Contributions to Discrete Mathematics

Volume 15, Number 1, Pages 90–101
ISSN 1715-0868

ON THE FIRST TWO ENTRIES OF THE f-VECTORS OF 6-POLYTOPES

JIN HONG KIM

ABSTRACT. In 1906, Steinitz gave a complete characterization of the first two entries of the f-vectors of 3-polytopes, while Grünbaum obtained a similar result for 4-polytopes in his well-known book published in 1967. Recently, Kusunoki and Murai and independently Pineda-Villavicencio, Ugon, and Yost completely determined the first two entries of the f-vectors of 5-polytopes. This paper can be regarded as a continuation of their works for 6-polytopes. To be more precise, let k denote the number of vertices of a 6-polytope. The aim of this paper is to show that, when the number of edges is greater than or equal to $\frac{7}{2}(k-1)$ and $k \geq 14$, we can completely characterize the first two entries of the f-vectors of 6-polytopes. As a consequence, for $7 \leq k \leq 15$ we also give a complete characterization of the first two entries of the f-vectors of 6-polytopes except for three cases $(12, 39)$, $(13, 43)$, and $(15, 47)$.

1. INTRODUCTION

Let P be a convex polytope of dimension d in the Euclidean space $\mathbb{R}^d$, and let $f_i(P)$ denote the number of i-dimensional faces of P for each $0 \leq i \leq d-1$. For the sake of simplicity, in this paper a convex polytope of dimension d will be often called a d-polytope. Let $f(P)$ be the f-vector of P given by

$$f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P)),$$

and, for $0 \leq i < j \leq d-1$, let

$$\mathcal{E}^d_{ij} := \{(f_i(P), f_j(P)) \mid P \text{ is a convex polytope of dimension } d\}$$

From now on, let

$$\mathcal{E}_d := \mathcal{E}^d_{01}.$$

It is one of the fundamental problems in convex geometry to find all possible f-vectors (in particular, $\mathcal{E}^d_{ij}$) of a convex d-polytope (refer to [8], [11], and [12]). For $d = 3$, it is well-known as in [10] that Steinitz obtained

Received by the editors February 19, 2018, and in revised form June 21, 2019 .

2000 *Mathematics Subject Classification.* 52B05, 52B12, 52B22.

Key words and phrases. convex 6-polytopes, f-vectors, number of vertices, number of edges.

all possible f-vectors of 3-polytopes (also refer to [4, Section 10.3]). That is, he has completely determined $\mathcal{E}_3$ by

$$\mathcal{E}_3 = \left\{ (v,e) \,\middle|\, \frac{3}{2}v \leq e \leq 3v - 6 \right\}. \tag{1.1}$$

It then will not be difficult to completely determine all possible f-vectors, not just $\mathcal{E}_3$, of 3-polytopes by using (1.1) together with Euler's formula.

On the other hand, for polytopes of dimension greater than or equal to 4, a complete characterization of f-vectors is still elusive. However, we remark that there are some works of Grünbaum, Barnette, Barnette and Reay, and Sjöberg and Ziegler for the determination of the sets $\mathcal{E}^4_{ij}$ (see [1], [2], [4], and [9]). In particular, for $d \geq 4$ it is easy to obtain

$$\frac{d}{2} f_0(P) \leq f_1(P) \leq \binom{f_0(P)}{2}. \tag{1.2}$$

In [4], Grünbaum proved that

$$\mathcal{E}_4 = \left\{ (v,e) \,\middle|\, 2v \leq e \leq \binom{v}{2} \right\} \setminus \{(6,12),(7,14),(8,17),(10,20)\}.$$

In higher dimensions, it is pretty much expected that the situation is more complicated (refer to [12]). Nonetheless, recently Kusunoki and Murai and independently Pineda-Villavicencio, Ugon, and Yost gave a complete characterization of $\mathcal{E}_5$ in [5, Theorem 1.2] and [7, Theorem 7.2], as follows.

