DECOMPOSITION OF THE COMPLETE BIPARTITE GRAPH WITH A 1-FACTOR REMOVED INTO PATHS AND STARS

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ABSTRACT. Let $P_k$ denote a path on $k$ vertices, and let $S_k$ denote a star with $k$ edges. For graphs $F$, $G$, and $H$, a decomposition of $F$ is a set of edge-disjoint subgraphs of $F$ whose union is $F$. A $(G,H)$-decomposition of $F$ is a decomposition of $F$ into copies of $G$ and $H$ using at least one copy of each. In this paper, necessary and sufficient conditions for the existence of the $(P_{k+1},S_k)$-decomposition of the complete bipartite graph with a 1-factor removed are given.

1. Introduction

Let $F$, $G$, and $H$ be graphs. A decomposition of $F$ is a set of edge-disjoint subgraphs of $F$ whose union is $F$. A $G$-decomposition of $F$ is a decomposition of $F$ into copies of $G$. If $F$ has a $G$-decomposition, we say that $F$ is $G$-decomposable and write $G | F$. A $(G,H)$-decomposition of $F$ is a decomposition of $F$ into copies of $G$ and $H$ using at least one copy of each. If $F$ has a $(G,H)$-decomposition, we say that $F$ is $(G,H)$-decomposable and write $(G,H) | F$.

For positive integers $m$ and $n$, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. A k-path, denoted by $P_k$, is a path on $k$ vertices. A k-star, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$. A k-cycle, denoted by $C_k$, is a cycle of length $k$. A spanning subgraph $H$ of a graph $G$ is a subgraph of $G$ with $V(H) = V(G)$. A 1-factor of $G$ is a spanning subgraph of $G$ in which each vertex of $G$ is incident with exactly one edge. Note that $K_{m,n}$ has a 1-factor if and only if $m = n$. Letting $I$ be a 1-factor of $K_{n,n}$, we use $K_{n,n} - I$ to denote $K_{n,n}$ with a 1-factor removed.

For positive integers $l$ and $n$ with $1 \leq l \leq n$, the crown $C_{n,l}$ is the bipartite graph with bipartition $(A,B)$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and edge set $\{a_i b_j : 1 \leq j - i \leq l\}$ with arithmetic...
modulo $n$. Note that $K_{n,n}$ is isomorphic to $C_{n,n}$, and $K_{n,n} - I$ is isomorphic to $C_{n,n-1}$ for any 1-factor $I$.

For a graph $G$ and a positive integer $\lambda$, we use $\lambda G$ to denote the multigraph obtained from $G$ by replacing each edge $e$ by $\lambda$ edges each having the same endpoints as $e$.

Decomposition into isomorphic paths has attracted considerable attention; see [7, 9, 10, 11, 14, 15, 18, 22, 25, 31, 33, 35]. Decomposition into $k$-stars has also attracted a fair share of interest; see [8, 20, 32, 34, 36, 37]. Abueida and Daven introduced the study of $(G, H)$-decompositions in [1], they investigated the $(K_k, S_k)$-decomposition of the complete graph $K_n$ in [2], and the $(C_4, E_2)$-decomposition of several graph products in [3], where $E_2$ denotes the 4-vertex graph having two disjoint edges. Abueida and O’Neil [5] settled the existence problem for $(C_k, S_{k-1})$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [23, 24] gave necessary and sufficient conditions for the existence of $(G_n, H_n)$-decompositions of $\lambda K_n$ and $\lambda K_{n,n}$, where $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$.

Recently, Lee [16], Lee [17], Lee and Lin [19], and Lin [21] established necessary and sufficient conditions for the existence of $(C_k, S_k)$-decompositions of the complete bipartite graph, the complete bipartite multigraph, the complete bipartite graph with a 1-factor removed, and the multicrown, respectively. Abueida and Lian [4] and Beggas et al. [6] investigated $(C_k, S_k)$-decompositions of the complete graph $K_n$ and $\lambda K_n$, giving some necessary or sufficient conditions for such decompositions to exist.

The problem of decomposing a graph into copies of a graph $G$ and copies of a graph $H$ where the number of copies of $G$ and the number of copies of $H$ are essential is also studied. Shyu gave necessary and sufficient conditions for the decomposition of $K_n$ into paths and stars (both with 3 edges) [26], paths and cycles (both with $k$ edges where $k = 3, 4$) [27, 28], and cycles and stars (both with 4 edges) [30]. Shyu [29] also gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths and stars both with 3 edges. Jeevadoss and Muthusamy [12, 13] considered the decomposability of $K_{m,n}$, $K_n$ and $\lambda K_{m,n}$ into paths and cycles having $k$ edges, giving some necessary or sufficient conditions for such decompositions to exist.

