is the initial square (and $\mathcal{B}_{m,n}$ does not have a hamiltonian cycle).

Case 1: Assume m is odd. Note that $\not{m}^- = \not{m}$ in this case (and we have $m - 1 - \not{m} = \not{m}$). Since $\mathcal{H}$ cannot contain the cycle $[(\not{m}, 0)](N^n)$, we know that some square in this cycle does not travel north in $\mathcal{H}$. This square must be in the terminal diagonal, so it is either σ_a or σ_b. It is therefore not the terminal square, so it must travel east.

From the preceding paragraph, we see that the square σ_b must exist in $\mathcal{B}_{m,n}$, so $b - \not{m} \le n-1$. Therefore

$$\begin{aligned} a - \not{m} + 1 &= (m+n-3) - b - \not{m} + 1 \\ &\ge (m+n-3) - (\not{m} + n - 1) - \not{m} + 1 \\ &= 0. \end{aligned}$$

Also,

$$a - \not{m} + 1 \le b - \not{m} + 1 \le (n-1) + 1 = n,$$

so $a - \not{m} + 1 \le n-1$ unless $a = b = \not{m} + n - 1$. But the alternative yields a contradiction:

$$m + n - 3 = a + b = 2(\not{m} + n - 1) = m + 2n - 3 > m + n - 3.$$

Therefore, the square $(\not{m} - 1, a - \not{m} + 1)$ exists (and is in S_a).

Suppose σ_b travels east. Since $\tau_+ E$ is the initial square, we see from Corollary 2.15 that $(\not{m} - 1, a - \not{m} + 1)$ travels east. If $a < b$, this implies that $\mathcal{H}$ contains the cycle

$$\sigma_b \xrightarrow{EN^{n-b+a+1}} (\not{m} - 1, a - \not{m} + 1) \xrightarrow{EN^{b-a-1}} \sigma_b.$$

On the other hand, if $a = b$, this implies that $\mathcal{H}$ contains the cycle

$$\sigma_b \xrightarrow{EN^{n+1}} (\not{m} - 1, b - \not{m} + 1) \xrightarrow{EN^{n-1}} \sigma_b.$$

In either case, we have a contradiction.

We may now assume that σ_b travels north. So it must be σ_a that travels east (and $\sigma_a \ne \sigma_b$, so $a \ne b$). From Corollary 2.15, we see that

$$(\not{m} - 1, a - \not{m} + 1) \text{ travels east and } (\not{m} + 1, b - \not{m} - 1) \text{ travels north.}$$

So $\mathcal{H}$ contains the cycle

$$\begin{aligned} &\sigma_a \xrightarrow{E} (\not{m} + 1, a - \not{m}) \xrightarrow{N^{b-a-1}} (\not{m} + 1, b - \not{m} - 1) \\ &\quad \xrightarrow{N^{n-b+a+2}} (\not{m} - 1, a - \not{m} + 1) \xrightarrow{E} (\not{m}, a - \not{m} + 1) \xrightarrow{N^{b-a-1}} \sigma_b \xrightarrow{N^{n-b+a}} \sigma_a. \end{aligned}$$

This is a contradiction.

Case 2: Assume m is even. Note that $\not{m}^- = \not{m} - 1$ in this case (and we have $m - 1 - \not{m} = \not{m} - 1$). Since $\mathcal{H}$ does not contain the cycle $[(\not{m}, 0)](N^{2n})$, we know that some square in this cycle does not travel north. In other words, there is a square (x, y) that does not travel north, such that $x \in \{\not{m} - 1, \not{m}\}$.

Assume, for the moment, that $b - \not{m} = n - 1$. Then σ_b is the only square that is in the intersection of the terminal diagonal with the cycle $[(\not{m}, 0)](N^{2n})$, so it must be σ_b that does not travel north. Since, by assumption, σ_b is not the terminal square, we conclude that σ_b travels east. Then Corollary 2.15 implies that $(\not{m} - 2, 0) = \sigma_b N E^{-1}$ also travels east. So $\mathcal{H}$ contains the cycle

$$\sigma_b \xrightarrow{E} (\not{m} + 1, n - 1) \xrightarrow{N} (\not{m} - 2, 0) \xrightarrow{E} (\not{m} - 1, 0) \xrightarrow{N^{2n-1}} \sigma_b.$$

This is a contradiction.

We may now assume $b - \not{m} \leq n - 2$, so

$$a = (m + n - 3) - b \geq (m + n - 3) - (n - 2 + \not{m}) = m - 1 - \not{m} = \not{m}^-.$$

Therefore S_a contains the square $(\not{m}^-, a - \not{m}^-) = \sigma_a$. Note that σ_a cannot travel north. (Otherwise, Corollary 2.15 implies that σ_b also travels north, contrary to the fact that at least one of these two squares does not travel north.) Since, by assumption, σ_a is not the terminal square, we conclude that σ_a travels east.

Since $\mathcal{H}$ does not contain the cycle $[\sigma_a](E, N^n)$, we conclude that $a \neq b$ and σ_b does not travel north. Therefore σ_b travels east. Then Corollary 2.15 tells us that $(\not{m} - 1, b - \not{m} + 1)$ and every square in S_a all travel east. So $\mathcal{H}$ contains the cycle

$$\begin{aligned}\sigma_b &\xrightarrow{E} (\not{m} + 1, b - \not{m}) \xrightarrow{N^{n-b+a+2}} (\not{m} - 2, a - \not{m} + 2) \\ &\xrightarrow{E} (\not{m} - 1, a - \not{m} + 2) \xrightarrow{N^{b-a-1}} (\not{m} - 1, b - \not{m} + 1) \\ &\xrightarrow{E} (\not{m}, b - \not{m} + 1) \xrightarrow{N^{n-b+a}} \sigma_a \xrightarrow{E} (\not{m}, a - \not{m} + 1) \xrightarrow{N^{b-a-1}} \sigma_b.\end{aligned}$$

This is a contradiction.

($\Leftarrow$) We use $(\ldots)^k$ to represent the concatenation of k copies of the sequence $(\ldots)$. (For example, $(N^3, E)^2 = (N, N, N, E, N, N, N, E)$.)

If σ_a exists (that is, if $a \geq \not{m}^-$), then we have the following hamiltonian path $\mathcal{H}_a$ from $\tau_+ E$ to σ_a (see Figure 5):

$$\begin{cases} [\tau_+ E]\big((N^{2n-1}, E)^{\not{m}^-}, N^{n-1}\big) & \text{if } n \text{ is odd,} \\ [\tau_+ E]\big((N^{2n-1}, E)^{\not{m}^-}, N^{2n-1}\big) & \text{if } n \text{ is even.} \end{cases}$$

Now assume σ_b exists (that is, $b \leq \not{m} + n - 1$) and $a \neq b$.

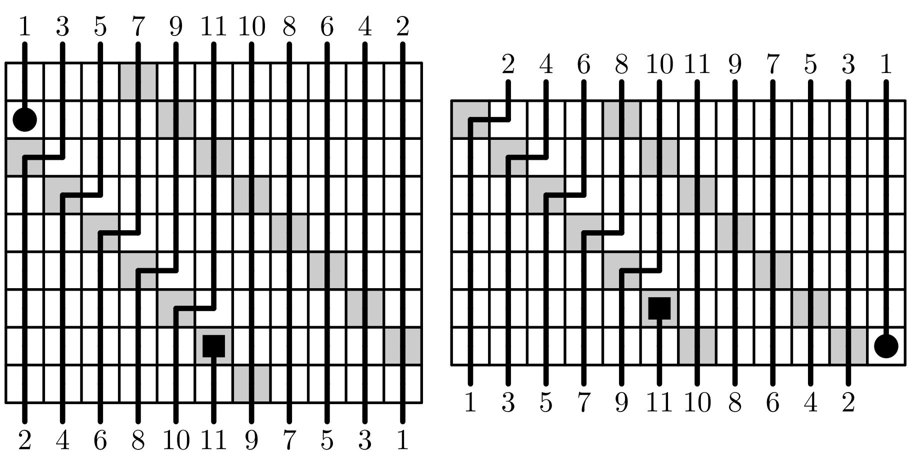

FIGURE 5. Illustrative examples of the hamiltonian path $\mathcal{H}_a$ from $\tau_+ E$ (●) to σ_a (■). (The terminal diagonal is shaded.)

