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# DECOMPOSITION OF THE COMPLETE BIPARTITE GRAPH WITH A 1-FACTOR REMOVED INTO PATHS AND STARS

JENQ-JONG LIN AND HUNG-CHIH LEE

ABSTRACT. Let $P_k$ denote a path on $k$ vertices, and let $S_k$ denote a star with $k$ edges. For graphs $F$, $G$, and $H$, a decomposition of $F$ is a set of edge-disjoint subgraphs of $F$ whose union is $F$. A $(G,H)$-decomposition of $F$ is a decomposition of $F$ into copies of $G$ and $H$ using at least one copy of each. In this paper, necessary and sufficient conditions for the existence of the $(P_{k+1}, S_k)$-decomposition of the complete bipartite graph with a 1-factor removed are given.

## 1. Introduction

Let $F$, $G$, and $H$ be graphs. A *decomposition* of $F$ is a set of edge-disjoint subgraphs of $F$ whose union is $F$. A *$G$-decomposition* of $F$ is a decomposition of $F$ into copies of $G$. If $F$ has a $G$-decomposition, we say that $F$ is *$G$-decomposable* and write $G|F$. A *$(G,H)$-decomposition* of $F$ is a decomposition of $F$ into copies of $G$ and $H$ using at least one copy of each. If $F$ has a $(G,H)$-decomposition, we say that $F$ is *$(G,H)$-decomposable* and write $(G,H)|F$.

For positive integers $m$ and $n$, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. A *$k$-path*, denoted by $P_k$, is a path on $k$ vertices. A *$k$-star*, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$. A *$k$-cycle*, denoted by $C_k$, is a cycle of length $k$. A *spanning subgraph* $H$ of a graph $G$ is a subgraph of $G$ with $V(H) = V(G)$. A *1-factor* of $G$ is a spanning subgraph of $G$ in which each vertex of $G$ is incident with exactly one edge. Note that $K_{m,n}$ has a 1-factor if and only if $m = n$. Letting $I$ be a 1-factor of $K_{n,n}$, we use $K_{n,n} - I$ to denote $K_{n,n}$ with a 1-factor removed.

For positive integers $l$ and $n$ with $1 \leq l \leq n$, the *crown* $C_{n,l}$ is the bipartite graph with bipartition $(A, B)$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and edge set $\{a_i b_j : 1 \leq j - i \leq l$ with arithmetic

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modulo $n\}$. Note that $K_{n,n}$ is isomorphic to $C_{n,n}$, and $K_{n,n}-I$ is isomorphic to $C_{n,n-1}$ for any 1-factor $I$.

For a graph $G$ and a positive integer $\lambda$, we use $\lambda G$ to denote the multigraph obtained from $G$ by replacing each edge $e$ by $\lambda$ edges each having the same endpoints as $e$.

Decomposition into isomorphic paths has attracted considerable attention; see [7, 9, 10, 11, 14, 15, 18, 22, 25, 31, 33, 35]. Decomposition into $k$-stars has also attracted a fair share of interest; see [8, 20, 32, 34, 36, 37]. Abueida and Daven introduced the study of $(G, H)$-decompositions in [1], they investigated the $(K_k, S_k)$-deomposition of the complete graph $K_n$ in [2], and the $(C_4, E_2)$-decomposition of several graph products in [3], where $E_2$ denotes the 4-vertex graph having two disjoint edges. Abueida and O'Neil [5] settled the existence problem for $(C_k, S_{k-1})$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [23, 24] gave necessary and sufficient conditions for the existence of $(G_n, H_n)$-decompositions of $\lambda K_n$ and $\lambda K_{n,n}$, where $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$.

Recently, Lee [16], Lee [17], Lee and Lin [19], and Lin [21] established necessary and sufficient conditions for the existence of $(C_k, S_k)$-decompositions of the complete bipartite graph, the complete bipartite multigraph, the complete bipartite graph with a 1-factor removed, and the multicrown, respectively. Abueida and Lian [4] and Beggas et al. [6] investigated $(C_k, S_k)$-decompositions of the complete graph $K_n$ and $\lambda K_n$, giving some necessary or sufficient conditions for such decompositions to exist.