Theorem 1.1. *Let*

$$L_5 := \left\{ \left(v, \left[\frac{5}{2}v + 1 \right] \right) \,\middle|\, v \geq 6 \right\},$$

and let $G_5 := \{(8,20),(9,25),(13,35)\}$. *Then the following identity holds:*

$$\mathcal{E}_5 = \left\{ (v,e) \,\middle|\, \frac{5}{2}v \leq e \leq \binom{v}{2} \right\} \setminus (L_5 \cup G_5).$$

Here $[x]$ denotes the Gauss function of x, i.e., the largest integer less than or equal to x.

Our main result of this paper is to give a partial characterization of $\mathcal{E}_6$, as follows.

Theorem 1.2. *The following statements hold.*

(1)

$$\left\{ (v,e) \,\middle|\, \frac{7}{2}(v-1) \leq e \leq \binom{v}{2}, v \geq 7 \right\}$$
$$\setminus \{(8,25),(9,28),(9,29),(10,32),(10,34),(11,36),(12,39),(13,43)\}$$
$$\subseteq \mathcal{E}_6.$$

(2) *If the number v of vertices of a* 6*-polytope is no less than* 14*, then* $\mathcal{E}_6$ *restricted to the collection of* 6*-polytopes whose number of edges is no less than* $\frac{7}{2}(v-1)$ *coincides with*

$$\left\{(v,e) \,\middle|\, \frac{7}{2}(v-1) \le e \le \binom{v}{2}, v \ge 14\right\}.$$

Theorem 1.2 provides some new results for the range of the number of edges of convex 6-polytopes with a given number of vertices. As a consequence, roughly speaking, the remaining values to be determined for convex 6-polytopes are between $3v$ and $3.5(v-1)$ and seems to be relatively narrow. In fact, by using the arguments in this paper we also provide an interesting result for the range of the numbers of convex d-polytopes with a given number of vertices for any $d \ge 6$ (see Corollary 2.10 for more details).

The results and techniques of Kusunoki and Murai in [5] and Pineda-Villavicencio, Ugon, and Yost in [6] play important roles in the proof of Theorem 1.2. Especially, the excess degree introduced by Pineda-Villavicencio, Ugon, and Yost in [6] turns out to be very useful when we want to decide whether or not a given 6-polytope is actually realized.

In addition, Theorem 1.2 partially answers a question given in [5], and it is reasonable to expect that we can obtain certain sporadic results further by the same techniques of this paper. However, it should be remarked that our main results of this paper also show that the problem of completely characterizing the first two entries of the f-vectors of 6-polytopes could be much more complicated compared to that of 5-polytopes.

We organize this paper as follows. In Section 2, we first give some preliminary results necessary for the proofs of our main results, and then complete the proof of Theorem 1.2. Section 3 is devoted to giving some partial results for 6-polytopes with a low number of vertices less than 16. To be more precise, by a case-by-case analysis we completely characterize the first two entries of the f-vectors of 6-polytopes with a low number of vertices except for a few cases.

2. Proof of Theorem 1.2: pyramids, truncations, and 6-polytopes

The aim of this section is to construct many new 6-polytopes from 5-polytopes by taking two well-known procedures of taking a pyramid over a 5-polytope and truncating a 6-polytope along a simple vertex, and give a proof of Theorem 1.2.

To do so, we first need the following lemma.

Lemma 2.1. *Let Q be a pyramid over a convex polytope P of dimension* 5 *such that*

$$\frac{5}{2}f_0(P) \le f_1(P) \le \binom{f_0(P)}{2}.$$

Then $(f_0(Q), f_1(Q))$ *satisfies*

$$\frac{7}{2}(f_0(Q)-1) \leq f_1(Q) \leq \binom{f_0(Q)}{2}.$$

Proof. It is easy to see

$$f_0(Q) = f_0(P)+1, \quad f_1(Q) = f_0(P) + f_1(P). \tag{2.1}$$

For simplicity, let $v = f_0(P)$ and $v' = f_0(Q)$, and let $e = f_1(P)$ and $e' = f_1(Q)$. It follows from (2.1) that we have

$$v' = v+1, \quad e' = v+e = v'-1+e.$$

Thus the inequality $\frac{5}{2}v \leq e \leq \binom{v}{2}$ implies the inequality

$$\frac{5}{2}(v'-1) \leq 1 - v' + e' \leq \binom{v'-1}{2},$$

which is in turn equivalent to

$$\frac{7}{2}(v'-1) \leq e' \leq \binom{v'-1}{2} + (v'-1) = \binom{v'}{2},$$

as desired. $\square$

As a consequence, it is immediate to see that the following corollary holds.