- If m is odd, then we have the following hamiltonian path $\mathcal{H}_b$ from $\tau_+ E$ to σ_b (see Figure 6):

$$[\tau_+ E]\big((N^{2n-1}, E)^{b-n+1},\\ (N^{b-a-1}, E, N^{2i-1}, E, N^{2n+a-b-2i+1}, E)_{i=1}^{\not{m}+n-b-1},\\ N^{b-a-1}\big).$$

- If m is even, then we have the following hamiltonian path $\mathcal{H}_b$ from $\tau_+ E$ to σ_b (see Figure 7):

$$[\tau_+ E]\big((N^{2n-1}, E)^{b-n+1},\\ (N^{b-a-1}, E, N^{2i-1}, E, N^{2n+a-b-2i+1}, E)_{i=1}^{\not{m}+n-b-2},\\ N^{b-a-1}, E, N^{n+a-b}, E, N^{b-a-1}\big).$$ □

We conclude this section by finding the terminal square in the exceptional case that is at the end of the statement of Proposition 3.1:

Lemma 3.3. *Let τ be any square in $\mathcal{B}_{m,m-2}$, with $m \geq 5$. There is a hamiltonian path $\mathcal{H}$ from $(m-2, 0)$ to τ in which all nonterminal diagonals travel north if and only if $\tau = (\not{m} - 1, \not{m}^- - 1)$.*

Proof. Let $n = m - 2$, $a = m - 3 = n - 1$ and $b = a + 1 = n$.

($\Rightarrow$) Since the initial square is $(m-2, 0) = (a, 0)E$, we know from Corollary 2.14 that S_a is a terminal subdiagonal. Then the other part of the terminal diagonal is $S_{m+n-3-a} = S_b$.

Since $\mathcal{H}$ does not contain the cycle $[(m-1, 0)](N^{2n})$, we know that $(0, n-1)$ does not travel north. From Corollary 2.15, this implies that the terminal square τ is somewhere in S_a, and that every square in S_b travels east.

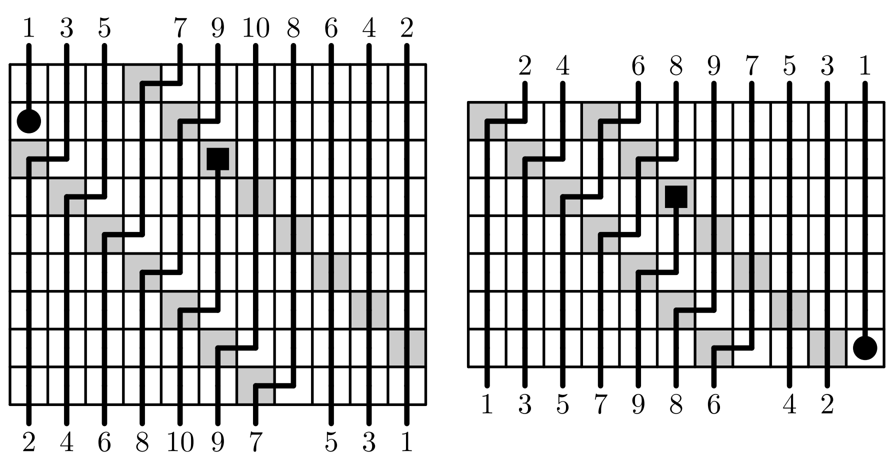

Figure 6. Illustrative examples of the hamiltonian path $\mathcal{H}_b$ from $\tau_+ E$ (●) to σ_b (■) when m is odd. (The terminal diagonal is shaded.)

There are no inner diagonals (since $b = a + 1$), so, by Corollary 2.20(1), we may let $\mathcal{H}'$ be the hamiltonian path from $(m-2, 0)$ to τ in which all nonterminal diagonals travel east.

Case 1: Assume m and n are odd. Since $\mathcal{H}'$ does not contain the cycle $[(0, \not{n})](E^m)$, we know that $(\not{n}, \not{n})$ does not travel east. We may also assume it is not the terminal square, for otherwise $\tau = (\not{n}, \not{n}) = (\not{m} - 1, \not{m}^- - 1)$ (since $\not{n} = \not{n}^-$ and $n = m - 2$), as desired. So $(\not{n}, \not{n})$ travels north. From Corollary 2.15, we conclude that $(\not{n} + 1, \not{n} - 1)$ also travels north. So $\mathcal{H}'$ contains the cycle

$$(\not{n}, \not{n}) \xrightarrow{N} (\not{n}, \not{n} + 1) \xrightarrow{E^{m+1}} (\not{n} + 1, \not{n} - 1) \xrightarrow{N} (\not{n} + 1, \not{n}) \xrightarrow{E^{m-1}} (\not{n}, \not{n}).$$

This is a contradiction.

Case 2: Assume m and n are even. Since $\mathcal{H}'$ does not contain the cycle $[(0, \not{n})](E^{2m})$, we know that $(\not{n}, \not{n}^-)$, $(\not{n}, \not{n})$, $(\not{n}^-, \not{n})$, and $(\not{n}^-, \not{n} + 1)$ do not all travel east. From Corollary 2.15, we conclude that $(\not{n}, \not{n}^-)$ does not travel east. We may also assume it is not the terminal square, for otherwise $\tau = (\not{n}, \not{n}^-) = (\not{m} - 1, \not{m}^- - 1)$, as desired. So $(\not{n}, \not{n}^-)$ travels north. Then $\mathcal{H}'$ contains the cycle

$$(\not{n}, \not{n}^-) \xrightarrow{N} (\not{n}, \not{n}) \xrightarrow{E^m} (\not{n}, \not{n}^-).$$

This is a contradiction.

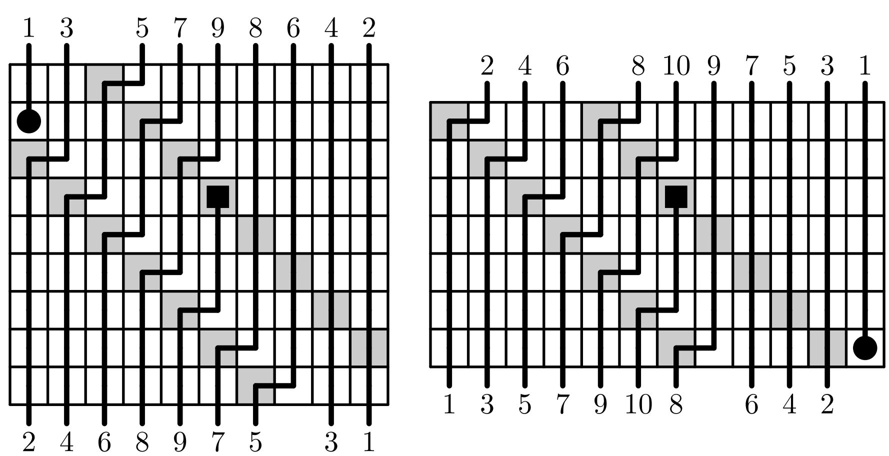

FIGURE 7. Illustrative examples of the hamiltonian path $\mathcal{H}_b$ from $\tau_+ E$ (●) to σ_b (■) when m is even. (The terminal diagonal is shaded.)

($\Leftarrow$) We have the following hamiltonian path from $(m-2,0)$ to $(\not{m}-1,\not{m}^{-}-1)$ in $\mathcal{B}_{m,n}$ when $n=m-2$ (see Figure 8):

$$\begin{cases} [(m-2,0)]\big((E,N^{2n+1-2i},E^2,N^{2i})_{i=1}^{\not{n}},E,N^n\big) & \text{if } m \text{ and } n \text{ are odd,} \\ [(m-2,0)](E,N^{2n+1-2i},E^2,N^{2i})_{i=1}^{\not{n}}\# & \text{if } m \text{ and } n \text{ are even} \end{cases}$$

(where # indicates deletion of the last term of the sequence). □

4. Reduction to diagonals that travel north

Definition 4.1. A diagonal $S_i \cup S_j$ of $\mathcal{B}_{m,n}$ with $i \le j$ is said to be *rowful* if $n-1 \le i \le j \le m-2$. (In other words, $S_i \cup S_j$ is rowful if S_i and S_j each contain a square from every row of the checkerboard.) The subdiagonals S_i and S_j of a rowful diagonal are also said to be *rowful*.