The problem of decomposing a graph into copies of a graph $G$ and copies of a graph $H$ where the number of copies of $G$ and the number of copies of $H$ are essential is also studied. Shyu gave necessary and sufficient conditions for the decomposition of $K_n$ into paths and stars (both with 3 edges) [26], paths and cycles (both with $k$ edges where $k = 3, 4$) [27, 28], and cycles and stars (both with 4 edges) [30]. Shyu [29] also gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths and stars both with 3 edges. Jeevadoss and Muthusamy [12, 13] considered the decomposability of $K_{m,n}$, $K_n$ and $\lambda K_{m,n}$ into paths and cycles having $k$ edges, giving some necessary or sufficient conditions for such decompositions to exist.

In this paper, we consider the existence of $(P_{k+1}, S_k)$-decompositions of the complete bipartite graph with a 1-factor removed, giving necessary and sufficient conditions.

## 2. Preliminaries

In this section we first collect some needed terminology and notations, and then present some results which are useful for our discussions to follow.

Let $G$ be a graph. The *degree* of a vertex $x$ of $G$, denoted by $\deg_G x$, is the number of edges incident with $x$. The vertex of degree $k$ in $S_k$ is the *center*, and any vertex of degree 1 is an *endvertex* of $S_k$. For $S \subseteq V(G)$ and $T \subseteq E(G)$, we use $G[S]$ and $G - T$ to denote the subgraph of $G$

induced by $S$ and the subgraph of $G$ obtained by deleting $T$, respectively. When $G_1, G_2, \ldots, G_t$ are graphs, not necessarily disjoint, we write $G_1 \cup G_2 \cup \cdots \cup G_t$ or $\bigcup_{i=1}^{t} G_i$ for the graph with vertex set $\bigcup_{i=1}^{t} V(G_i)$ and edge set $\bigcup_{i=1}^{t} E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^{t} G_i$ expresses the decomposition of $G$ into $G_1, G_2, \ldots, G_t$. Let $v_1 v_2 \ldots v_k$ and $(v_1, v_2, \ldots, v_k)$ denote the $k$-path and the $k$-cycle through vertices $v_1, v_2, \ldots, v_k$ in order, respectively, and let $(x; y_1, y_2, \ldots, y_k)$ denote the $k$-star with center $x$ and endvertices $y_1, y_2, \ldots, y_k$.

For the edge $a_i b_j$ in $C_{n,n-1}$, the *label* of $a_i b_j$ is $j - i \pmod{n}$. For example, in $C_{8,7}$ the labels of $a_1 b_6$ and $a_7 b_3$ are 5 and 4, respectively. Note that each vertex of $C_{n,n-1}$ is incident with exactly one edge with label $i$ for $i \in \{1, 2, \ldots, n-1\}$. Let $H$ be a subgraph of $C_{n,n-1}$ (recall that $C_{n,n-1}$ has partite sets $\{a_0, a_1, \ldots, a_{n-1}\}$ and $\{b_0, b_1, \ldots, b_{n-1}\}$). When $r$ is a nonnegative integer, $H_{+r}$ denotes the graph with vertex set $\{a_i : a_i \in V(H)\} \cup \{b_{j+r} : b_j \in V(H)\}$ and edge set $\{a_i b_{j+r} : a_i b_j \in E(H)\}$, and $H + r$ denotes the subgraph of $C_{n,n-1}$ with vertex set $\{a_{i+r} : a_i \in V(H)\} \cup \{b_{j+r} : b_j \in V(H)\}$ and edge set $\{a_{i+r} b_{j+r} : a_i b_j \in E(H)\}$ where the subscripts of $a$ and $b$ are taken modulo $n$. In particular, $H_{+0} = H + 0 = H$.

The following results are essential to our proof.