Corollary 2.2. *The following inclusion holds:*

$$\left\{(v,e) \mid \frac{7}{2}(v-1) \leq e \leq \binom{v}{2}, v \geq 7\right\} \setminus (M_6 \cup G_6) \subset \mathcal{E}_6,$$

where

$$M_6 = \left\{\left(v, \left[\frac{7}{2}v - \frac{5}{2}\right]\right) \,\Big|\, v \geq 7\right\},$$
$$G_6 = \{(9,28),(10,34),(14,48)\}.$$

Proof. By Lemma 2.1, it suffices to note that the pair

$$\left(v, \left[\frac{5}{2}v+1\right]\right), \quad v \geq 6$$

has to be changed to the pair

$$\left(v, \left[\frac{7}{2}v - \frac{5}{2}\right]\right), \quad v \geq 7$$

after taking the pyramid over a 5-polytope. Note also that G_5 should be changed to G_6 after the procedure of taking the pyramid over a 5-polytope. $\square$

Recall that the *degree* $\deg a$ of a vertex a of a convex d-polytope P is the number of edges of P that contain a and that a vertex a is *simple* if $\deg a = d$. Let us denote by $V(P)$ the *vertex set* of a convex polytope P. Then the following lemma holds.

Lemma 2.3. *Let P be a convex polytope of dimension 6 such that*

$$f_1(P) \leq \frac{7}{2} f_0(P) - \alpha, \quad \alpha > 0. \tag{2.2}$$

Then there exists at least one simple vertex of P.

Proof. For the proof, note that the degree $\deg a$ of every vertex a of P is at least 6. Suppose on the contrary that $\deg a$ is greater than 6 for all vertices a of P. Then it follows from (2.2) and the well-known identity

$$f_1(P) = \frac{1}{2} \sum_{a \in V(P)} \deg a$$

that we have

$$\frac{7}{2} f_0(P) \leq f_1(P) \leq \frac{7}{2} f_0(P) - \alpha, \quad \alpha > 0.$$

But clearly this is a contradiction. Therefore, there should exist at least one vertex a of P whose degree $\deg a$ is equal to 6. This completes the proof of Lemma 2.3. □

By Theorem 1.1, there does not exist a convex 5-polytope in L_5. Thus, simply by taking a pyramid over a convex 5-polytope P in L_5 we cannot obtain a convex 6-polytope Q such that

$$\left(f_0(Q), \left[\frac{7}{2} f_0(Q) - \frac{5}{2} \right] \right) \in \mathcal{E}_6, \quad f_0(Q) \geq 7.$$

However, it turns out that the following lemma holds.

Lemma 2.4. *The following inclusion holds:*

$$\left\{ \left(v, \left[\frac{7}{2} v - \frac{5}{2} \right] \right) \;\middle|\; \textit{even } v \geq 14 \right\} \subset \mathcal{E}_6.$$

Proof. For the proof, note that for any odd integer $v \geq 9$ we have $(v, \frac{7}{2}v - \frac{1}{2}) \in \mathcal{E}_6$ by Corollary 2.2. That is, there is a 6-polytope Q such that $f_1(Q) = \frac{7}{2} f_0(Q) - \frac{1}{2}$ for any odd $f_0(Q) \geq 9$. Note also that by Lemma 2.3 there is a simple vertex a of Q. Thus we can take the truncation $\mathrm{tr}(Q, a)$ of Q at the simple vertex a.