Lemma 4.4 below shows that if a rowful diagonal travels east, then it basically just stretches the checkerboard to make it wider (see Figure 9). Proposition 4.6 uses this observation to show that finding a hamiltonian path between any two given squares of $\mathcal{B}_{m,n}$ reduces to the problem of finding a hamiltonian path in a smaller checkerboard, such that all nonterminal diagonals travel north.

Warning 4.2. The subdiagonal S_{m-1} is *not* rowful, even though it contains a square from every row of the checkerboard (if $m \ge n$), because it is a constituent of the diagonal $S_{n-2} \cup S_{m-1}$, which is not rowful.

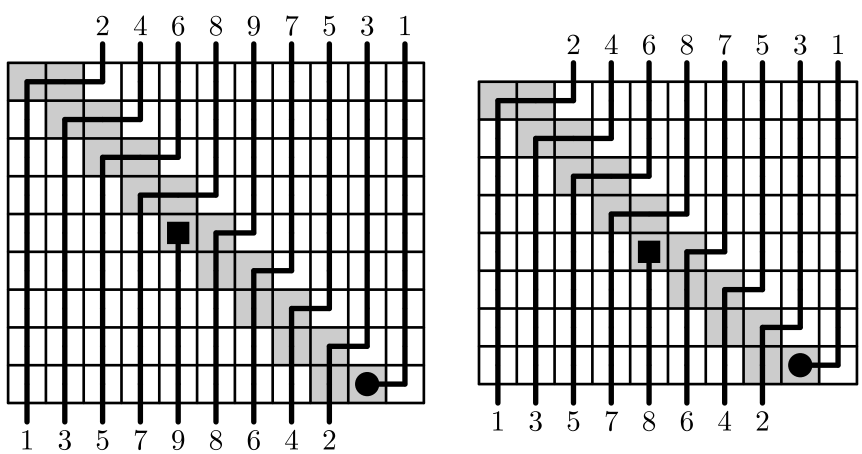

Figure 8. Illustrative examples (with m odd on the left, and m even on the right) of the hamiltonian path from $(m-2,0)$ (●) to $(\not{m}-1,\not{m}^{-}-1)$ (■) in $\mathcal{B}_{m,m-2}$, such that all nonterminal diagonals travel north. (The terminal diagonal is shaded.)

Notation 4.3. For $i, j \in \mathbb{N}$, define $\Delta_{i,j} \colon \mathbb{N} \to \{0, 1, 2\}$ by

$$\Delta_{i,j}(k) = \bigl|\{i, j\} \cap \{0, 1, 2 \ldots, k-1\}\bigr|.$$

Then, for each square (p, q) of $\mathcal{B}_{m,n}$, we let

$$\Delta^{\square}_{i,j}(p, q) = \bigl(p - \Delta_{i,j}(p+q), q\bigr).$$

Lemma 4.4. *Suppose*

- *τ_0 and τ are two squares of $\mathcal{B}_{m,n}$ that are in the same diagonal, and*
- *$S_i \cup S_j$ is a rowful diagonal of $\mathcal{B}_{m,n}$ that is not the diagonal containing τ_0 and τ.*

Then there is a hamiltonian path $\mathcal{H}$ from $\tau_0 E$ to τ in $\mathcal{B}_{m,n}$, such that $S_i \cup S_j$ travels east, if and only if there is a hamiltonian path $\mathcal{H}'$ from $\bigl(\Delta^{\square}_{i,j}(\tau_0)\bigr)E$ to $\Delta^{\square}_{i,j}(\tau)$ in $\mathcal{B}_{m-\Delta_{i,j}(m+n),n}$.

More precisely, if σ is any square of $\mathcal{B}_{m,n}$ that is not in $S_i \cup S_j$, then the square $\Delta^{\square}_{i,j}(\sigma)$ travels the same direction in $\mathcal{H}'$ as the square σ travels in $\mathcal{H}$.

Proof. Assume, for the moment, that $j \neq i+1$ (so $S_i E \cap S_j = \emptyset$). Define a digraph $\mathcal{B}'$ from $\mathcal{B}_{m,n}$ by

(1) replacing each directed edge $\sigma \to \phi$, such that $\phi \in S_i \cup S_j$, with a directed edge from σ to ϕE, and
(2) deleting all the squares in $S_i \cup S_j$ (and the incident edges).

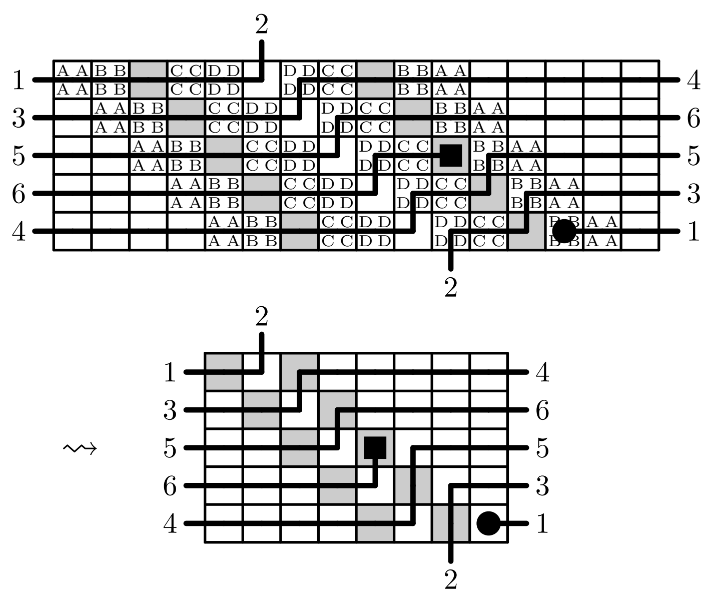

FIGURE 9. The diagonals marked $\substack{\mathrm{A\,A}\\ \mathrm{A\,A}}$, $\substack{\mathrm{B\,B}\\ \mathrm{B\,B}}$, $\substack{\mathrm{C\,C}\\ \mathrm{C\,C}}$, $\substack{\mathrm{D\,D}\\ \mathrm{D\,D}}$ are rowful and travel east. Removing them yields a hamiltonian path in a smaller checkerboard. (As usual, the terminal diagonal is shaded.)

It is clear that hamiltonian paths in $\mathcal{B}'$ correspond to hamiltonian paths in $\mathcal{B}_{m,n}$ such that $S_i \cup S_j$ travels east. Since the digraph $\mathcal{B}'$ is isomorphic to $\mathcal{B}_{m-\Delta_{i,j}(m+n),n}$ (via the map $\Delta^{\square}_{i,j}$), the desired conclusion is immediate.

If $j = i+1$, then the definition of $\mathcal{B}'$ needs a slight modification: instead of considering only a directed edge $\sigma \to \phi$, one needs to allow for the possibility of a longer path. Namely, if there is a path $\sigma \to \phi \xrightarrow{E} \alpha$ with $\phi \in S_i$ and $\alpha \in S_j$, then, instead of inserting the edge σ to ϕE (which cannot exist in $\mathcal{B}'$, because $\phi E = \alpha$ is one of the squares deleted in (2)), one inserts the edge $\sigma \to \alpha E$, because $\mathcal{H}$ must proceed from σ to αE (via ϕ and α) if it travels from σ to ϕ. □

When $m = n$, it was proved in [2, Prop. 3.3] that if some inner diagonal travels east in a hamiltonian path, then all inner diagonals must travel east. That is not always true when $m \neq n$, but we have the following weaker statement:

Lemma 4.5 (cf. [2, Prop. 3.3])**.** *Let $\mathcal{H}$ be a hamiltonian path in $\mathcal{B}_{m,n}$ with $m \geq n$. If D is any inner diagonal that travels east in $\mathcal{H}$, then either D is rowful, or all inner diagonals travel east.*

Proof. By repeated application of Lemma 4.4, we may assume that all rowful diagonals travel north. In this situation, we wish to show that if some inner diagonal $S_i \cup S_j$ travels east, then all inner diagonals travel east. Assume j is minimal (or, equivalently, that i is maximal), such that $S_i \cup S_j$ is an inner diagonal that travels east, and $i \leq j$. This means $S_{i+1}, S_{i+2}, \ldots, S_{j-1}$ all travel north. From the first sentence of the proof, we know that $S_i \cup S_j$ is not rowful, so $j \geq m-1$. Therefore, we may let $\sigma = (m-1, j-m+1)$. Since $\sigma \in S_j$ and the first coordinate of σ is $m-1$, we see that $\sigma E N^{-1} \in S_i$, so $\sigma E \in S_{i+1}$. So $\mathcal{H}$ contains the cycle $[\sigma](E, N^{2(j-m)+3})$. This is a contradiction. □

The following result essentially reduces the proof of Theorem 1.3 to the special case considered in Section 3, where all nonterminal diagonals travel north. (Although the diagonals travel east in conclusion (3b) of the following proposition, passing to the transpose yields a hamiltonian path in which all nonterminal diagonals travel north, because the checkerboard $\mathcal{B}_{m',n}$ is square in this case.)