**Proposition 2.1** ([20])**.** *Let $\lambda$, $k$, $l$, and $n$ be positive integers. $\lambda C_{n,l}$ is $S_k$-decomposable if and only if $k \le l$ and $\lambda n l \equiv 0 \pmod{k}$.*

**Proposition 2.2** ([37])**.** *For integers $m$ and $n$ with $m \ge n \ge 1$, the graph $K_{m,n}$ is $S_k$-decomposable if and only if $m \ge k$ and*

$$\begin{cases} m \equiv 0 \pmod{k} & \text{if } n < k, \\ mn \equiv 0 \pmod{k} & \text{if } n \ge k. \end{cases}$$

**Proposition 2.3** ([31])**.** *Let $k$, $l$, and $n$ be positive integers. $C_{n,l}$ is $P_{k+1}$-decomposable if and only if $nl \equiv 0 \pmod{k}$ and*

$$k \le \begin{cases} 2 & \text{if } l = n = 2, \\ 2n - 3 & \text{if } l \text{ is even and } n \ge 3, \\ l & \text{if } l \text{ is odd.} \end{cases}$$

**Proposition 2.4** ([22])**.** *Let $k$, $m$, and $n$ be positive integers. There exists a $P_{k+1}$-decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod{k}$ and one of the following cases occurs.*

| *Case* | $k$ | $m$ | $n$ | *Conditions* |
|---|---|---|---|---|
| *1* | *even* | *even* | *even* | $k \le 2m, k \le 2n$, *not both equalities* |
| *2* | *even* | *even* | *odd* | $k \le 2m - 2, k \le 2n$ |
| *3* | *even* | *odd* | *even* | $k \le 2m, k \le 2n - 2$ |
| *4* | *odd* | *even* | *even* | $k \le 2m - 1, k \le 2n - 1$ |
| *5* | *odd* | *even* | *odd* | $k \le 2m - 1, k \le n$ |
| *6* | *odd* | *odd* | *even* | $k \le m, k \le 2n - 1$ |
| *7* | *odd* | *odd* | *odd* | $k \le m, k \le n$ |

## 3. Main results

Since $K_{n,n} - I$ is isomorphic to the crown $C_{n,n-1}$ for any 1-factor $I$ of $K_{n,n}$, $K_{n,n} - I$ is replaced by $C_{n,n-1}$ in the following discussions. We first give necessary conditions for a $(P_{k+1}, S_k)$-decomposition of $C_{n,n-1}$.

**Lemma 3.1.** *If $C_{n,n-1}$ is $(P_{k+1}, S_k)$-decomposable, then $k \le n-1$ and $n(n-1) \equiv 0 \pmod{k}$.*

*Proof.* Since the maximum size of a star in $C_{n,n-1}$ is $n-1$, $k \le n-1$ is necessary. Since $C_{n,n-1}$ has $n(n-1)$ edges and each subgraph in a decomposition has $k$ edges, $k$ must divide $n(n-1)$. □

We now show that the necessary conditions are also sufficient. Since $P_{k+1} = S_k$ for $k = 1, 2$, the result holds for $k = 1, 2$ by Proposition 2.1. So it remains to consider the case $k \ge 3$. The proof is divided into cases $n \ge 2k+1$, $n = 2k$, $2k-1 \ge n \ge k+2$, and $n = k+1$, which are treated in Lemmas 3.2, 3.3, 3.7, and 3.8, respectively.

**Lemma 3.2.** *Let $k$ and $n$ be positive integers with $n \ge 2k+1$. If $n(n-1) \equiv 0 \pmod{k}$, then $C_{n,n-1}$ is $(P_{k+1}, S_k)$-decomposable.*

*Proof.* Let $n - 1 = qk + r$ where $q$ and $r$ are integers with $0 \le r < k$. From the assumption $n \ge 2k+1$, we have $q \ge 2$. Note that

$$\begin{aligned} C_{n,n-1} &= C_{qk+r+1,qk+r} \\ &= C_{(q-1)k+1,(q-1)k} \cup C_{k+r+1,k+r} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k}. \end{aligned}$$