In this paper, we consider the existence of $(P_{k+1}, S_k)$-decompositions of the complete bipartite graph with a 1-factor removed, giving necessary and sufficient conditions.

2. Preliminaries

In this section we first collect some needed terminology and notations, and then present some results which are useful for our discussions to follow.

Let $G$ be a graph. The degree of a vertex $x$ of $G$, denoted by $\text{deg}_G(x)$, is the number of edges incident with $x$. The vertex of degree $k$ in $S_k$ is the center, and any vertex of degree 1 is an endvertex of $S_k$. For $S \subseteq V(G)$ and $T \subseteq E(G)$, we use $G[S]$ and $G - T$ to denote the subgraph of $G$. 
induced by $S$ and the subgraph of $G$ obtained by deleting $T$, respectively. When $G_1, G_2, \ldots, G_t$ are graphs, not necessarily disjoint, we write $G_1 \cup G_2 \cup \cdots \cup G_t$ or $\bigcup_{i=1}^t G_i$ for the graph with vertex set $\bigcup_{i=1}^t V(G_i)$ and edge set $\bigcup_{i=1}^t E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^t G_i$ expresses the decomposition of $G$ into $G_1, G_2, \ldots, G_t$. Let $v_1v_2\ldots v_k$ and $(v_1, v_2, \ldots, v_k)$ denote the $k$-path and the $k$-cycle through vertices $v_1, v_2, \ldots, v_k$ in order, respectively, and let $(x ; y_1, y_2, \ldots, y_k)$ denote the $k$-star with center $x$ and endvertices $y_1, y_2, \ldots, y_k$.

For the edge $a_ib_j$ in $C_{n,n-1}$, the label of $a_ib_j$ is $j-i \pmod n$. For example, in $C_{8,7}$ the labels of $a_1b_0$ and $a_7b_3$ are 5 and 4, respectively. Note that each vertex of $C_{n,n-1}$ is incident with exactly one edge with label $i$ for $i \in \{1, 2, \ldots, n-1\}$. Let $H$ be a subgraph of $C_{n,n-1}$ (recall that $C_{n,n-1}$ has partite sets $\{a_0, a_1, \ldots, a_{n-1}\}$ and $\{b_0, b_1, \ldots, b_{n-1}\}$). When $r$ is a nonnegative integer, $H_{+r}$ denotes the graph with vertex set $\{a_i : a_i \in V(H)\} \cup \{b_{j+r} : b_j \in V(H)\}$ and edge set $\{a_ib_{j+r} : a_i b_j \in E(H)\}$, and $H + r$ denotes the subgraph of $C_{n,n-1}$ with vertex set $\{a_{i+r} : a_i \in V(H)\} \cup \{b_{j+r} : b_j \in V(H)\}$ and edge set $\{a_{i+r}b_{j+r} : a_i b_j \in E(H)\}$ where the subscripts of $a$ and $b$ are taken modulo $n$. In particular, $H_{+0} = H_{+0} = H$.

The following results are essential to our proof.

**Proposition 2.1** ([20]). Let $\lambda$, $k$, $l$, and $n$ be positive integers. $\lambda C_{n,l}$ is $S_k$-decomposable if and only if $k \leq l$ and $\lambda nl \equiv 0 \pmod k$.

**Proposition 2.2** ([37]). For integers $m$ and $n$ with $m \geq n \geq 1$, the graph $K_{m,n}$ is $S_k$-decomposable if and only if $m \geq k$ and

$$\begin{align*}
m &\equiv 0 \pmod k & \text{if } n < k, \\
nm &\equiv 0 \pmod k & \text{if } n \geq k.
\end{align*}$$

**Proposition 2.3** ([31]). Let $k$, $l$, and $n$ be positive integers. $C_{n,l}$ is $P_{k+1}$-decomposable if and only if $nl \equiv 0 \pmod k$ and

$$k \leq \begin{cases} 2 & \text{if } l = n = 2, \\
2n - 3 & \text{if } l \text{ is even and } n \geq 3, \\
l & \text{if } l \text{ is odd.}
\end{cases}$$

**Proposition 2.4** ([22]). Let $k$, $m$, and $n$ be positive integers. There exists a $P_{k+1}$-decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod k$ and one of the following cases occurs:

<table>
<thead>
<tr>
<th>Case</th>
<th>$k$</th>
<th>$m$</th>
<th>$n$</th>
<th>Conditions</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>even</td>
<td>even</td>
<td>$k \leq 2m, k \leq 2n$, not both equalities</td>
</tr>
<tr>
<td>2</td>
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<td>even</td>
<td>odd</td>
<td>$k \leq 2m - 2, k \leq 2n$</td>
</tr>
<tr>
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<td>odd</td>
<td>even</td>
<td>$k \leq 2m, k \leq 2n - 2$</td>
</tr>
<tr>
<td>4</td>
<td>odd</td>
<td>even</td>
<td>even</td>
<td>$k \leq 2m - 1, k \leq 2n - 1$</td>
</tr>
<tr>
<td>5</td>
<td>odd</td>
<td>even</td>
<td>odd</td>
<td>$k \leq 2m - 1, k \leq n$</td>
</tr>
<tr>
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<td>odd</td>
<td>even</td>
<td>$k \leq m, k \leq 2n - 1$</td>
</tr>
<tr>
<td>7</td>
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<td>odd</td>
<td>odd</td>
<td>$k \leq m, k \leq n$</td>
</tr>
</tbody>
</table>
3. Main results

Since \( K_{n,n} - I \) is isomorphic to the crown \( C_{n,n-1} \) for any 1-factor \( I \) of \( K_{n,n} \), \( K_{n,n} - I \) is replaced by \( C_{n,n-1} \) in the following discussions. We first give necessary conditions for a \((P_{k+1}, S_k)\)-decomposable of \( C_{n,n-1} \).

**Lemma 3.1.** If \( C_{n,n-1} \) is \((P_{k+1}, S_k)\)-decomposable, then \( k \leq n - 1 \) and \( n(n-1) \equiv 0 \pmod{k} \).

**Proof.** Since the maximum size of a star in \( C_{n,n-1} \) is \( n - 1 \), \( k \leq n - 1 \) is necessary. Since \( C_{n,n-1} \) has \( n(n-1) \) edges and each subgraph in a decomposition has \( k \) edges, \( k \) must divide \( n(n-1) \). \( \square \)

We now show that the necessary conditions are also sufficient. Since \( P_{k+1} = S_k \) for \( k = 1, 2 \), the result holds for \( k = 1, 2 \) by Proposition 2.1. So it remains to consider the case \( k \geq 3 \). The proof is divided into cases \( n \geq 2k + 1, n = 2k, 2k - 1 \geq n \geq k + 2 \), and \( n = k + 1 \), which are treated in Lemmas 3.2, 3.3, 3.7, and 3.8, respectively.

**Lemma 3.2.** Let \( k \) and \( n \) be positive integers with \( n \geq 2k + 1 \). If \( n(n-1) \equiv 0 \pmod{k} \), then \( C_{n,n-1} \) is \((P_{k+1}, S_k)\)-decomposable.

**Proof.** Let \( n - 1 = qk + r \) where \( q \) and \( r \) are integers with \( 0 \leq r < k \). From the assumption \( n \geq 2k + 1 \), we have \( q \geq 2 \). Note that

\[
C_{n,n-1} = C_{qk+r+1,qk+r} = C_{(q-1)k+1,(q-1)k} \cup C_{k+r+1,k+r} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k}.
\]

Clearly, \( |E(C_{(q-1)k+1,(q-1)k})|, |E(K_{(q-1)k,k+r})|, \) and \( |E(K_{k+r,(q-1)k})| \) are multiples of \( k \). Thus, \( (k + r + 1)(k + r) \equiv 0 \pmod{k} \) from the assumption \( n(n-1) \equiv 0 \pmod{k} \). By Propositions 2.1 and 2.2, \( C_{(q-1)k+1,(q-1)k} \) together with \( K_{(q-1)k,k+r} \) and \( K_{k+r,(q-1)k} \) are \( S_k \)-decomposable. By Proposition 2.3, \( C_{k+r+1,k+r} \) is \( P_{k+1} \)-decomposable. Hence, \( C_{n,n-1} \) is \((P_{k+1}, S_k)\)-decomposable. \( \square \)

**Lemma 3.3.** If \( k \) is an integer with \( k \geq 3 \), then the crown \( C_{2k,2k-1} \) is \((P_{k+1}, S_k)\)-decomposable.