It is well-known that the truncation $\mathrm{tr}(Q, a)$ satisfies

$$f_0(\mathrm{tr}(Q, a)) = f_0(Q) + 5, \quad f_1(\mathrm{tr}(Q, a)) = f_1(Q) + \binom{6}{2} = f_1(Q) + 15.$$

Thus we can obtain a 6-polytope $P := \mathrm{tr}(Q, a)$ such that

$$f_1(P) - 15 = \frac{7}{2}(f_0(P) - 5) - \frac{1}{2}.$$

As a consequence, we have

$$f_1(P) = \frac{7}{2} f_0(P) - 3 = \left[\frac{7}{2} f_0(P) - \frac{5}{2} \right]$$

for any even $f_0(P) \geq 14$. This completes the proof of Lemma 2.4. □

For the case of odd $f_0(P) > 13$, Lemma 2.5 below also holds. Before giving a proof of Lemma 2.5, we first need to recall the notion of the *excess degree* (or *excess number*) $\bar{\zeta}(P)$ of a polytope P of dimension d introduced in [6]:

$$\bar{\zeta}(P) = 2e - dv = \sum_{a \in V(P)} (\deg a - d).$$

It has been shown in [6, Theorem 7 (3)] that for $l := v - d \leq d$ we have

$$\bar{\zeta}(P) \geq (l-1)(d-l).$$

Lemma 2.5. *The following inclusion holds:*

$$\left\{ \left(v, \left\lceil \frac{7}{2}v - \frac{5}{2} \right\rceil \right) \;\middle|\; \text{odd } v \geq 15 \right\} \subset \mathcal{E}_6.$$

Proof. Let R be a 5-polytope such that

$$f_1(R) = \frac{5}{2} f_0(R) + \frac{7}{2}, \quad \text{odd } f_0(R) > 7.$$

Note that such a polytope R exists by Theorem 1.1. Since the excess degree $\bar{\zeta}(R)$ of R is equal to

$$2f_1(R) - 5f_0(R) = 7 < f_0(R),$$

there should be a simple vertex in R.

Now, let Q be a pyramid of dimension 6 over R. Since we have

$$f_0(Q) = f_0(R) + 1 \text{ and } f_1(Q) = f_1(R) + f_0(R),$$

$f_1(Q)$ satisfies

$$f_1(Q) = \frac{7}{2} f_0(Q).$$

Observe that the polytope Q still has a simple vertex a with its degree equal to 6. By truncating Q at the simple vertex a, we can obtain a 6-polytope P such that

$$f_1(P) = \frac{7}{2} f_0(P) - \frac{5}{2} = \left\lceil \frac{7}{2} f_0(P) - \frac{5}{2} \right\rceil$$

with odd $f_0(P) \geq 15$, as required. $\square$

By Lemmas 2.4 and 2.5, it still remains to determine whether or not the following pairs

$$\underbrace{(8,25), (10,32), (12,39)}_{v \text{ even}}, \underbrace{(9,29), (11,36), (13,43)}_{v \text{ odd}} \in M_6$$

are elements of $\mathcal{E}_6$. For some of these cases, we prove the following result in Section 3.

Lemma 2.6. *The following properties hold.*

(1) $(9,29) \notin \mathcal{E}_6$.
(2) $(10,32) \notin \mathcal{E}_6$.
(3) $(11,36) \notin \mathcal{E}_6$.

Proof. For the proofs of (1), (2), and (3), see the proofs of Lemmas 3.2 and 3.3 in Section 3. □

Remark 2.7. We do not know whether or not $(12, 39)$ and $(13, 43)$ are actually realized as convex 6-polytopes. It seems very difficult to completely determine those cases for $\mathcal{E}_6$.

On the other hand, in order to deal with the elements of G_6 we first show that by a result in [7] the following lemma holds.

Lemma 2.8. *For any $v \geq 7$, we have*

$$(v, 3v + 1) \notin \mathcal{E}_6.$$

Proof. By a main result of [7], the excess degree $\bar{\zeta} = 2f_1 - df_0$ of a d-polytope with (f_0, f_1) cannot have any natural number between 0 and $d - 2$. Since

$$0 < 2(3v + 1) - 6v = 2 < 4 = 6 - 2 = d - 2,$$

any pair $(v, 3v + 1)$ does not belong to $\mathcal{E}_6$, as desired. □

The following lemma holds.