Proposition 4.6. *Assume*

- $S_a \cup S_b$ *is a diagonal of* $\mathcal{B}_{m,n}$*, with* $m \geq n$*,* $a \leq b$*, and* $a+b \neq m+n-2$*,*
- (x,y) *is a square in* $S_a \cup S_b$*,*
- (p,q) *is a square of* $\mathcal{B}_{m,n}$ *with* $p+q-1 \in \{a,b\}$*,*
- $o = \max\big(a-n+1, 0\big)$*,*
- $e \in \mathbb{N}$*,*
- $e_1 = \begin{cases} 0 & \text{if } p+q-1 = a, \\ e & \text{if } p+q-1 = b, \end{cases}$

- $e_2 = \begin{cases} 0 & \text{if } x + y = a, \\ e & \text{if } x + y = b, \end{cases}$
- $m' = m - 2o - e$.

There is a hamiltonian path $\mathcal{H}$ from (p, q) to (x, y) in $\mathcal{B}_{m,n}$, such that exactly e rowful inner subdiagonals travel east, if and only if

(1) $0 \le e \le \max\bigl(\min(m - n, b - a - 1), 0\bigr)$,

(2) *e is even if $m + n$ is even, and*

(3) *there is a hamiltonian path $\mathcal{H}'$ from $(p - o - e_1, q)$ to $(x - o - e_2, y)$ in $\mathcal{B}_{m',n}$, such that either*

  (a) *all nonterminal diagonals travel north in $\mathcal{H}'$, and $m' \ge n$, or*

  (b) *all nonterminal diagonals travel east in $\mathcal{H}'$, and $m' = n \ge a+3$.*

Proof. ($\Rightarrow$) Note, first, that $\max\bigl(\min(m - n, b - a - 1), 0\bigr)$ is the number of rowful inner subdiagonals, so (1) is obvious. In addition, if $m + n$ is even, then every diagonal is the union of two distinct subdiagonals (which means that the subdiagonals counted by e come in pairs), so e must be even. This establishes (2).

By Corollary 2.20(1), we may assume that all outer diagonals travel east. The definition of o implies that it is the number of rowful outer diagonals. Furthermore, for any outer diagonal $S_i \cup S_j$, we have $i < a \le b < j$ (and, by assumption, we have $p + q \in \{a + 1, b + 1\}$ and $x + y \in \{a, b\}$). Therefore

$$\Delta_{i,j}(p, q) = \Delta_{i,j}(x, y) = |\{i\}| = 1.$$

Therefore, by repeated application of Lemma 4.4, we conclude that there is a hamiltonian path $\mathcal{H}'$ from $(p - o, q)$ to $(x - o, y)$ in $\mathcal{B}_{m-2o,n}$, such that exactly e rowful inner subdiagonals travel east in $\mathcal{H}'$.

The definitions of e_1 and e_2 imply that

$$e_1 = \sum \Delta_{i,j}(p, q) \quad \text{and} \quad e_2 = \sum \Delta_{i,j}(x, y),$$

where the sums are taken over all pairs (i, j) such that $S_i \cup S_j$ is a rowful diagonal that travels east (with $i \le j$). Therefore, repeated application of Lemma 4.4 to $\mathcal{H}'$ yields a hamiltonian path $\mathcal{H}''$ from $(p - o - e_1, q)$ to $(x - o - e_2, y)$ in $\mathcal{B}_{m-2o-e,n}$, such that no rowful inner diagonals travel east.

- If $a \le n-1$, then $o = 0$ and $e \le m-n$, so $m' \ge m-2(0)-(m-n) = n$.
- If $a \ge n$, then $o = a - n + 1$ and $e \le \max(b - a - 1, 0) \le b - a$, so

$$\begin{aligned} m' &= m - 2o - e \ge m - 2(a - n + 1) - (b - a) \\ &= m + 2n - 2 - (a + b) = n + 1 > n. \end{aligned}$$

In either case, we have $m' \ge n$.

The terminal diagonal of $\mathcal{H}''$ is $S_{a'} \cup S_{b'}$, where

$$\begin{cases} a' = a - o & \text{if } x + y = a, \\ b' = b - o - e & \text{if } x + y = b. \end{cases}$$

By definition, we have $o \geq a - n + 1$. Therefore:

- If $x + y = a$, then
$$a' = a - o \leq a - (a - n + 1) = n - 1,$$
so $(0, n-1)$ is not in an outer diagonal of $\mathcal{H}'$.
- If $x + y = b$, then
$$\begin{aligned} m' - 2 &= m - 2o - e - 2 = (m - o - e) - o - 2 \\ &\leq (m - o - e) - (a - n + 1) - 2 = b', \end{aligned}$$
so $(m' - 2, 0)$ is not in an outer diagonal of $\mathcal{H}'$.

In either case, Corollary 2.20(2) tells us that changing all of the outer diagonals of $\mathcal{H}''$ to travel north yields a hamiltonian path $\mathcal{H}'''$ with the same endpoints. If no inner diagonal travels east in $\mathcal{H}''$, then all nonterminal diagonals travel north in $\mathcal{H}'''$, so $\mathcal{H}'''$ is a hamiltonian path as described in conclusion (3a).

We may now assume some inner diagonal $S_i \cup S_j$ travels east in $\mathcal{H}''$. From the definition of $\mathcal{H}''$, we know $S_i \cup S_j$ is not rowful. So Lemma 4.5 tells us that all inner diagonals travel east. Since we have already assumed (near the start of the proof) that all outer diagonals travel east, this implies that all nonterminal diagonals travel east.

Note that there are no rowful nonterminal diagonals in $\mathcal{B}_{m',n}$. (All inner diagonals travel east, but, by the definition of $\mathcal{H}''$, no rowful diagonal travels east.) Since, by the assumption of the preceding paragraph, some inner diagonal travels east, this implies there must be at least one inner diagonal that is not rowful. So $a \leq n - 3$.

All that remains is to show that $m' = n$. Suppose not, which means $n < m'$. So $S_{n-1} \cup S_{m'-2}$ is a rowful diagonal, and $n - 1 \leq m' - 2$. This is not the terminal diagonal $S_a \cup S_b$, because $a \leq n - 3$. This contradicts the fact that there are no rowful nonterminal diagonals.

($\Leftarrow$) We need to construct a hamiltonian path $\mathcal{H}$ from (p, q) to (x, y). (For convenience, we will call a diagonal of $\mathcal{B}_{m,n}$ *inner* if it would be inner with respect to such a hamiltonian path.) From (1), we know that e is no more than the number of rowful inner subdiagonals, so we may choose a set $\mathcal{E}$ of $\lceil e/2 \rceil$ rowful inner diagonals. Furthermore:

(1) If e is even, we choose each diagonal in $\mathcal{E}$ to be the union of two distinct subdiagonals.

(2) If e is odd, then (2) tells us that $m + n$ is also odd, so we may choose $\mathcal{E}$ to contain the diagonal $S_{(m+n-3)/2}$ that consists of only one subdiagonal.

Then the diagonals in $\mathcal{E}$ constitute precisely e subdiagonals.

Now, applying Lemma 4.4($\Leftarrow$) (repeatedly) to the $\lceil e/2 \rceil$ rowful inner diagonals in $\mathcal{E}$ and to all o rowful outer diagonals of $\mathcal{B}_{m,n}$ yields a hamiltonian path $\mathcal{H}$ from (p, q) to (x, y) in $\mathcal{B}_{m,n}$.