**Proof.** Let \( A_1 = \{a_0, a_1, \ldots, a_{k-1}\} \), \( A_2 = \{a_k, a_{k+1}, \ldots, a_{2k-1}\} \), \( B = \{b_0, b_1, \ldots, b_{2k-1}\} \), and let \( G_i = C_{2k,2k-1}[A_i \cup B] \) for \( i = 1, 2 \). Note that \( C_{2k,2k-1} = G_1 \cup G_2 \). It is easy to check that \( G_1 \) is isomorphic to \( G_2 \) with isomorphism \( f \) such that \( f(a_i) = a_{i+k} \) and \( f(b_j) = b_{j+k} \) where the subscripts of \( a \) and \( b \) are taken modulo \( 2k \) for \( i \in \{0, 1, \ldots, k-1\} \) and \( j \in \{0, 1, \ldots, 2k-1\} \). Hence it is sufficient to show that \( G_1 \) is \((P_{k+1}, S_k)\)-decomposable. We distinguish two cases according to the parity of \( k \).

**Case 1:** \( k \) is even.

Define a \((k+1)\)-path \( P = b_1a_0b_2a_1 \ldots b_{k/2}a_{k/2-1}b_{k/2} \). Note that \( P_{2+2i} \), where the subscripts of \( b \) are taken modulo \( 2k-1 \), is a \((k+1)\)-path for \( i \in \{0, 1, \ldots, k-2\} \). We can see that \( P \cup P_{2+2} \cup P_{4+2} \cup \ldots \cup P_{(2+2)(k-2)} \) is the subgraph of \( G_1 \) consisting of all edges between \( \{a_0, a_1, \ldots, a_{k/2-1}\} \)
and $B - \{b_{2k-1}\}$. Moreover, $P_{2i} + k/2$, where the subscripts of $b$ are taken modulo $2k - 1$, is also a $(k + 1)$-path, and $G_1[A_1 \cup B - \{b_{2k-1}\}] = \bigcup_{i=0}^{k-2} (P_{2i} \cup (P_{2i} + k/2))$. Since $(b_{2k-1}; a_0, a_1, \ldots, a_{k-1})$ is a $k$-star, $G_1$ is $(P_{k+1}, S_k)$-decomposable.

**Case 2:** $k$ is odd.

Observe that $G_1 = C_{k,k-1} \cup K_{k,k}$. Since $k - 1$ is even and $k \leq 2k - 3$ for $k \geq 3$, Proposition 2.3 implies that $C_{k,k-1}$ is $P_{k+1}$-decomposable. By Proposition 2.2, $K_{k,k}$ is $S_k$-decomposable. Hence $G_1$ is $(P_{k+1}, S_k)$-decomposable.

\( \square \)

Before plunging into the proof of the next case, we need the following results.

**Proposition 3.4** ([20]). Let \( \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\} \) be the vertex set of the multiconnected \( \lambda C_{n,t} \). If there exist positive integers $p$ and $q$ so that $q < p \leq l$ and $\lambda nq \equiv 0 \pmod{p}$, then there exists a spanning subgraph $G$ of $\lambda C_{n,t}$ with $\deg_{G} b_j = \lambda q$ for $0 \leq j \leq n - 1$, and $G$ has an $S_p$-decomposition.

The following is trivial.

**Lemma 3.5.** If $W$ is a graph consisting of a $k$-cycle $C$ and a $t$-star $S$ such that $S$ has its center in $C$ and has at least one endvertex not in $C$, then $W$ can be decomposed into a $(k + 1)$-path and a $t$-star.

**Lemma 3.6.** Let $k$ and $r$ be positive integers with $r < k - 1$ and let $t = (r + 1)r/k$. Suppose that $t$ is an integer with $t \geq 2$. If $G$ is a bipartite graph with bipartition $(A_1, B_1 \cup B')$, where $|A_1| = |B_1| = k$ and $|B'| = r + 1$ such that $G[A_1 \cup B_1] = C_{k,k-1}$ and $G[A_1 \cup B'] = K_{k,r+1}$, then $G$ can be decomposed into $t - 1$ copies of $P_{k+1}$ and $k$ copies of $S_k$ with centers at distinct vertices in $A_1$, and $r + 1$ copies of $S_{k-r}$ with centers at distinct vertices in $B'$.