Lemma 2.9. *The following properties hold.*

(1) $(8, 25), (9, 28) \notin \mathcal{E}_6$.
(2) $(10, 34) \notin \mathcal{E}_6$.
(3) $(14, 48) \in \mathcal{E}_6$.

Proof. (1) By Lemma 2.8, $(8, 25) = (8, 3 \cdot 8 + 1)$ and $(9, 28) = (9, 3 \cdot 9 + 1)$ are not an element of $\mathcal{E}_6$.

(2) Suppose that $(10, 34)$ is realized as a convex 6-polytope P. Then we have $l = 4$ and $d = 6$, and the excess degree satisfies

$$\bar{\zeta}(P) = 2e - 6v = 8 \geq (l - 1)(d - l) = 6.$$

So we cannot apply [6, Theorem 7 (3)] to show that $(10, 34) \notin \mathcal{E}_6$. However, it follows from [6, Theorem 19] that in this case either P is a triplex with $f_1(P) = 33$ or its excess degree $\bar{\zeta}(P) = 8$ would be at least $3 \cdot 6 - 8 = 10$. Hence, we have $(10, 34) \notin \mathcal{E}_6$.

(3) Let Q be the prism $\Delta^5 \times [-1, 1]$ of dimension 6 over the 5-simplex Δ^5. Then we have $(f_0(Q), f_1(Q)) = (12, 36)$. Now, let P be the 6-polytope obtained by adding two pyramids over each of two 5-simplices $\Delta^5 \times \{\pm 1\}$. Then we have

$$(f_0(P), f_1(P)) = (14, 48),$$

as desired. □

Now, we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. By combining Corollary 2.2 with Lemmas 2.4, 2.5, 2.6, and 2.9, it is immediately seen that Theorem 1.2 holds. □

By repeatedly applying the procedure of taking the pyramid, staring from a convex polytope in $\mathcal{E}_6$ it is straightforward to obtain the following corollary.

Corollary 2.10. *For each dimension $d \geq 6$, the following inclusion holds:*

$$Z_d := \left\{ (v, e) \,\middle|\, \frac{1}{2}\left((2d-5)v - (d^2 - 4d - 5)\right) \leq e \leq \binom{v}{2}, v \geq d+8 \right\} \subset \mathcal{E}_d.$$

Proof. We prove the corollary by induction on d. To do so, for each $n \geq 6$ let Q be a convex polytope obtained by taking the pyramid over a polytope P in Z_d, and let $(v, e) = (f_0(P), f_1(P))$ (resp. $(v', e') = (f_0(Q), f_1(Q))$. Then, clearly we have

$$v' = v + 1, \ e' = e + v.$$

Thus it is easy to see that the inequality

$$\frac{1}{2}\left((2d-5)v - (d^2 - 4d - 5)\right) \leq e \leq \binom{v}{2}, \quad v \geq d+8$$

is equivalent to

$$\begin{aligned} &\frac{1}{2}\left((2d-5)(v'-1) - (d^2 - 4d - 5)\right) + v' - 1 \leq e' \\ &\qquad\qquad\qquad\qquad \leq \binom{v'-1}{2} + v' - 1, \quad v' \geq d+9. \end{aligned}$$

That is, we have the inequality

$$\frac{1}{2}\left((2d-3)v' - (d^2 - 2d - 8)\right) \leq e' \leq \binom{v'}{2}, \quad v' \geq (d+1) + 8$$

for Z_{d+1}. This implies that Z_{d+1} is a subset of $\mathcal{E}_{d+1}$, as desired. $\square$

3. 6-polytopes with the number of vertices ≤ 15

The aim of this section is to completely characterize the first two entries of 6-polytopes for 6-polytopes with the number of vertices up to 15, except for the cases $(12, 39)$, $(13, 43)$, and $(15, 47)$. Here the number 15 is taken simply because it is a relatively small number that is manageable. This section may be regarded as an appendix to Section 2.