Cleary, $|E(C_{(q-1)k+1,(q-1)k})|$, $|E(K_{(q-1)k,k+r})|$, and $|E(K_{k+r,(q-1)k})|$ are multiples of $k$. Thus, $(k+r+1)(k+r) \equiv 0 \pmod{k}$ from the assumption $n(n-1) \equiv 0 \pmod{k}$. By Propositions 2.1 and 2.2, $C_{(q-1)k+1,(q-1)k}$ together with $K_{(q-1)k,k+r}$ and $K_{k+r,(q-1)k}$ are $S_k$-decomposable. By Proposition 2.3, $C_{k+r+1,k+r}$ is $P_{k+1}$-decomposable. Hence, $C_{n,n-1}$ is $(P_{k+1}, S_k)$-decomposable. □

**Lemma 3.3.** *If $k$ is an integer with $k \ge 3$, then the crown $C_{2k,2k-1}$ is $(P_{k+1}, S_k)$-decomposable.*

*Proof.* Let $A_1 = \{a_0, a_1, \ldots, a_{k-1}\}$, $A_2 = \{a_k, a_{k+1}, \ldots, a_{2k-1}\}$, $B = \{b_0, b_1, \ldots, b_{2k-1}\}$, and let $G_i = C_{2k,2k-1}[A_i \cup B]$ for $i = 1, 2$. Note that $C_{2k,2k-1} = G_1 \cup G_2$. It is easy to check that $G_1$ is isomorphic to $G_2$ with isomorphism $f$ such that $f(a_i) = a_{i+k}$ and $f(b_j) = b_{j+k}$ where the subscripts of $a$ and $b$ are taken modulo $2k$ for $i \in \{0, 1, \ldots, k-1\}$ and $j \in \{0, 1, \ldots, 2k-1\}$. Hence it is sufficient to show that $G_1$ is $(P_{k+1}, S_k)$-decomposable. We distinguish two cases according to the parity of $k$.

*Case 1*: $k$ is even.

Define a $(k+1)$-path $P = b_1 a_0 b_2 a_1 \ldots b_{k/2} a_{k/2-1} b_{k/2+1}$. Note that $P_{+2i}$, where the subscripts of $b$ are taken modulo $2k-1$, is a $(k+1)$-path for $i \in \{0, 1, \ldots, k-2\}$. We can see that $P \cup P_{+2} \cup P_{+4} \cup \cdots \cup P_{+2(k-2)}$ is the subgraph of $G_1$ consisting of all edges between $\{a_0, a_1, \ldots, a_{k/2-1}\}$

and $B - \{b_{2k-1}\}$. Moreover, $P_{+2i} + k/2$, where the subscripts of $b$ are taken modulo $2k-1$, is also a $(k+1)$-path, and $G_1[A_1 \cup B - \{b_{2k-1}\}] = \bigcup_{i=0}^{k-2}(P_{+2i} \cup (P_{+2i} + k/2))$. Since $(b_{2k-1}; a_0, a_1, \dots, a_{k-1})$ is a $k$-star, $G_1$ is $(P_{k+1}, S_k)$-decomposable.

*Case 2*: $k$ is odd.

Observe that $G_1 = C_{k,k-1} \cup K_{k,k}$. Since $k-1$ is even and $k \leq 2k-3$ for $k \geq 3$, Proposition 2.3 implies that $C_{k,k-1}$ is $P_{k+1}$-decomposable. By Proposition 2.2, $K_{k,k}$ is $S_k$-decomposable. Hence $G_1$ is $(P_{k+1}, S_k)$-decomposable.

□

Before plunging into the proof of the next case, we need the following results.

**Proposition 3.4** ([20]). *Let $\{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$ be the vertex set of the multicrown $\lambda C_{n,l}$. If there exist positive integers $p$ and $q$ so that $q < p \leq l$ and $\lambda nq \equiv 0 \pmod{p}$, then there exists a spanning subgraph $G$ of $\lambda C_{n,l}$ with $\deg_G b_j = \lambda q$ for $0 \leq j \leq n-1$, and $G$ has an $S_p$-decomposition.*

The following is trivial.