**Proof.** Let $A_1 = \{a_0, a_1, \ldots, a_{k-1}\}$, $B_1 = \{b_0, b_1, \ldots, b_{k-1}\}$, and $B' = \{b_k, b_{k+1}, \ldots, b_{k+r}\}$. Letting $F = G[A_1 \cup B_1]$, $H = G[A_1 \cup B']$, and $p = \left\lfloor \frac{(t-1)k}{2} \right\rfloor$. Define a $2k$-cycle $C = (b_1, a_0, b_2, a_1, \ldots, b_{k-1}, a_{k-2}, b_0, a_{k-1})$, and for even $t$ define a $(k + 1)$-path $P$ according to the parity of $k$ as follows:

\[
P = \begin{cases} 
2p+1 & a_0 b_2 + a_1 b_3 + \ldots + b_{2k} a_{k+1} & \text{if } k \text{ is even}, \\
2p+2 & a_0 b_2 + a_1 b_3 + \ldots + b_{2k} a_{k+1} & \text{if } k \text{ is odd}.
\end{cases}
\]

It is easy to check that $C, C_{22}, \ldots, C_{2(p-1)}$ and $P$, where the subscripts of $b$ are taken modulo $k$, are edge-disjoint in $F$.

Define a subgraph $W$ of $F$ as follows:

\[
W = \begin{cases} 
\bigcup_{i=0}^{t-1} C_{2i} & \text{if } t \text{ is odd,} \\
\bigcup_{i=0}^{t-1} C_{2i} \cup P & \text{if } t \text{ is even.}
\end{cases}
\]

Since $C_{2k}$ can be decomposed into two copies of $P_{k+1}$ and $2p = t - 1$ for odd $t$ as well as $2p + 1 = t - 1$ for even $t$, $W$ can be decomposed into $t - 1$ copies of $P_{k+1}$. When $t$ is odd, $\deg_{F-E(W)} a_i = k - 2p - 1$. When $t$ is even,
\[ \deg_{F-E(W)} a_i = \begin{cases} 
2k-2p-3 & \text{if } i \in \{0,1,\ldots,\lceil k/2 \rceil -2\}, \\
nk-2p-\delta & \text{if } i = \lceil k/2 \rceil -1, \\
nk-2p-1 & \text{if } i \in \{\lceil k/2 \rceil, \lceil k/2 \rceil +1, \ldots, k-1\}, 
\end{cases} \]

where

\[ \delta = \begin{cases} 
2 & \text{if } k \text{ is odd,} \\
3 & \text{if } k \text{ is even.} 
\end{cases} \]

For \( i = 0,1,\ldots,k-1 \), let \( X_i = (F - E(W))[\{a_i\} \cup B_1] \). Clearly \( X_i \) is a star with center \( a_i \), and for odd \( t \), we have \( X_i = S_{k-2p-1} \), and for even \( t \), we have

\[ X_i = \begin{cases} 
S_{k-2p-3} & \text{if } i \in \{0,1,\ldots,\lceil k/2 \rceil -2\}, \\
nS_{k-2p-\delta} & \text{if } i = \lceil k/2 \rceil -1, \\
nS_{k-2p-1} & \text{if } i \in \{\lceil k/2 \rceil, \lceil k/2 \rceil +1, \ldots, k-1\}. 
\end{cases} \]

In the remainder of the proof, we will show that \( H \) can be decomposed into \( r+1 \) copies of \( S_{k-r} \) with centers in \( B' \), \( k \) copies of \( S_{2p+1} \) with centers in \( A_1 \) for odd \( t \), \( \lceil k/2 \rceil -1 \) copies of \( S_{2p+3} \) with centers in \( \{a_0,a_1,\ldots,a_{\lceil k/2 \rceil -2}\} \), and a copy of \( S_{2p+\delta} \) with center \( a_{\lceil k/2 \rceil -1} \), together with \( k - \lceil k/2 \rceil \) copies of \( S_{2p+1} \) with centers in \( \{a_{\lceil k/2 \rceil},a_{\lceil k/2 \rceil +1},\ldots,a_{k-1}\} \) for even \( t \).

Now we show the required star-decomposition of \( H \) by orienting the edges of \( H \). For any vertex \( x \) of \( H \), the outdegree \( \deg^+_H x \) (indegree \( \deg^-_H x \), respectively) of \( x \) in an orientation of \( H \) is the number of arcs incident from (to, respectively) \( x \). It is sufficient to show that there exists an orientation of \( H \) such that

\[ \text{(3.1)} \quad \deg^+_H b_j = k - r, \]

where \( j \in \{k,k+1,\ldots,k+r\} \), and for odd \( t \)

\[ \text{(3.2)} \quad \deg^+_H a_i = 2p + 1 \]

where \( i \in \{0,1,\ldots,k-1\} \), and for even \( t \)

\[ \text{(3.3)} \quad \deg^+_H a_i = \begin{cases} 
n2p+3 & \text{if } i \in \{0,1,\ldots,\lceil k/2 \rceil -2\}, \\
n2p+\delta & \text{if } i = \lceil k/2 \rceil -1, \\
n2p+1 & \text{if } i \in \{\lceil k/2 \rceil, \lceil k/2 \rceil +1, \ldots, k-1\}. 
\end{cases} \]