To do so, we first introduce the set L_6 analogous to L_5, as follows.

$$L_6 := \left\{ (v, 3v+1) \,\middle|\, v \geq 8 \right\}.$$

For the notational simplicity, we also set

$$\begin{aligned} K_6 := \{&(8,24),(9,27),(9,29),(10,30),(10,32),(10,34),(11,33),(11,36), \\ &(12,38),(12,39),(13,43),(14,42),(14,44),(15,45),(15,47)\}, \end{aligned}$$

$$X := \left\{ (v, e) \,\middle|\, 3v \leq e \leq \binom{v}{2} \right\} \setminus (L_6 \cup K_6).$$

Let

$$X_k := \{(v, e) \in X \mid v = k\}$$

for each $k \in \mathbb{Z}$. If $k \geq 16$, we have

$$X_k = \left\{ (k, e) \,\middle|\, 3k \leq e \leq \binom{k}{2} \right\} \setminus \{(k, 3k+1)\}.$$

It would be interesting to note the following lemma which enables us to inductively characterize large classes of $\mathcal{E}_6$ and shows some way of treating cases with (v,e) such that $3v \leq e \leq \frac{7}{2}v - 1$.

Lemma 3.1. *For $k \geq 16$, if $X_k \subset \mathcal{E}_6$, then $X_{k+5} \subset \mathcal{E}_6$.*

Proof. Let P be a convex polytope of dimension 6 with $f_0(P) = k$ and

$$f_1(P) \leq \frac{7}{2} f_0(P) - 1 = \frac{7}{2}k - 1.$$

Then there exists a simple vertex of P by Lemma 2.3. As mentioned above, by truncating at the simple vertex, we can always make a new convex polytope Q of dimension 6 with

$$f_0(Q) = k+5, \quad f_1(Q) = f_1(P) + 15.$$

Thus, if $X_k \subset \mathcal{E}_6$, then we have

$$\begin{aligned} \mathcal{E}_6 \supset & \left\{ (k+5, e+15) \,|\, 3k \leq e \leq \frac{7}{2}k - 1 \right\} \\ & - \left\{ \left(k+5, \left[\frac{7}{2}(k+5) - \frac{5}{2} \right] \right), (k+5, 3(k+5)+1) \right\} \\ = & \left\{ (k+5, e) \,|\, 3(k+5) \leq e \leq \frac{7}{2}((k+5)-1) \right\} \\ & - \left\{ \left(k+5, \left[\frac{7}{2}(k+5) - \frac{5}{2} \right] \right), (k+5, 3(k+5)+1) \right\}. \end{aligned} \tag{3.1}$$

It then follows from (3.1) and Theorem 1.2 that we have

$$X_{k+5} \subset \mathcal{E}_6,$$

as desired. ☐

For simplicity, from now on let k denote the number of vertices of a 6-polytope. Then, the following series of lemmas hold.

Lemma 3.2. *If $k = 8$, 9, or 10, then $\mathcal{E}_6$ coincides with X_k.*

Proof. It is easy to obtain that

$$\begin{aligned} X_8 &= \{(8,e) \,|\, 24 \leq e \leq 28, e \neq 24, 25\}, \\ X_9 &= \{(9,e) \,|\, 27 \leq e \leq 36, e \neq 27, 28, 29\}, \\ X_{10} &= \{(10,e) \,|\, 30 \leq e \leq 45, e \neq 31, 32, 34\}. \end{aligned}$$

First, we show that $(8,24) \notin \mathcal{E}_6$. To see it, suppose on the contrary that $(8,24) \in \mathcal{E}_6$. In this case, $l = 2 \leq 6 = d$. But then we have the excess degree

$$\bar{\zeta} := 2e - dv = 2 \cdot 24 - 6 \cdot 8 = 0 \geq (l-1) \cdot (d-l) = 1 \cdot 4.$$

This is a contradiction. Since $(8,25) \in L_6$, by Theorem 1.2 this completes the proof for the case X_8. The proofs of the cases for X_9 and X_{10} are same as in the case of X_8. ☐