**Lemma 3.5.** *If $W$ is a graph consisting of a $k$-cycle $C$ and a $t$-star $S$ such that $S$ has its center in $C$ and has at least one endvertex not in $C$, then $W$ can be decomposed into a $(k+1)$-path and a $t$-star.*

**Lemma 3.6.** *Let $k$ and $r$ be positive integers with $r < k-1$ and let $t = (r+1)r/k$. Suppose that $t$ is an integer with $t \geq 2$. If $G$ is a bipartite graph with bipartition $(A_1, B_1 \cup B')$, where $|A_1| = |B_1| = k$ and $|B'| = r+1$ such that $G[A_1 \cup B_1] = C_{k,k-1}$ and $G[A_1 \cup B'] = K_{k,r+1}$, then $G$ can be decomposed into $t-1$ copies of $P_{k+1}$ and $k$ copies of $S_k$ with centers at distinct vertices in $A_1$, and $r+1$ copies of $S_{k-r}$ with centers at distinct vertices in $B'$.*

*Proof.* Let $A_1 = \{a_0, a_1, \dots, a_{k-1}\}$, $B_1 = \{b_0, b_1, \dots, b_{k-1}\}$, and $B' = \{b_k, b_{k+1}, \dots, b_{k+r}\}$. Letting $F = G[A_1 \cup B_1]$, $H = G[A_1 \cup B']$, and $p = \lfloor (t-1)/2 \rfloor$. Define a $2k$-cycle $C = (b_1, a_0, b_2, a_1, \dots, b_{k-1}, a_{k-2}, b_0, a_{k-1})$, and for even $t$ define a $(k+1)$-path $P$ according to the parity of $k$ as follows:

$$P = \begin{cases} b_{2p+1}a_0b_{2p+2}a_1 \dots b_{2p+k/2}a_{k/2-1}b_{2p+k/2+1} & \text{if } k \text{ is even,} \\ b_{2p+1}a_0b_{2p+2}a_1 \dots b_{2p+(k+1)/2}a_{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

It is easy to check that $C, C_{+2}, \dots, C_{+2(p-1)}$ and $P$, where the subscripts of $b$ are taken modulo $k$, are edge-disjoint in $F$.

Define a subgraph $W$ of $F$ as follows:

$$W = \begin{cases} \bigcup_{i=0}^{p-1} C_{+2i} & \text{if } t \text{ is odd,} \\ \bigcup_{i=0}^{p-1} C_{+2i} \cup P & \text{if } t \text{ is even.} \end{cases}$$

Since $C_{2k}$ can be decomposed into two copies of $P_{k+1}$ and $2p = t-1$ for odd $t$ as well as $2p+1 = t-1$ for even $t$, $W$ can be decomposed into $t-1$ copies of $P_{k+1}$. When $t$ is odd, $\deg_{F-E(W)} a_i = k - 2p - 1$. When $t$ is even,

$$\deg_{F-E(W)} a_i = \begin{cases} k-2p-3 & \text{if } i \in \{0,1,\ldots,\lceil k/2\rceil-2\}, \\ k-2p-\delta & \text{if } i = \lceil k/2\rceil-1, \\ k-2p-1 & \text{if } i \in \{\lceil k/2\rceil, \lceil k/2\rceil+1,\ldots,k-1\}, \end{cases}$$

where

$$\delta = \begin{cases} 2 & \text{if } k \text{ is odd}, \\ 3 & \text{if } k \text{ is even}. \end{cases}$$

For $i = 0, 1, \ldots, k-1$, let $X_i = (F - E(W))[\{a_i\} \cup B_1]$. Clearly $X_i$ is a star with center $a_i$, and for odd $t$, we have $X_i = S_{k-2p-1}$, and for even $t$, we have

$$X_i = \begin{cases} S_{k-2p-3} & \text{if } i \in \{0,1,\ldots,\lceil k/2\rceil-2\}, \\ S_{k-2p-\delta} & \text{if } i = \lceil k/2\rceil-1, \\ S_{k-2p-1} & \text{if } i \in \{\lceil k/2\rceil, \lceil k/2\rceil+1,\ldots,k-1\}. \end{cases}$$