Note that all rowful inner diagonals travel north in $\mathcal{H}'$. (Namely, either all nonterminal diagonals travel north, or $m' = n$, in which case, there are no rowful inner diagonals, so the claim is vacuously true.) Therefore, the diagonals in $\mathcal{E}$ are the only rowful inner diagonals that travel east in $\mathcal{H}$. So exactly e rowful inner subdiagonals travel east, as desired. □

5. The general case

In this section, we utilize Proposition 4.6 and the results of Section 3 to determine which pairs of squares in $\mathcal{B}_{m,n}$ are joined by a hamiltonian path $\mathcal{H}$, and use this information to establish Theorems 1.3 and 1.4. First, Proposition 5.2 sharply restricts the possibilities for the initial square (perhaps after replacing $\mathcal{H}$ with its inverse). Then Propositions 5.4 to 5.6 determine the terminal squares of the hamiltonian paths (if any) that start at each of these potential initial squares. Theorems 1.3 and 1.4 are straightforward consequences of these much more detailed results.

Proposition 4.6 will be employed several times in this section. To facilitate this, we fix the following notation:

Notation 5.1. Given a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_{m,n}$ (with $m \geq n \geq 3$), we let:

- $S_a \cup S_b$ be the terminal diagonal of $\mathcal{H}$, with $a \leq b$,
- (x, y) be the terminal square of $\mathcal{H}$ (so $(x, y) \in S_a \cup S_b$),
- (p, q) be the initial square of $\mathcal{H}$ (so $p + q - 1 \in \{a, b\}$ by Corollary 2.14),
- $o = \max\big(a - n + 1, 0\big)$,
- e be the number of rowful inner subdiagonals that travel east in $\mathcal{H}$,
- $e_1 = \begin{cases} 0 & \text{if } p + q - 1 = a, \\ e & \text{if } p + q - 1 = b, \end{cases}$
- $e_2 = \begin{cases} 0 & \text{if } x + y = a, \\ e & \text{if } x + y = b, \end{cases}$
- $m' = m - 2o - e$,
- $p' = p - o - e_1$,
- $x' = x - o - e_2$,
- $\mathcal{H}'$ be the hamiltonian path from (p', q) to (x', y) in $\mathcal{B}_{m',n}$ that is provided by Proposition 4.6,
- $S_{a'} \cup S_{b'}$ be the terminal diagonal of $\mathcal{H}'$, with $a' \leq b'$, so

$$a' = a - o \quad \text{and} \quad b' = b - o - e,$$

- τ_+ be the southeasternmost square of S_b in $\mathcal{B}_{m,n}$,
- τ'_+ be the southeasternmost square of $S_{b'}$ in $\mathcal{B}_{m',n}$.

Proposition 5.2. *Assume $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_{m,n}$ with $m \geq n \geq 3$. Then the initial square of either $\mathcal{H}$ or the inverse of $\mathcal{H}$ is $\tau_+ E$, unless all inner diagonals travel east, in which case, the initial square (of either $\mathcal{H}$ or $\widetilde{\mathcal{H}}$) might also be of the form $(p, 0)$, with $1 \leq p \leq \lfloor (m+n)/2 \rfloor - 1$.*

Proof. We consider the two possibilities presented in Proposition 4.6(3) as separate cases.

Case 1: Assume all nonterminal diagonals travel north in $\mathcal{H}'$. From the conclusion of Proposition 3.1, we see that there are three possibilities to consider (perhaps after replacing $\mathcal{H}$ with its inverse, which also replaces $\mathcal{H}'$ with its inverse).

Subcase 1.1: Assume the initial square of $\mathcal{H}'$ is $\tau'_+ E$. Then the initial square of $\mathcal{H}$ is $\tau_+ E$.

Subcase 1.2: Assume $m' = n = a' + 2 = b' + 1$ and the initial square of the transpose of $\mathcal{H}'$ is $\tau'_+ E$. Since $\tau'_+ = (b', 0) = (m' - 1, 0)$, we have $\tau'_+ E = (0, n - 1)$, so the initial square of $\mathcal{H}'$ is the transpose of this, namely $(n - 1, 0)$. This means $p' = n - 1$ and $q = 0$, so the initial square of $\mathcal{H}$ is

$$(p, q) = (p' + o + e_1, q) = (n - 1 + o + e_1, 0),$$

which is obviously of the form $(p, 0)$. (Also, since $a' \neq b'$, we have $a \neq b$, so $a < (m + n - 3)/2$, which means $a \leq \lfloor (m + n - 4)/2 \rfloor = \lfloor (m + n)/2 \rfloor - 2$. Therefore $p = p + q = a + 1 \leq \lfloor (m + n)/2 \rfloor - 1$.) Furthermore, since $a' + 2 = b' + 1$, we have $a' + 1 = b'$, so $\mathcal{H}'$ has no inner diagonals. This implies that every inner diagonal of $\mathcal{H}$ travels east.

Subcase 1.3: Assume $a' + 1 = b' = n = m' - 2$ and the initial square of $\mathcal{H}'$ is $(n, 0)$. Then $e_1 = 0$ (since $p' + q - 1 = n - 0 - 1 = a'$) and the initial square of $\mathcal{H}$ is $(n + o, 0)$, which is of the form $(p, 0)$. (Also, since $a' \neq b'$, we have $p \leq \lfloor (m + n)/2 \rfloor - 1$, as in the preceding subcase.) Furthermore, since $a' + 1 = b'$, we know that $\mathcal{H}'$ has no inner diagonals, so every inner diagonal of $\mathcal{H}$ travels east.

Case 2: Assume all nonterminal diagonals travel east in $\mathcal{H}'$, and $m' = n \geq a' + 3$. Since $m' = n$, we may let $(\mathcal{H}')^*$ be the transpose of $\mathcal{H}'$. All nonterminal diagonals travel north in $(\mathcal{H}')^*$ (and $n \notin \{a + 1, a + 2\}$), so Proposition 3.1 tells us (perhaps after replacing $\mathcal{H}$ with its inverse) that the initial square $(\iota')^*$ of $(\mathcal{H}')^*$ is $\tau'_+ E$. We have

$$b' = m' + n - (a' + 3) \geq m' + 0 = m',$$

so $\tau'_+ = (m' - 1, b' - m' + 1)$, which means $\tau'_+ E = (0, a' + 1)$. Therefore the initial square of $\mathcal{H}'$ is the transpose of this, namely $(a' + 1, 0)$. So the initial square of $\mathcal{H}$ is $(a' + 1 + o, 0) = (a + 1, 0)$, which is of the form $(p, 0)$. (Also, since $a' \leq n - 3$, we have $a' \neq b'$, so $p \leq \lfloor (m + n)/2 \rfloor - 1$, as in the preceding subcases.) □

It is important to note that the possibilities for the square $\tau_+ E$ can be described quite precisely:

Lemma 5.3. *If $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_{m,n}$, with $m \geq n$, then either:*

(1) *$\tau_+ E = (0, q)$ with $1 \leq q \leq n-1$, or*

(2) *$\tau_+ E = (p, 0)$, with $\lceil (m+n-1)/2 \rceil \leq p \leq m-1$ (and $m \neq n$).*

Thus, Proposition 5.2 tells us that the initial square of $\mathcal{H}$ is of the form $(0, q)$ or $(p, 0)$ (perhaps after passing to the inverse). We will find all of the corresponding terminal squares in Propositions 5.4 to 5.6.

Proposition 5.4. *Assume $\iota = (0, q)$ with $1 \leq q \leq n-1$ and $m \geq n \geq 3$. There is a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_{m,n}$ from ι to (x, y) if and only if either*

(1) *$x + y = q - 1 \geq \not{n}^-$ and $\not{n}^- \leq x \leq \not{m}^-$, or*

(2) *$x + y = m + n - q - 2$ and $\not{m} \leq x \leq m - \not{n}^+$ (and $q \geq \not{n}^-$).*

Proof. ($\Rightarrow$) Note that $a = a' = q - 1$,

$$b = m + n - 3 - a = m + n - q - 2,$$

$o = e_1 = 0$, and $\tau'_+ = (m' - 1, n - q - 1)$. Also note that, since $a \leq n - 2$, the largest possible value of e is $m - n$. Proposition 4.6(3) gives us two cases to consider.