Lemma 3.3. *If $k = 11$, then $\mathcal{E}_6$ coincides with X_{11}.*

Proof. Note that $X_{11} = \{(11, e) \mid 33 \leq e \leq 55, e \neq 33, 34, 36\}$. But, for case $(11, 33)$, $l = 5 \leq d = 6$, and we have $\bar{\zeta}(P) = 0$. But $(l-1)(d-l)$ is equal to 4. So, we have $(11, 33) \notin \mathcal{E}_6$. Moreover, $(11, 36) \notin \mathcal{E}_6$. Indeed, in the case of $(11, 36)$ its excess degree is equal to 6. So any 6-polytope P with $(f_0(P), f_1(P)) = (11, 36)$ cannot be a triplex by [6, Theorem 7 (iii)]. Moreover, in that case ($d = 6$ and $l = 5$) it follows from [6, Theorem 18] that the excess degree of P should be at least

$$(l-1)(d-k) + 2(l-3) = 8,$$

which is a contradiction. Since $(11, 34) \in L_6$, it follows from Theorem 1.2 that X_{10} is the same as $\mathcal{E}_6$ for $k = 11$. □

Lemma 3.4. *If $k = 12$ and $(12, 39) \notin \mathcal{E}_6$, then $\mathcal{E}_6$ coincides with X_{12}.*

Proof. In this case, $X_{12} = \{(12, e) \mid 36 \leq e \leq 66, e \neq 37, 38, 39\}$. We can show that $(12, 36) \in \mathcal{E}_6$. Indeed, let P be the triplex that is the prism over the 5-simplex. Then we have $(f_0(P), f_1(P)) = (12, 36)$. On the other hand, we can show that $(12, 38) \notin \mathcal{E}_6$ by using the same argument as in the case of $(11, 36)$. Since $(12, 37) \in L_6$, it follows from Theorem 1.2 that, when $(12, 39) \notin \mathcal{E}_6$, X_{12} is the same as $\mathcal{E}_6$ for $k = 12$. □

Lemma 3.5. *If $k = 13$ and $(13, 43) \notin \mathcal{E}_6$, then $\mathcal{E}_6$ coincides with X_{13}.*

Proof. Note that $X_{13} = \{(13, e) \mid 39 \leq e \leq 78, e \neq 40, 43\}$. We first show that $(13, 39) \notin \mathcal{E}_6$. To prove it, suppose that $(13, 39)$ is realized as a convex 6-polytope P. Then its excess degree $\bar{\zeta}(P)$ is equal to zero. Thus P is a simple polytope. Hence it follows from [3, Example 1.30 (3)] (or see also [3, Theorem 1.29] and [3, Theorem 1.37]) that we have

$$39 = f_1(P) \geq \binom{d}{1} \cdot k - \binom{d+1}{2} = \binom{6}{1} \cdot 13 - \binom{7}{2} = 57$$

with $d = 6$ and $v = 13$. This is a contradiction.

On the other hand, it is easy to see that $(13, 41) \in \mathcal{E}_6$. Indeed, note that by Lemma 3.1 $(8, 26)$ is realized as a 6-polytope P. Since 26 is less than $\frac{7}{2} \cdot 8 - 1 = 27$, there is a simple vertex a of P. By taking the truncation of P at a, we can obtain a 6-polytope Q with $(f_0(Q), f_1(Q)) = (13, 41)$.

Since $(13, 40) \in L_6$, it follows from Theorem 1.2 that, when $(13, 43) \notin \mathcal{E}_6$, X_{13} is the same as $\mathcal{E}_6$ for $k = 13$. □

Lemma 3.6. *If $k = 14$, then $\mathcal{E}_6$ coincides with X_{14}.*

Proof. Note that $X_{14} = \{(14, e) \mid 42 \leq e \leq 91, e \neq 43, 44\}$. By a similar argument as in the case of $(13, 39)$, we can show that $(14, 42) \notin \mathcal{E}_6$. On the other hand, it is easy to see that $(14, 45) \in \mathcal{E}_6$. Indeed, note that by Theorem 1.2 $(9, 30)$ is realized as a convex 6-polytope Q. Moreover, by Lemma 2.3 there is a simple vertex a of Q. Thus, if we truncate the vertex a from Q, we obtain a convex polytope P such that $(f_0(P), f_1(P)) = (14, 45)$.

