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BARNES-TYPE BOOLE POLYNOMIALS

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ABSTRACT. In this paper we consider Barnes-type Boole polynomials and give some interesting properties and identities of those polynomials which are derived from the fermionic p-adic integrals on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$,and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of all $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim [5, 8, 10] to be

$$
\begin{aligned}
(1.1) \qquad I(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x + p^N \mathbb{Z}_p) \\
&= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x).
\end{aligned}
$$

By Equation (1.1), we get

$$
(1.2) \qquad I(f_1) + I(f) = 2f(0),
$$

where $f_1(x) = f(x + 1)$; see [10, 9, 7] for details.

The Boole polynomials are defined in [15] by the generating function to be

$$
(1.3) \qquad \sum_{n=0}^{\infty} Bl_n(x|\lambda) \frac{t^n}{n!} = \frac{1}{1 + (1 + t)^\lambda} (1 + t)^x,
$$

while the Peters polynomials are given in [15] by

$$
(1.4) \qquad \left(\frac{1}{1 + (1 + t)^w} \right)^r (1 + t)^x = \sum_{n=0}^{\infty} S_n(x; w, r) \frac{t^n}{n!}.
$$

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As is known, the Barnes-type Euler polynomials are defined by the generating function to be

$$\frac{2^r}{(e^{w_1 t}+1)(e^{w_2 t}+1)\dots(e^{w_r t}+1)}e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x|w_1,\dots,w_r)\frac{t^n}{n!}, \tag{1.5}$$

where $w_1,\dots,w_r\in\mathbb{Z}_p$; for details see [7, 11]. When $x=0$, the numbers $E_n^{(r)}(0|w_1,\dots,w_r)=E_n^{(r)}(w_1,\dots,w_r)$ are called the Barnes-type Euler numbers of order r.

The Changhee polynomials of order r are given by

$$\left(\frac{2}{t+2}\right)^r(1+t)^x=\sum_{n=0}^{\infty}Ch_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x=0$, $Ch_n^{(r)}(0)=Ch_n^{(r)}$ are called the Changhee numbers of order r in [1–19]. The Stirling numbers of the first kind are defined by, for $n\geq 0$

$$(x)_n=x(x-1)\dots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l, \tag{1.7}$$

while the Stirling numbers of the second kind are given by

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}\frac{S_2(l,n)t^l}{l!}; \tag{1.8}$$

see [6, 15] for details.

In this paper, we consider the Barnes-type Boole polynomials and investigate some properties and identities of those polynomials which are derived from the fermionic p-adic integrals on $\mathbb{Z}_p$.

2. Barnes-type Boole polynomials

For $w_1,\dots,w_r,x\in\mathbb{Z}_p$, we consider the Barnes-type Boole polynomials for $r\in\mathbb{N}$ as follows:

$$\begin{aligned}&\frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x+w_1x_1+\cdots+w_rx_r}\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r)\\&=(1+t)^x\left(\frac{1}{1+(1+t)^{w_1}}\right)\cdots\left(\frac{1}{1+(1+t)^{w_r}}\right)\\&=\sum_{n=0}^{\infty}Bl_n(x|w_1,\dots,w_r)\frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

When $x=0$, $Bl_n(0|w_1,\dots,w_r)=Bl_n(w_1,\dots,w_r)$ are called the Barnes-type Boole numbers. Note that we have

$$Bl_n(x|\underbrace{w,\cdots,w}_{r-\text{times}})=S_n(x;w,r) \tag{2.2}$$

and

$$Bl_n(x|\underbrace{1,\ldots,1}_{r-\text{times}}) = \frac{1}{2^r}Ch_n^{(r)}(x). \tag{2.3}$$

It is then not difficult to show that

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x+w_1x_1+\cdots+w_rx_r}\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r) \\ &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+w_1x_1+\cdots+w_rx_r)_n\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r)\frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

Therefore, by Equations (2.1) and (2.4), we obtain the following theorem:

Theorem 2.1. *For $n \geq 0$, we have*

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+w_1x_1+\cdots+w_rx_r)_n\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r) \\ &= 2^r Bl_n(x|w_1,\ldots,w_r). \end{aligned}$$

From Equation (1.2), we can derive the following equation:

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x+w_1x_1+\cdots+w_rx_r)t}\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r) \\ &= \frac{2^r}{(e^{w_1t}+1)\ldots(e^{w_rt}+1)}e^{xt} = \sum_{n=0}^{\infty}E_n^{(r)}(x|w_1,\ldots,w_r)\frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

Thus, by Equation (2.5) we get for all $n \geq 0$

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+w_1x_1+\cdots+w_rx_r)^n\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r) \\ &= E_n^{(r)}(x|w_1,\ldots,w_r), \ (n \geq 0). \end{aligned} \tag{2.6}$$