Case 1: Assume all nonterminal diagonals travel north in $\mathcal{H}'$. Proposition 3.2 tells us there are (at most) two possibilities for the terminal square (x', y).

Subcase 1.1: Assume $x' + y = a$ and $x' = \lfloor (m' - 1)/2 \rfloor$. We have $x + y = a = q - 1$ and (since $o = e_2 = 0$)

$$x = x' = \lfloor (m' - 1)/2 \rfloor = \lfloor (m - e - 1)/2 \rfloor.$$

The smallest possible value of e is 0, so $x \leq \not{m}^-$. Conversely, since the largest possible value of e is $m - n$, we have $x \geq \lfloor (m - (m-n) - 1)/2 \rfloor = \not{n}^-$. (Therefore $q - 1 \geq x \geq \not{n}^-$.)

Subcase 1.2: Assume $x' + y = b' \neq a$ and $x' = \lfloor m'/2 \rfloor$. We have

$$x + y = b = m + n - q - 2$$

and

$$x = x' + e = \lfloor m'/2 \rfloor + e = \lfloor (m - e)/2 \rfloor + e = \lfloor (m + e)/2 \rfloor.$$

The smallest possible value of e is 0, so $x \geq \not{m}$. Conversely, since the largest possible value of e is $m - n$, we have

$$x \leq \lfloor (2m - n)/2 \rfloor = m - \lceil n/2 \rceil = m - \not{n}^+.$$

Therefore

$$m + n - q - 2 = x + y \leq (m - \not{n}^+) + (n - 1),$$

so $q \geq \not{n}^+ - 1 = \not{n}^-$.

Case 2: Assume all nonterminal diagonals travel east in $\mathcal{H}'$ and $m' = n \geq a + 3$. Since $m' = n$, we may let $(\mathcal{H}')^*$ be the transpose of $\mathcal{H}'$. Then, let $\widetilde{(\mathcal{H}')^*}$ be the inverse of $(\mathcal{H}')^*$, and note that all nonterminal diagonals travel north in both $(\mathcal{H}')^*$ and $\widetilde{(\mathcal{H}')^*}$. Since $n \notin \{a+1, a+2\}$, Proposition 3.1 tells us that $\tau'_+ E = (0, q)$ is the initial square of either $(\mathcal{H}')^*$ or $\widetilde{(\mathcal{H}')^*}$. Since the initial square of $(\mathcal{H}')^*$ is $(q, 0) \neq (0, q)$, we conclude that $\tau'_+ E$ is the initial square of $\widetilde{(\mathcal{H}')^*}$. The terminal square of this hamiltonian path is the inverse $(m' - q - 1, m' - 1)$ of the initial square of $(\mathcal{H}')^*$.

Also, we see from Proposition 3.2 that the terminal square of $\widetilde{(\mathcal{H}')^*}$ is $\bigl(\lfloor m'/2 \rfloor, b' - \lfloor m'/2 \rfloor\bigr)$ (since $(m' - q - 1) + (m' - 1) = b'$). Therefore, we must have $m' - q - 1 = \lfloor m'/2 \rfloor$, so $q = \lfloor (m'-1)/2 \rfloor$. Also, the terminal square of $\mathcal{H}'$ is the transpose of the inverse of the initial square $(0, q)$ of $\widetilde{(\mathcal{H}')^*}$, so

$$x' = m' - 1 - q = \lfloor m'/2 \rfloor \quad \text{and} \quad y = m' - 1 - 0 = m' - 1.$$

This means that we are in precisely the situation considered in Subcase 1.2, so the argument there verifies that the conditions of (2) are satisfied.

($\Leftarrow$) We assume the notation of Proposition 4.6 (with $p = 0$, because the initial square is $(0, q)$).

(1) Since $q \leq n - 1$ and $x + y = q - 1$, we have $a = q - 1 = x + y$, $o = 0$, and $e_1 = e_2 = 0$. Also, since

$$\lfloor (m - 0 - 1)/2 \rfloor = \not{m}^- \geq x \quad \text{and} \quad \lfloor (m - (m - n) - 1)/2 \rfloor = \not{n}^- \leq x,$$

there exists $e \in \{0, 1, \ldots, m - n\}$, such that $\lfloor (m - e - 1)/2 \rfloor = x$. (That is, $\lfloor (m' - 1)/2 \rfloor = x$.) Furthermore, we may assume e is even if $m + n$ is even (because the two extremes 0 and $m - n$ are even in this case). Then Proposition 3.2 provides a hamiltonian path $\mathcal{H}'$ in $\mathcal{B}_{m',n}$ from $(0, q)$ to (x, y), such that all nonterminal diagonals travel north. So Proposition 4.6 yields a hamiltonian path from $(0, q)$ to (x, y) in $\mathcal{B}_{m,n}$.

(2) We have $x + y = b$, $o = 0$, $e_1 = 0$, and $e_2 = e$. Also, since

$$\lfloor (m + 0)/2 \rfloor = \not{m} \leq x \quad \text{and} \quad \lfloor (m + (m - n))/2 \rfloor = m - \not{n}^+ \geq x,$$

there exists $e \in \{0, 1, \ldots, m - n\}$, such that $\lfloor (m + e)/2 \rfloor = x$. Furthermore, we may assume e is even if $m + n$ is even. Then Proposition 3.2 provides a hamiltonian path $\mathcal{H}'$ in $\mathcal{B}_{m',n}$ from $(0, q)$ to $\bigl(\lfloor (m - e)/2 \rfloor, y\bigr) = (x - e, y)$, such that all nonterminal diagonals travel north. So Proposition 4.6 yields a hamiltonian path from $(0, q)$ to (x, y) in $\mathcal{B}_{m,n}$. □

Proposition 5.5. *Assume* $1 \leq p \leq \lfloor (m + n)/2 \rfloor - 1$ *and* $m \geq n \geq 3$. *There is a hamiltonian path* $\mathcal{H}$ *in* $\mathcal{B}_{m,n}$ *from* $(p, 0)$ *to* (x, y)*, such that all inner diagonals travel east, if and only if either*

(1) $x + y = p - 1 \geq \not{n}^-$, *and* $y = \not{n}^-$, *or*

(2) $x + y = m + n - p - 2$, $y = \not{n}$, *and* $\not{n}^- \leq p \leq n - 1$.

Proof. We prove only ($\Rightarrow$), but the argument can be reversed. We may assume that all nonterminal diagonals travel east in $\mathcal{H}$ (see Corollary 2.20(1)). Note that $q = 0$, $a = p - 1$, and $e_1 = 0$. We consider two cases.

Case 1: Assume $p \leq n - 1$. Then $\tau_+ = (m - 1, n - p - 1)$ and $\tau_+ E = (0, p)$. We have $a = p - 1 \leq n - 2$, so $e = m - n$ and $o = 0$, so

$$m' = m - e = m - (m - n) = n.$$

Therefore, we may let $(\mathcal{H}')^*$ be the transpose of $\mathcal{H}'$. The initial square of $(\mathcal{H}')^*$ is $(0,p) = \tau'_+ E$, and all nonterminal diagonals travel north in $(\mathcal{H}')^*$, so Proposition 3.2 tells us there are only two possible terminal squares. (Note that, since we are considering the transpose, the role of x in Proposition 3.2 is played by y here.)

Subcase 1.1: Assume $x' + y = a$ and $y = \lfloor (m'-1)/2 \rfloor$. We have $x + y = x' + y = a = p - 1$ and $y = \not{n}^-$. Since $x + y = p - 1$, this implies $p - 1 \geq \not{n}^-$.

Subcase 1.2: Assume $x' + y = b'$ and $y = \lfloor m'/2 \rfloor$. We have

$$x + y = b = m + n - 3 - a = m + n - p - 2$$

and $y = \not{n}$. Since $x + y = m + n - p - 2$ and $x \leq m - 1$, this implies $p \geq \not{n}^-$.