Finally, note that $(14,43) \in L_6$, so that $(14,43) \notin \mathcal{E}_6$ by Lemma 2.6. As a consequence, it follows from Theorem 1.2 that X_{13} is the same as $\mathcal{E}_6$ for $k = 14$. $\square$

Lemma 3.7. *If $k = 15$ and $(15,47) \notin \mathcal{E}_6$, then $\mathcal{E}_6$ coincides with X_{15}.*

Proof. In this case, we have $X_{15} = \{(15,e) \,|\, 45 \le e \le 105, e \ne 45, 46, 47\}$. Note first that $(15,46) \in L_6$. Moreover, it is not difficult to show that $(15,45), (15,46) \notin \mathcal{E}_6$, and $(15,48) \in \mathcal{E}_6$. In particular, for the case $(15,48)$, note first that by Theorem 1.2 $(10,33)$ is realized as a convex 6-polytope Q and that by Lemma 2.3 there is a simple vertex a of Q. Thus, if we truncate the vertex a from Q, we obtain a convex polytope P such that $(f_0(P), f_1(P)) = (15,48)$, as desired. This together with Theorem 1.2 completes the proof of Lemma 3.7. $\square$

Combining all of the results above, we can state the following theorem.

Theorem 3.8. *Let k denote the number of vertices of a 6-polytope. Assume that $7 \le k \le 15$. Then $\mathcal{E}_6$ restricted to the collection of 6-polytopes with $7 \le k \le 15$ coincides with*

$$\bigcup_{k=7}^{15} X_k,$$

provided that $(12,39)$, $(13,43)$, and $(15,47)$ are not elements of $\mathcal{E}_6$.

Remark 3.9. We do not know whether or not $(15,47)$ is actually realizable as a convex 6-polytope. As in the case of Remark 2.7, it seems to be very difficult to completely determine this case for $\mathcal{E}_6$.

Remark 3.10. By similar techniques in this section, it is possible to determine the first two entries of the f-vectors of many 6-polytopes even with more than 15 vertices and thus obtain some sporadic results.

Acknowledgement

The author is very grateful to the reviewer for providing valuable comments.

References

1. D. W. Barnette, *The projection of the f-vectors of 4-polytopes onto the (E,S)-plane*, Discrete Math. **10** (1974), 201–209.
2. D. W. Barnette and J. R. Reay, *Projection of f-vectors of four-polytopes*, J. Comb. Theory, Ser. A **15** (1973), 200–209.
3. V. Buchstaber and T. Panov, *Torus actions and their applications in topology and combinatorics*, Univ, Lecture Series **24**, Amer. Math. Soc., Providence, 2002.
4. B. Grünbaum, *Convex polytopes*, Grad. Texts in Math. **221**, Springer, New York, 2003.
5. T. Kusunoki and S. Murai, *The numbers of edges of 5-polytopes*, Annals of Combinatorics **29** (2019), 89–101.
6. G. Pineda-Villavicencio, J. Ugon, and D. Yost, *Lower bound theorems for general polytopes*, European Journal of Combinatorics **79** (2019), 27–45.

7. ———, *The excess degree of a polytope*, SIAM J. Discrete M. **32** (2018), 2011–2046.
8. H. Sjöberg and G. M. Ziegler, *Semi-algebraic sets of f-vectors*, preprint (2017); arXiv:1711.01864v1.
9. ———, *Characterizing face and flag vector pairs for polytopes*, To appear in Discrete and Computational Geometry; arXiv:1803.04801v1.
10. E. Steinitz, *Über die Eulerschen Polyederrelationen*, Archiv der Mathematik und Physik **11** (1906), 86–88.
11. G. M. Ziegler, *Lectures on polytopes*, Grad. Texts in Math. **152**, Springer, New York, 1995.
12. ———, *Additive structures on f-vector sets of polytopes*, preprint (2017); arXiv:1709.02021v2.

Department of Mathematics Education, Chosun University, 309 Pilmun-daero, Dong-gu, Gwangju 61452, Republic of Korea
E-mail address: `jinhkim11@gmail.com`