We first consider the edges oriented outward from \( A_1 \) according to the parity of \( t \). When \( t \) is odd, the edges \( a_ib_{k+1}(2p+1)i, a_ib_{k+1}(2p+1)i+1, \ldots, a_ib_{k+(2p+1)i+2p} \) are all oriented outward from \( a_i \) for \( i \in \{0,\ldots,k-1\} \). Let \( \beta = (2p+3)(\lceil k/2 \rceil -1) \). When \( t \) is even, the edges \( a_ib_{k+1}(2p+3)i, a_ib_{k+1}(2p+3)i+1, \ldots, a_ib_{k+(2p+3)i+2p} \) for \( i \in \{0,1,\ldots,\lceil k/2 \rceil -2\} \), the edges \( a_{\lceil k/2 \rceil -1}b_{k+\beta}, a_{\lceil k/2 \rceil -1}b_{k+\beta+1}, \ldots, a_{\lceil k/2 \rceil -1}b_{k+\beta+2+2p} \) and the edges \( a_{\lceil k/2 \rceil -1}b_{k+1(2p+1)i+\beta+2p+\delta}, a_{\lceil k/2 \rceil -1}b_{k+1(2p+1)i+\beta+2p+\delta+1}, \ldots, a_{\lceil k/2 \rceil -1}b_{k+1(2p+1)i+\beta+2p+\delta +4p+\delta} \) for \( i \in \{\lceil k/2 \rceil, \lceil k/2 \rceil +1, \ldots, k-1\} \) are all oriented outward from \( a_i \). The subscripts of \( b \) are taken modulo \( r+1 \) in the set of numbers \( \{k,k+1,\ldots,k+r\} \). Note that for odd \( t \) we orient \( 2p+1 \) edges from each \( a_i \), and for even \( t \) we orient at most
2p + 3 edges from $a_i$. Since $2p + 1 ≤ 2(t - 1)/2 + 1 = t < r$ for odd $t$ and $2p + 3 ≤ 2(t - 2)/2 + 3 = t + 1 < r + 1$ for even $t$, this guarantees that there are enough edges for the above orientation. Finally, the edges which are not oriented yet are all oriented from $B'$ to $A_1$.

From the construction of the orientation, it is easy to see that (3.2) and (3.3) are satisfied, and for all $b_w, b_w' \in B'$, we have

$$\left| \deg_H b_w - \deg_H b_w' \right| \leq 1.$$  

So, we only need to check (3.1).

Since $\deg_H^+ b_w + \deg_H^- b_w = k$ for $b_w \in B'$, it follows from (3.4) that $|\deg_H^+ b_w - \deg_H^- b_w'| \leq 1$ for $b_w, b_w' \in B'$. Further, note that for odd $t$,

$$\sum_{i=0}^{k-1} \deg_H^+ a_i = (2p + 1)k = (2(t - 1)/2 + 1)k = tk,$$

and for even $t$,

$$\sum_{i=0}^{k-1} \deg_H^+ a_i = (2p + 3)\left(\lceil k/2 \rceil - 1\right) + 2p + \delta + (2p + 1)(k - \lceil k/2 \rceil)$$

$$= (2p + 1)(k - 1) + 2(\lceil k/2 \rceil - 1 + p) + \delta$$

$$= \begin{cases} 
(2p + 1)(k - 1) + 2(\lceil k/2 \rceil - 1 + p) + 2 & \text{if } k \text{ is odd}, \\
(2p + 1)(k - 1) + 2(k/2 - 1 + p) + 3 & \text{if } k \text{ is even}, 
\end{cases}$$

$$= (2p + 2)k$$

$$= (2(t - 2)/2 + 2)k$$

$$= tk$$

Thus

$$\sum_{w=k}^{k+r} \deg_H^+ b_w = |E(K_{k,r+1})| - \sum_{i=0}^{k-1} \deg_H^+ a_i$$

$$= k(r + 1) - tk$$

$$= k(r + 1) - r(r + 1)$$

$$= (k - r)(r + 1)$$

Therefore, $\deg_H^+ b_w = k - r$ for $b_w \in B'$. This proves (3.1). Hence there exists the required decomposition $\mathcal{D}$ of $H$. Let $X'_i$ be the star with center at $a_i$ in $\mathcal{D}$ for $i \in \{0, 1, \ldots, k - 1\}$. Clearly $X_i + X'_i$ is a $k$-star. This completes the proof.