Case 2: Assume $p \geq n$. All inner diagonals are rowful (since $a = p - 1 \geq n - 1$), and they all travel east in $\mathcal{H}$ (by assumption), so $\mathcal{H}'$ has no inner diagonals, which means $b' \leq a' + 1$. Also, since $p \leq \lfloor (m+n)/2 \rfloor - 1$, we have

$$a = p - 1 \leq \left\lfloor \frac{m+n-4}{2} \right\rfloor < \frac{m+n-3}{2} \leq b.$$

Thus $a \neq b$, so $a' \neq b'$. Therefore $b' = a' + 1$.

Since $a = p - 1 \geq n - 1$, we have $o = a - n + 1 = p - n$, which means $p' = n$, so $a' = n - 1$. Since $b' = a' + 1$, this implies $m' = n + 2$. Then $m' \neq n$, so Proposition 4.6(3) tells us that all nonterminal diagonals travel north in $\mathcal{H}'$. Then Lemma 3.3 tells us that $y = \lfloor (m'-1)/2 \rfloor - 1 = \not{n}^-$. We also have $x + y = a = p - 1$ (and $p - 1 \geq n - 1 \geq \not{n}^-$). □

Proposition 5.6. *Assume $\lfloor (m+n)/2 \rfloor \leq p \leq m - 1$, and $m > n \geq 3$. There is a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_{m,n}$ from $(p,0)$ to (x,y) if and only if either*

(1) *$x + y = m + n - p - 2$, $m - p + \not{n}^- \leq x \leq \not{m}^-$, and $p \neq (m+n-1)/2$, or*

(2) *$x + y = p - 1$ and $\not{m} \leq x \leq p - \not{n}^+$, or*

(3) *$x = \not{m}^-$, $y = \not{n}^-$, and $p = (m+n-1)/2$ (so $m + n$ is odd).*

Proof. ($\Rightarrow$) Note that $b = p - 1$, $o = a - n + 1$ (since $b \leq m - 2$), $a' = n - 1$, $q = 0$, $e_1 = e$, and $\tau'_+ = (b', 0)$. The largest possible value of e is $2p - m - n$, unless $m + n$ is odd and $p = (m+n-1)/2$, in which case the only value of e is 0. As usual, Proposition 4.6(3) gives us two cases to consider.

Case 1: Assume all nonterminal diagonals travel north in $\mathcal{H}'$. Proposition 3.2 tells us there are (at most) two possibilities for the terminal square (x', y) of $\mathcal{H}'$.

Subcase 1.1: Assume $x' + y = a'$ and $x' = \lfloor (m'-1)/2 \rfloor$. We have $x + y = a = m + n - 3 - b = m + n - p - 2$. Also,

$$x = x' + o + e_2 = \lfloor (m'-1)/2 \rfloor + o + 0 = \lfloor (m - e - 1)/2 \rfloor.$$

The smallest possible value of e is 0, so $x \le \not{m}^-$. Conversely, if $p \ne (m+n-1)/2$, then the largest possible value of e is $2p-m-n$, so $x \ge m-p+\not{n}^-$. However, if $p=(m+n-1)/2$, then $e=0$, so $x=\not{m}^-$ and

$$y = a - x = m + n - p - 2 - \not{m}^- = \not{n}^-.$$

Subcase 1.2: Assume $x'+y=b'$, $x'=\lfloor m'/2\rfloor$, and $a' \ne b'$. We have $x+y=b=p-1$. Also, since $a' \ne b'$, we have $a \ne b$, so $b \ne (m+n-3)/2$, which means $p \ne (m+n-1)/2$. Furthermore,

$$x = x' + o + e = \lfloor m'/2 \rfloor + o + e = \lfloor (m+e)/2 \rfloor.$$

The smallest possible value of e is 0, so $x \ge \not{m}$. Conversely, since the largest possible value of e is $2p-m-n$, we have $x \le p - \not{n}^+$.

Case 2: Assume all nonterminal diagonals travel east in $\mathcal{H}'$, and $m' = n \ge a'+3$. To see that this is impossible, recall that the initial square of $\mathcal{H}$ is $(b+1,0) = \tau_+ E$, so the initial square of $\mathcal{H}'$ is $(b'+1,0) = \tau'_+ E$. This implies $b'+1 \le m'-1$. Therefore $b' \le m'-2$. However, we must have $b' \ge m'-1$, since $m'=n$. This is a contradiction.

($\Leftarrow$) We assume the notation of Proposition 4.6 (with $q=0$, because the initial square is $(p,0)$). Note that $b = p-1 \ge \lceil (m+n-3)/2 \rceil$, so $e_1 = e$ and $o = a-n+1$. Also, we have $b-a-1 = 2p-m-n$.

(1) We have $x+y=a$, so $e_2=0$. Since

$$\lfloor (m-0-1)/2 \rfloor = \not{m}^- \ge x$$

and

$$\lfloor (m-(2p-m-n)-1)/2 \rfloor = m-p+\not{n}^- \le x,$$

there is some e, such that $0 \le e \le 2p-m-n$ and $\lfloor (m-e-1)/2 \rfloor = x$. Furthermore, we may assume e is even if $m+n$ is even. Proposition 3.2 provides a hamiltonian path $\mathcal{H}'$ in $\mathcal{B}_{m',n}$ from $(p-o-e,0)$ to $(\lfloor (m'-1)/2\rfloor, y')$, for some y', such that the terminal square is on the lower subdiagonal of the terminal diagonal. So Proposition 4.6 yields a hamiltonian path from $(p,0)$ to $(\lfloor (m'-1)/2 \rfloor + o, y') = (x,y')$ in $\mathcal{B}_{m,n}$. Since (x,y') is on the lower subdiagonal S_a of the terminal diagonal, we must have $y'=y$.

(2) We have $x+y=b$, so $e_2=e$. Since

$$\lfloor (m+0)/2 \rfloor = \not{m} \le x$$

and

$$\lfloor \big(m+(b-a-1)\big)/2 \rfloor = \lfloor \big(m+(2p-m-n)\big)/2 \rfloor = p - \not{n}^+ \ge x,$$

there is some e, such that $0 \le e \le b-a+1$ and $\lfloor (m+e)/2 \rfloor = x$. Furthermore, we may assume e is even if $m+n$ is even. Proposition 3.2 provides a hamiltonian path $\mathcal{H}'$ in $\mathcal{B}_{m',n}$ from $(p-o-e,0)$ to $(\lfloor m'/2 \rfloor, y')$, for some y', such that the terminal square is on the upper subdiagonal of the terminal diagonal. So Proposition 4.6 yields a hamiltonian path from $(p,0)$ to $(\lfloor (m'-1)/2 \rfloor + o, y')$ in $\mathcal{B}_{m,n}$. Since (x,y') is on the upper subdiagonal S_b of the terminal diagonal, we must have $y'=y$.

(3) Since $b = p - 1 = (m + n - 3)/2$, we have $a = b$. Let $e = 0$. Proposition 3.2 provides a hamiltonian path $\mathcal{H}'$ in $\mathcal{B}_{m',n}$ from $(p - o, 0)$ to $(\lfloor (m' - 1)/2 \rfloor, y')$, for some y', such that $(\lfloor (m' - 1)/2 \rfloor, y')$ is on the terminal diagonal S_{b-o}. So Proposition 4.6 yields a hamiltonian path from $(p, 0)$ to $(\lfloor (m' - 1)/2 \rfloor + o, y') = (\not{m}^-, y') = (x, y')$ in $\mathcal{B}_{m,n}$. Since (x, y') must be on the terminal diagonal S_b, we have $y' = y$. □

Proof of Theorems 1.3 and 1.4. It is immediate from Proposition 2.4 that a square σ is the initial square of a hamiltonian path if and only if its inverse $\widetilde{\sigma}$ is the terminal square of a hamiltonian path. Therefore, Theorems 1.3 and 1.4 are logically equivalent: the squares listed in one theorem are simply the inverses of the squares listed in the other. So it suffices to prove Theorem 1.4. That is, we wish to show that the terminal squares (and the inverses of the initial squares) listed in Propositions 5.4 to 5.6 combine to give precisely the squares listed in Theorem 1.4.

Case 1: The inverses of the initial squares in Proposition 5.4. The set of initial squares is $\{\, (0, q) \mid \not{n}^- \le q \le n - 1 \,\}$. Their inverses form the set $\{\, (m - 1, y) \mid 0 \le y \le n - 1 - \not{n}^- \,\}$. Since $n - 1 - \not{n}^- = \not{n}$, these are precisely the squares in Theorem 1.4(1).