In the remainder of the proof, we will show that $H$ can be decomposed into $r+1$ copies of $S_{k-r}$ with centers in $B'$, $k$ copies of $S_{2p+1}$ with centers in $A_1$ for odd $t$, $\lceil k/2\rceil - 1$ copies of $S_{2p+3}$ with centers in $\{a_0, a_1, \ldots, a_{\lceil k/2\rceil-2}\}$, and a copy of $S_{2p+\delta}$ with center $a_{\lceil k/2\rceil-1}$, together with $k - \lceil k/2\rceil$ copies of $S_{2p+1}$ with centers in $\{a_{\lceil k/2\rceil}, a_{\lceil k/2\rceil+1}, \ldots, a_{k-1}\}$ for even $t$.

Now we show the required star-decomposition of $H$ by orienting the edges of $H$. For any vertex $x$ of $H$, the *outdegree* $\deg_H^+ x$ (*indegree* $\deg_H^- x$, respectively) of $x$ in an orientation of $H$ is the number of arcs incident from (to, respectively) $x$. It is sufficient to show that there exists an orientation of $H$ such that

$$\deg_H^+ b_j = k - r, \tag{3.1}$$

where $j \in \{k, k+1, \ldots, k+r\}$, and for odd $t$

$$\deg_H^+ a_i = 2p+1 \tag{3.2}$$

where $i \in \{0, 1, \ldots, k-1\}$, and for even $t$

$$\deg_H^+ a_i = \begin{cases} 2p+3 & \text{if } i \in \{0,1,\ldots,\lceil k/2\rceil-2\}, \\ 2p+\delta & \text{if } i = \lceil k/2\rceil-1, \\ 2p+1 & \text{if } i \in \{\lceil k/2\rceil, \lceil k/2\rceil+1,\ldots,k-1\}. \end{cases} \tag{3.3}$$

We first consider the edges oriented outward from $A_1$ according to the parity of $t$. When $t$ is odd, the edges $a_ib_{k+(2p+1)i}, a_ib_{k+(2p+1)i+1}, \ldots, a_ib_{k+(2p+1)i+2p}$ are all oriented outward from $a_i$ for $i \in \{0, \ldots, k-1\}$. Let $\beta = (2p+3)(\lceil k/2\rceil - 1)$. When $t$ is even, the edges $a_ib_{k+(2p+3)i}, a_ib_{k+(2p+3)i+1}, \ldots, a_ib_{k+(2p+3)i+2p+2}$ for $i \in \{0, 1, \ldots, \lceil k/2\rceil - 2\}$, the edges $a_{\lceil k/2\rceil-1}b_{k+\beta}$, $a_{\lceil k/2\rceil-1}b_{k+\beta+1}, \ldots, a_{\lceil k/2\rceil-1}\, b_{k+\beta+2p+\delta-1}$, and the edges $a_ib_{k+(2p+1)i+\beta+2p+\delta}$, $a_ib_{k+(2p+1)i+\beta+2p+\delta+1}, \ldots, a_ib_{k+(2p+1)i+\beta+4p+\delta}$ for $i \in \{\lceil k/2\rceil, \lceil k/2\rceil+1, \ldots, k-1\}$ are all oriented outward from $a_i$. The subscripts of $b$ are taken modulo $r+1$ in the set of numbers $\{k, k+1, \ldots, k+r\}$. Note that for odd $t$ we orient $2p+1$ edges from each $a_i$, and for even $t$ we orient at most

$2p+3$ edges from $a_i$. Since $2p+1 \le 2(t-1)/2+1 = t < r$ for odd $t$ and $2p+3 \le 2(t-2)/2+3 = t+1 < r+1$ for even $t$, this guarantees that there are enough edges for the above orientation. Finally, the edges which are not oriented yet are all oriented from $B'$ to $A_1$.