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we observe that

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+w_1x_1+\cdots+w_rx_r)_n\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r) \\ &= \lim_{N\to\infty}\sum_{x_1,\ldots,x_r=0}^{dP^N-1}(-1)^{x_1+\cdots+x_r}(x+w_1x_1+\cdots+w_rx_r)_n \\ &= \sum_{a_1,\ldots,a_r=0}^{d-1}(-1)^{a_1+\cdots+a_r}\int_{\mathbb{Z}_p}\cdots \\ &\qquad \cdots\int_{\mathbb{Z}_p}\left(x+\sum_{l=1}^{r}w_l(a_l+dx_l)\right)_n\mathrm{d}\mu_{-1}(x_1)\cdots\mathrm{d}\mu_{-1}(x_r) \end{aligned} \tag{2.7}$$

$$= 2^r \sum_{a_1,\ldots,a_r=0}^{d-1} (-1)^{a_1+\cdots+a_r} Bl_n(x + w_1a_1 + \cdots + w_ra_r|dw_1, \cdots, dw_r).$$

Therefore, by Equation (2.7), we obtain the following theorem:

Theorem 2.2. *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$\begin{aligned} &Bl_n(x|w_1, \ldots, w_r) \\ &= \sum_{a_1,\ldots,a_r=0}^{d-1} (-1)^{a_1+\cdots+a_r} Bl_n(x + w_1a_1 + \cdots + w_ra_r|dw_1, \cdots, dw_r). \end{aligned}$$

It then follows from Equation (2.1) that we get

$$\begin{aligned} &\sum_{n=0}^{\infty} 2^r Bl_n(x|w_1, \ldots, w_r)\frac{1}{n!}(e^t - 1)^n \\ &= \left(\frac{2}{e^{w_1t}+1}\right)\cdots\left(\frac{2}{e^{w_rt}+1}\right)e^{tx} \\ &= \sum_{n=0}^{\infty} E_n^{(r)}(x|w_1, \ldots, w_r)\frac{t^n}{n!}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} 2^r Bl_n(x|w_1, \ldots, w_r)\frac{(e^t - 1)^n}{n!} \\ &= \sum_{n=0}^{\infty} 2^r Bl_n(x|w_1, \ldots, w_r) \sum_{m=n}^{\infty} S_2(m, n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(2^r \sum_{n=0}^{m} Bl_n(x|w_1, \ldots, w_r)S_2(m, n)\right)\frac{t^m}{m!}. \end{aligned} \tag{2.9}$$

Thus, by Equations (2.8) and (2.9), we derive the following theorem:

Theorem 2.3. *For $m \geq 0$, we have*

$$\sum_{n=0}^{m} Bl_n(x|w_1, \ldots, w_r)S_2(m, n) = \frac{1}{2^r}E_m^{(r)}(x|w_1, \ldots, w_r).$$

Observe that from Theorem 2.1 we have

(2.10)

$$\begin{aligned} &2^r Bl_n(x|w_1, \ldots, w_r) \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x + w_1x_1 + \cdots + w_rx_r)_n \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\ &= \sum_{l=0}^{n} S_1(n, l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x + w_1x_1 + \cdots + w_rx_r)^l \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \end{aligned}$$

$$= \sum_{l=0}^{n} S_1(n,l) E_l^{(r)}(x|w_1,\ldots,w_r).$$

Thus by Equation (2.10) we obtain the following theorem:

Theorem 2.4. *For $n \geq 0$ we have*

$$2^r Bl_n(x|w_1,\ldots,w_r) = \sum_{l=0}^{n} S_1(n,l) E_l^{(r)}(x|w_1,\ldots,w_r).$$

We now consider the Barnes-type Boole polynomials of the second kind. These polynomials are given by the equation

$$\begin{aligned}
&\widehat{Bl}_n(x|w_1,\ldots,w_r) \\
&= \frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (-w_1x_1 - w_2x_2 - \cdots - w_rx_r + x)_n \, \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r),
\end{aligned} \tag{2.11}$$

where $n \geq 0$.

From Equation (2.11), we can derive the generating function as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{Bl}_n(x|w_1,\ldots,w_r)\frac{t^n}{n!} \\
&= \frac{1}{2^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+t)^{-w_1x_1-\cdots-w_rx_r+x} \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= \left(\frac{(1+t)^{w_1}}{(1+t)^{w_1}+1}\right)\left(\frac{(1+t)^{w_2}}{(1+t)^{w_2}+1}\right)\cdots\left(\frac{(1+t)^{w_r}}{(1+t)^{w_r}+1}\right)(1+t)^x \\
&= \prod_{l=1}^{r}\left(\frac{(1+t)^{w_l}}{(1+t)^{w_l}+1}\right)(1+t)^x.
\end{aligned} \tag{2.12}$$

From here we observe that

$$\begin{aligned}
&\prod_{l=1}^{r}\left(\frac{(1+t)^{w_l}}{(1+t)^{w_l}+1}\right)(1+t)^x \\
&= \left(\frac{1}{(1+t)^{w_1}+1}\right)\left(\frac{1}{(1+t)^{w_2}+1}\right)\cdots\left(\frac{1}{(1+t)^{w_r}+1}\right)(1+t)^{w_1+\cdots+w_r+x} \\
&= \sum_{n=0}^{\infty} Bl_n(x+w_1+\cdots+w_r|w_1,\ldots,w_r)\frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

Therefore, from Equations (2.12) and (2.13), we obtain the following theorem:

Theorem 2.5. *For $n \geq 0$, we have*

$$\widehat{Bl}_n(x|w_1,\ldots,w_r) = Bl_n(w_1+\cdots+w_r+x|w_1,\ldots,w_r).$$

From Equation (2.12), we derive that

$$\begin{aligned}
&\sum_{n=0}^{\infty} 2^r \widehat{Bl}_n(x|w_1,\dots,w_r)\frac{1}{n!}(e^t-1)^n \\
&= \left(\frac{2}{e^{w_1 t}+1}\right)\cdots\left(\frac{2}{e^{w_r t}+1}\right)e^{(x+w_1+\cdots+w_r)t} \\
&= \sum_{n=0}^{\infty} E_n^{(r)}(x+w_1+\cdots+w_r|w_1,\dots,w_r)\frac{t^n}{n!},
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{Bl}_n(x|w_1,\dots,w_r)\left(\frac{(e^t-1)^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \frac{\widehat{Bl}_n(x|w_1,\dots,w_r)}{n!}\left(n!\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}\right) \\
&= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \widehat{Bl}_n(x|w_1,\dots,w_r)S_2(m,n)\right)\frac{t^m}{m!}.
\end{aligned} \tag{2.15}$$

Therefore, by Equations (2.14) and (2.15) we obtain the following theorem:

Theorem 2.6. *For $m \geq 0$, we have*

$$\begin{aligned}
&\frac{1}{2^r}E_m^{(r)}(x+w_1+\cdots+w_r|w_1,\dots,w_r) \\
&= \sum_{n=0}^{m} \widehat{Bl}_n(x|w_1,\dots,w_r)S_2(m,n).
\end{aligned}$$

From Equation (2.11), we derive

$$\begin{aligned}
&2^r\widehat{Bl}_n(x|w_1,\dots,w_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-w_1x_1-\cdots-w_rx_r+x)_n \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-(w_1x_1+\cdots+w_rx_r)+x)^l \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l)(-1)^l E_l^{(r)}(-x|w_1,\dots,w_r).
\end{aligned} \tag{2.16}$$

For $n \geq 0$, the rising factorial sequence is defined by

$$x^{\underline{n}} = x(x+1)\dots(x+n-1) = (-1)^n(-x)_n = \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix} x^l, \tag{2.17}$$

where

$$\begin{bmatrix} n \\ l \end{bmatrix} = (-1)^{n-l} S_1(n,l).$$

Therefore, by Equation (2.17), we obtain the following theorem:

Theorem 2.7. *For $n \geq 0$, we have*

$$2^r(-1)^n \widehat{Bl}_n(x|w_1,\ldots,w_r) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} E_l^{(r)}(-x|w_1,\ldots,w_r).$$

We now observe that

(2.18)

$$\begin{aligned}
&2^r(-1)^n \frac{Bl_n(x|w_1,\ldots,w_r)}{n!} \\
&= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x + w_1x_1 + \cdots + w_rx_r}{n} \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-x - w_1x_1 - \cdots - w_rx_r + n - 1}{n} \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \\
&\qquad \cdots \int_{\mathbb{Z}_p} \binom{-x - w_1x_1 - \cdots - w_rx_r}{m} \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= \sum_{m=1}^{n} \frac{\binom{n-1}{m-1}}{m!} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x - w_1x_1 - \cdots - w_rx_r)_m \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r) \\
&= 2^r \sum_{m=1}^{n} \frac{\binom{n-1}{m-1}}{m!} \widehat{B}l_m(-x|w_1,\ldots,w_r).
\end{aligned}$$

Therefore, by Theorem 2.4 and Equation (2.18), we obtain the following theorem:

Theorem 2.8. *For $n \geq 1$, we have*

$$\begin{aligned}
&\frac{(-1)^n}{n!} \sum_{l=0}^{n} S_1(n,l) E_l^{(r)}(x|w_1,\ldots,w_r) \\
&= 2^r \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{B}l_m(-x|w_1,\ldots,w_r)}{m!}.
\end{aligned}$$

From Equation (2.11), we note that

$$\begin{aligned}
&2^r(-1)^n \frac{\widehat{Bl}_n(x|w_1,\ldots,w_r)}{n!} \\
&= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x - w_1x_1 - \cdots - w_rx_r}{n} \mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r)
\end{aligned}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{w_1x_1 + \cdots + w_rx_r - x + n - 1}{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$
$$= \sum_{m=0}^{n} \binom{n-1}{m-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{w_1x_1 + \cdots + w_rx_r - x}{m} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$
$$= \sum_{m=1}^{n} \frac{\binom{n-1}{m-1}}{m!} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (w_1x_1 + \cdots + w_rx_r - x)_m d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$
$$= 2^r \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{Bl_m(-x|w_1, \ldots, w_r)}{m!}.$$

Therefore we obtain the following theorem:

Theorem 2.9. *For $n \geq 1$, we have*

$$\frac{1}{n!} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} E_l^{(r)}(-x|w_1, \ldots, w_r)$$
$$= 2^r \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{Bl_m(-x|w_1, \ldots, w_r)}{m!}.$$

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