\textbf{Lemma 3.7.} Let $k$ and $n$ be integers with $3 ≤ k < n - 1 < 2k - 1$. If $n(n - 1) ≡ 0 \pmod{k}$, then $C_{n,n-1}$ is $(P_{k+1}, S_k)$-decomposable.

\textbf{Proof.} Let $n - 1 = k + r$. From the assumption $k < n - 1 < 2k - 1$, we have $0 < r < k - 1$. Let $t = (r + 1)r/k$. Since $k \mid n(n - 1)$, we have $k \mid (r + 1)r$, \ldots
which implies that $t$ is a positive integer. The proof is divided into two parts according to the value of $t$.

**Case 1**: $t = 1$.

When $t = 1$, $k = (r + 1)r$. This implies that $k$ is even, and hence $k \geq 4$; in turn, we have $r \geq 2$. Let $A_0 = \{a_0, a_1, \ldots, a_{k/2-1}\}$ and $B_0 = \{b_0, b_1, \ldots, b_{k/2-1}\}$. We distinguish two subcases.

**Subcase 1.1**: $r = 2$.

For $r = 2$, $k = (r + 1)r = 6$, and $n = k + r + 1 = 9$. Let $H_1 = C_{9,8}[A_0 \cup B]$, and $H_2 = C_{9,8}[(A - A_0) \cup B]$. Clearly $C_{9,8} = H_1 \cup H_2$. Let $P = b_0 a_0 b_2 a_1 b_4 a_2 b_1$, we have $\deg_{H_1 - E(P)} a_i = 6$ for $i \in \{0, 1, 2\}$. This implies $H_1 - E(P)$ is $S_6$-decomposable. Let $F_i = (a_i; a_3, a_4, \ldots, a_8)$ for $i \in \{0, 1\}$ and $G_j = (a_j; b_2, b_3, b_j+1, b_j+2, \ldots, b_8)$ for $j \in \{3, 4, \ldots, 8\}$. It is easy to check that $F_i$ and $G_j$ are 6-stars and $\{F_0, F_1, G_3, G_4, \ldots, G_8\}$ is an $S_6$-decomposition of $H_2$.

**Subcase 1.2**: $r \geq 3$.

We will show that $C_{n,n-1}$ can be decomposed into copies of $C_k$ together with a copy of $P_k$. Let $D_0 = C_{n,n-1}[A_0 \cup B_0]$, $D_1 = C_{n,n-1}[(A - A_0) \cup B_0]$, and $D_2 = C_{n,n-1}[A \cup (B - B_0)]$. Clearly $C_{n,n-1}$ is isomorphic to $C_k/2/k-1$, $D_1$ is isomorphic to $K_k/2+r+1/k-2$, and $D_2$ is isomorphic to $C_k/2+r+1/k-2$ and $K_{k/2,k/2+r+1}$. Let $C = (a_0, b_{k/2-1}, a_1, b_0, a_2, b_1, \ldots, a_{k/2-1}, b_{k/2-2})$ and $D = D_0 - E(C)$. Trivially $C$ is a $k$-cycle in $D_0$ and $D = C_{k/2,k/2-3}$. Note that $0 < r - 2 < k/2 - r - 1 < k/2 - 3$ for $r \geq 3$ and $(k/2)(r - 2) = r(r + 1)(r - 2)/2 = r(k/2 - r - 1)$. Therefore, Proposition 3.4 implies that there exists a spanning subgraph $X$ of $D$ such that $\deg_X b_j = r - 2$ for $0 \leq j \leq k/2 - 1$, and $X$ has an $S_k/2-r-1$-decomposition $D$ with $|D| = r$. Furthermore, each $S_k/2-r-1$ has its center in $A_0$ since $\deg_X b_j = r - 2 < k/2 - r - 1$. Suppose that the centers of $(k/2-r-1)$-stars in $D$ are $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$. Let $S(w)$ be the $(k/2-r-1)$-star with center $a_{i_w}$ in $D$ and let $Y = D - E(X) \cup D_1$. Note that $\deg_Y b_j = (k/2 - 3 - (r - 2)) + (k/2 + r + 1) = k$ for $0 \leq j \leq k/2 - 1$. Hence $Y$ is $S_k$-decomposable. For $w \in \{1, 2, \ldots, r\}$, define $S'(w) = D_2[a_{i_w} \cup (B - B_0)]$ and $Z = D_2 - E(\bigcup_{w=1}^{r} S'(w))$. Clearly $S'(w)$ is a $(k/2 + r + 1)$-star with center $a_{i_w}$ in $D_2$, and $S(w) \cup S'(w)$ is a $k$-star. Moreover, $\deg_Z b_j = k + r - r = k$ for $k/2 \leq j \leq k + r$. Thus $Z$ is $S_k$-decomposable. By Lemma 3.5, $C \cup S(1) \cup S'(1)$ can be decomposed into a $(k + 1)$-path and a $k$-star.