From the construction of the orientation, it is easy to see that (3.2) and (3.3) are satisfied, and for all $b_w, b_{w'} \in B'$, we have

$$|\deg_H^- b_w - \deg_H^- b_{w'}| \le 1. \tag{3.4}$$

So, we only need to check (3.1).

Since $\deg_H^+ b_w + \deg_H^- b_w = k$ for $b_w \in B'$, it follows from (3.4) that $|\deg_H^+ b_w - \deg_H^+ b_{w'}| \le 1$ for $b_w, b_{w'} \in B'$. Further, note that for odd $t$,

$$\sum_{i=0}^{k-1} \deg_H^+ a_i = (2p+1)k = (2(t-1)/2+1)k = tk,$$

and for even $t$,

$$\begin{aligned}
\sum_{i=0}^{k-1} \deg_H^+ a_i &= (2p+3)(\lceil k/2\rceil - 1) + 2p + \delta + (2p+1)(k - \lceil k/2\rceil) \\
&= (2p+1)(k-1) + 2(\lceil k/2\rceil - 1 + p) + \delta \\
&= \begin{cases} (2p+1)(k-1) + 2((k+1)/2 - 1 + p) + 2 & \text{if } k \text{ is odd,} \\ (2p+1)(k-1) + 2(k/2 - 1 + p) + 3 & \text{if } k \text{ is even,} \end{cases} \\
&= (2p+2)k \\
&= (2(t-2)/2+2)k \\
&= tk
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{w=k}^{k+r} \deg_H^+ b_w &= |E(K_{k,r+1})| - \sum_{i=0}^{k-1} \deg_H^+ a_i \\
&= k(r+1) - tk \\
&= k(r+1) - r(r+1) \\
&= (k-r)(r+1)
\end{aligned}$$

Therefore, $\deg_H^+ b_w = k - r$ for $b_w \in B'$. This proves (3.1). Hence there exists the required decomposition $\mathscr{D}$ of $H$. Let $X_i'$ be the star with center at $a_i$ in $\mathscr{D}$ for $i \in \{0, 1, \ldots, k-1\}$. Clearly $X_i + X_i'$ is a $k$-star. This completes the proof. □

**Lemma 3.7.** *Let $k$ and $n$ be integers with $3 \le k < n-1 < 2k-1$. If $n(n-1) \equiv 0 \pmod{k}$, then $C_{n,n-1}$ is $(P_{k+1}, S_k)$-decomposable.*

*Proof.* Let $n-1 = k+r$. From the assumption $k < n-1 < 2k-1$, we have $0 < r < k-1$. Let $t = (r+1)r/k$. Since $k \mid n(n-1)$, we have $k \mid (r+1)r$, which implies that $t$ is a positive integer. The proof is divided into two parts according to the value of $t$.

*Case 1*: $t = 1$.

When $t = 1$, $k = (r+1)r$. This implies that $k$ is even, and hence $k \geq 4$; in turn, we have $r \geq 2$. Let $A_0 = \{a_0, a_1, \ldots, a_{k/2-1}\}$ and $B_0 = \{b_0, b_1, \ldots, b_{k/2-1}\}$. We distinguish two subcases.

*Subcase 1.1*: $r = 2$.

For $r = 2$, $k = (r+1)r = 6$, and $n = k + r + 1 = 9$. Let $H_1 = C_{9,8}[A_0 \cup B]$ and $H_2 = C_{9,8}[(A - A_0) \cup B]$. Clearly $C_{9,8} = H_1 \cup H_2$. Let $P = b_1 a_0 b_2 a_1 b_3 a_2 b_4$, we have $\deg_{H_1 - E(P)} a_i = 6$ for $i \in \{0, 1, 2\}$. This implies $H_1 - E(P)$ is $S_6$-decomposable. Let $F_i = (b_i; a_3, a_4, \ldots, a_8)$ for $i \in \{0, 1\}$ and $G_j = (a_j; b_2, \ldots, b_{j-1}, b_{j+1}, b_{j+2}, \ldots, b_8)$ for $j \in \{3, 4, \ldots, 8\}$. It is easy to check that $F_i$ and $G_j$ are 6-stars and $\{F_0, F_1, G_3, G_4, \ldots, G_8\}$ is an $S_6$-decomposition of $H_2$.