Case 2: The inverses of the initial squares in Propositions 5.5 and 5.6. The combined set of these initial squares is $\{\, (p, 0) \mid \not{n}^- \le p \le m - 1 \,\}$. Their inverses form the set $\{\, (x, n-1) \mid 0 \le x \le m - 1 - \not{n}^- \,\}$. Since $m - 1 - \not{n}^- = m - \not{n}^+$, these are precisely the squares in Theorem 1.4(2).

Case 3: The terminal squares in Proposition 5.5, and also 5.6(3) when m is even and n is odd. The terminal squares in Proposition 5.5(1) have $y = \not{n}^-$, so $x = p - 1 - \not{n}^- = p - \not{n}^+$. Since p can take on any value from $\not{n}^+$ to $\lfloor (m + n)/2 \rfloor - 1$, this means that x ranges from 0 to

$$\lfloor (m+n)/2 \rfloor - 1 - \not{n}^+ = \begin{cases} \not{m} - 2 & \text{if } m \text{ is even and } n \text{ is odd,} \\ \not{m} - 1 & \text{otherwise.} \end{cases}$$

Thus, these are precisely the squares listed in Theorem 1.4(3), except that the square $(\not{m} - 1, \not{n}^-)$ is missing when m is even and n is odd. Fortunately, in this case, the missing square is precisely the square listed in Proposition 5.6(3).

The terminal squares in Proposition 5.5(2) have $y = \not{n}$, so x ranges from

$$m + n - (n - 1) - 2 - \not{n} = m - \not{n} - 1$$

to

$$m + n - \not{n}^- - 2 - \not{n} = m - 1.$$

Thus, these are precisely the squares listed in Theorem 1.4(4).

Case 4: The terminal squares in Propositions 5.4(1) and 5.6(1), and also 5.6(3) when m is odd and n is even. The terminal squares in Proposition 5.4(1) are:

$$\{ (x,y) \mid \not{n}^- \leq x \leq \not{m}^-,\ x+y \leq n-2 \}.$$

Since $x \geq \not{n}^-$, we have $y \leq \not{n} - 1$. So these are the squares listed in Theorem 1.4(5) that satisfy $x + y \leq n - 2$.

Now consider Proposition 5.6(1). Since $x+y = m+n-p-2$, the constraint $x \geq m - p + \not{n}^-$, can be replaced with

$$y \leq m + n - p - 2 - (m - p + \not{n}^-) = n - \not{n}^- - 2 = \not{n} - 1.$$

Also, the range $\lceil (m+n)/2 \rceil \leq p \leq m-1$ means that $x+y$ is allowed to take any value from $n-1$ to

$$m + n - \lceil (m+n)/2 \rceil - 2 = \lfloor (m+n)/2 \rfloor - 2.$$

Since $x + y \geq n - 1$ and $y \leq \not{n} - 1$, we have $x \geq (n-1) - (\not{n} - 1) > \not{n}^-$. Thus, the terminal squares in Proposition 5.4(2) are:

$$\{ (x,y) \mid \not{n}^- \leq x \leq \not{m}^-,\ y \leq \not{n} - 1,\, n - 1 \leq x + y \leq \lfloor (m+n)/2 \rfloor - 2 \}.$$

Therefore, the union of these two sets consists of precisely the squares (x,y) listed in Theorem 1.4(5) that satisfy

$$x + y \leq \lfloor (m+n)/2 \rfloor - 2.$$

However, any square (x,y) listed in Theorem 1.4(5) satisfies $x + y \leq \not{m}^- + \not{n} - 1$. Since

$$\lfloor (m+n)/2 \rfloor - 2 \geq \frac{m}{2} + \frac{n}{2} - \frac{5}{2} \geq \not{m}^- + (\not{n} - 1) - 1,$$

we conclude that in order for a square (x,y) of Theorem 1.4(5) to be missing from the union, equality must hold throughout (so m is odd and n is even) and we must have $x = \not{m}^-$ and $y = \not{n} - 1 = \not{n}^-$. This is precisely the square listed in Proposition 5.6(3). So these three sets together constitute the squares listed in Theorem 1.4(5).

Case 5: The terminal squares in Proposition 5.4(2) and Proposition 5.6(2). First, we consider Proposition 5.6(2). Since $x + y = p - 1$, the constraint $x \leq p - \not{n}^+$ can be replaced with $y \geq \not{n}^-$. Also, since $x + y \leq m - 2$, this implies $x \leq m - 2 - \not{n}^- < m - \not{n}^+$. Observe that there is no harm in replacing the inequality $x < m - \not{n}^+$ with $x \leq m - \not{n}^+$. (If $x = m - \not{n}^+$, then $y \geq \not{n}^-$ implies $x + y \geq m - 1$, which contradicts $x + y \leq m - 2$.) Therefore, the set of terminal squares is

$$\{ (x,y) \mid \not{m} \leq x \leq m - \not{n}^+,\, \not{n}^- \leq y \leq n-1,\ \lfloor (m+n)/2 \rfloor - 1 \leq x+y \leq m-2 \}.$$

We now consider Proposition 5.4(2). The constraint $\not{n}^- \leq q \leq n-1$ means that $x + y$ is allowed to take any value from $m - 1$ to $m + \not{n} - 1$. However, since $x \leq m - \not{n}^+$ and $y \leq n - 1$, the upper bound is redundant.

Also, since $x \le m - \not{n}^+$ and $x + y \ge m - 1$, we must have $y \ge \not{n}^-$. Therefore, the set of terminal squares is

$$\{\, (x, y) \mid \not{m} \le x \le m - \not{n}^+,\ \not{n}^- \le y \le n - 1,\ m - 1 \le x + y \,\}.$$

Therefore, the union of these two sets consists of precisely the squares (x, y) listed in Theorem 1.4(6) that satisfy

$$x + y \ge \lfloor (m + n)/2 \rfloor - 1.$$

However, it is easy to see that every one of the squares satisfies this condition (since $x \ge \not{m}$ and $y \ge \not{n}^-$), so we conclude that these two sets constitute the squares listed in Theorem 1.4(6). □

Remark 5.7. The above results assume $n \ge 3$. For completeness, we state, without proof, the analogous results for $n = 1, 2$. It was already pointed out in Remark 1.5 that every square is both an initial square and a terminal square in these cases, but we now provide a precise list of the pairs of squares that can be joined by a hamiltonian path.

(1) Assume $n = 1$. Then every square in the board is of the form $(*, 0)$. There is a hamiltonian path from $(p, 0)$ to $(x, 0)$ if and only if $(p, 0) = (x, 0)E$. More precisely, every hamiltonian path is obtained by removing an edge from the hamiltonian cycle (E^m).

(2) Assume $m \ge n = 2$. Figure 10 lists the initial square (p, q) and terminal square (x, y) of every hamiltonian path in $\mathcal{B}_{m,2}$.

	initial square (p, q)	terminal square (x, y)	restrictions, if any
A_2	(p, q)	$(p, q)E^{-1}$	
B_2	$(0, 1)$	$(0, 0)$	
C_2	$(1, 0)$	$(m - 2, 1)$	
D_2	$(m - 1, 1)$	$(m - 1, 0)$	
E_2	$(0, 1)$	$(m - 2, 1)$	$m \ge 3$
F_2	$(1, 0)$	$(m - 1, 0)$	m is odd
G_2	$(1, 0)$	$(m - 1, 0)$	$m \ge 4$
H_2	$(p, 1)$	$(m - 2 - p, 1)$	$m \ge 4$ and $\not{m}^+ \le p \le m - 2$
I_2	$(p, 0)$	$(m - p, 0)$	$m \ge 4$ and $p \ge \not{m}^+ + 1$
J_2	$(p, 0)$	$(p - 2, 1)$	$m \ge 5$ and $p \notin \{1, \not{m}, \not{m} + 1\} \smallsetminus \{\not{m}^-\}$

FIGURE 10. Endpoints of the hamiltonian paths in $\mathcal{B}_{m,n}$ when $n = 2$ (and $m \ge 2$).

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Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, T1K 6R4, Canada
E-mail address: `mccarthyd@uleth.ca`
E-mail address: `Dave.Morris@uleth.ca, http://people.uleth.ca/~dave.morris/`