**Case 2**: $t \geq 2$.

Let $A_1 = \{a_0, a_1, \ldots, a_{k-1}\}$, $A' = \{a_k, a_{k+1}, \ldots, a_{k+r}\}$, $B_1 = \{b_0, b_1, \ldots, b_{k-1}\}$, and $B' = \{b_k, b_{k+1}, \ldots, b_{k+r}\}$. Moreover, let $G = C_{n,n-1}$\[A_1 \cup B_1], F = C_{n,n-1}[A' \cup B_1], and H = C_{n,n-1}[A' \cup B']. Clearly C_{n,n-1} = G \cup F \cup H. Note that G is isomorphic to C_{k,k-1} \cup K_{k,r+1}, H is isomorphic to C_{r+1,k}, and F is isomorphic to K_{r+1,k}. Proposition 2.2 implies that K_{r+1,k} is S_k-decomposable. By Lemma 3.6, there exists a decomposition
Lemma 3.8. If $k$ is an integer with $k \geq 3$, then $C_{k+1,k}$ is $(P_{k+1}, S_k)$-decomposable.

Proof. The proof is divided into two parts according to the parity of $k$.

Case 1: $k$ is odd.

Note that $C_{k+1,k}$ can be decomposed into $C_{k,k-1}$ and two copies of $S_k$.

Since $k-1$ is even and $k \leq 2k-3$ for $k \geq 3$, Proposition 2.3 implies that $C_{k,k-1}$ is $P_{k+1}$-decomposable. Hence $C_{k+1,k}$ is $(P_{k+1}, S_k)$-decomposable.

Case 2: $k$ is even.

Let $A_1 = \{a_0, a_1, \ldots, a_{k/2-1}\}$ and $A_2 = \{a_{k/2}, a_{k/2+1}, \ldots, a_k\}$. For $i \in \{1, 2\}$, define $G_i = C_{k+1,k}[A_i \cup B]$. Clearly $C_{k+1,k} = G_1 \cup G_2$. We will show that $G_1$ is $P_{k+1}$-decomposable and $G_2$ is $S_k$-decomposable.

Let $P = b_1a_0b_2a_1 \ldots b_{k/2}a_{k/2-1}b_{k/2+1}$. Note that $P$ is a $(k+1)$-path containing all of the edges incident with the vertices in $A_1$, having labels 1 and 2. Furthermore, $P_{+2i}$ is a $(k+1)$-path containing all of the edges incident with the vertices in $A_1$, having labels $2i+1$ and $2i+2$ for $i = 0, 1, 2, \ldots, k/2-1$. Hence $P \cup P_{+2} \cup P_{+4} \cup \cdots \cup P_{+(k-2)}$ is a subgraph of $C_{k+1,k}$ consisting of all edges incident with the vertices in $A_1$, that is, $\bigcup_{i=0}^{k/2-1} P_{+2i} = G_1$. Therefore, $G_1$ is $P_{k+1}$-decomposable.

Let $Q_i = G_2[a_i \cup B]$ for $i \in \{k/2, k/2+1, \ldots, k\}$. It is easy to check that $Q_i = S_k$ and $\bigcup_{i=k/2}^{k} Q_i = G_2$. Hence $G_2$ is $S_k$-decomposable. This completes the proof.

Now, we are ready for the main result. It is obtained by combining Lemmas 3.1–3.3, 3.7, and 3.8.

Theorem 3.9. Let $k$ and $n$ be positive integers and let $I$ be a 1-factor of $K_{n,n}$. The graph $K_{n,n} - I$ is $(P_{k+1}, S_k)$-decomposable if and only if $k \leq n-1$ and $n(n-1) \equiv 0 \pmod{k}$.

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