*Subcase 1.2*: $r \geq 3$.

We will show that $C_{n,n-1}$ can be decomposed into copies of $S_k$ together with a copy of $P_{k+1}$. Let $D_0 = C_{n,n-1}[A_0 \cup B_0]$, $D_1 = C_{n,n-1}[(A - A_0) \cup B_0]$, and $D_2 = C_{n,n-1}[A \cup (B - B_0)]$. Clearly $C_{n,n-1} = D_0 \cup D_1 \cup D_2$. Note that $D_0$ is isomorphic to $C_{k/2,k/2-1}$, $D_1$ is isomorphic to $K_{k/2+r+1,k/2}$, and $D_2$ is isomorphic to $C_{k/2+r+1,k/2+r} \cup K_{k/2,k/2+r+1}$. Let $C = (a_0, b_{k/2-1}, a_1, b_0, a_2, b_1, \ldots, a_{k/2-1}, b_{k/2-2})$ and $D = D_0 - E(C)$. Trivially $C$ is a $k$-cycle in $D_0$ and $D = C_{k/2,k/2-3}$. Note that $0 < r - 2 < k/2 - r - 1 < k/2 - 3$ for $r \geq 3$ and $(k/2)(r-2) = r(r+1)(r-2)/2 = r(k/2 - r - 1)$. Therefore, Proposition 3.4 implies that there exists a spanning subgraph $X$ of $D$ such that $\deg_X b_j = r - 2$ for $0 \leq j \leq k/2 - 1$, and $X$ has an $S_{k/2-r-1}$-decomposition $\mathscr{D}$ with $|\mathscr{D}| = r$. Furthermore, each $S_{k/2-r-1}$ has its center in $A_0$ since $\deg_X b_j = r - 2 < k/2 - r - 1$. Suppose that the centers of $(k/2-r-1)$-stars in $\mathscr{D}$ are $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$. Let $S(w)$ be the $(k/2 - r - 1)$-star with center $a_{i_w}$ in $\mathscr{D}$ and let $Y = D - E(X) \cup D_1$. Note that $\deg_Y b_j = (k/2 - 3 - (r-2)) + (k/2 + r + 1) = k$ for $0 \leq j \leq k/2 - 1$. Hence $Y$ is $S_k$-decomposable. For $w \in \{1, 2, \ldots, r\}$, define $S'(w) = D_2[\{a_{i_w}\} \cup (B - B_0)]$ and $Z = D_2 - E(\bigcup_{w=1}^{r} S'(w))$. Clearly $S'(w)$ is a $(k/2 + r + 1)$-star with center $a_{i_w}$ in $D_2$, and $S(w) \cup S'(w)$ is a $k$-star. Moreover, $\deg_Z b_j = k + r - r = k$ for $k/2 \leq j \leq k + r$. Thus $Z$ is $S_k$-decomposable. By Lemma 3.5, $C \cup S(1) \cup S'(1)$ can be decomposed into a $(k+1)$-path and a $k$-star.

*Case 2*: $t \geq 2$.

Let $A_1 = \{a_0, a_1, \ldots, a_{k-1}\}$, $A' = \{a_k, a_{k+1}, \ldots, a_{k+r}\}$, $B_1 = \{b_0, b_1, \ldots, b_{k-1}\}$, and $B' = \{b_k, b_{k+1}, \ldots, b_{k+r}\}$. Moreover, let $G = C_{n,n-1}[A_1 \cup B]$, $F = C_{n,n-1}[A' \cup B_1]$, and $H = C_{n,n-1}[A' \cup B']$. Clearly $C_{n,n-1} = G \cup F \cup H$. Note that $G$ is isomorphic to $C_{k,k-1} \cup K_{k,r+1}$, $H$ is isomorphic to $C_{r+1,r}$, and $F$ is isomorphic to $K_{r+1,k}$, Proposition 2.2 implies that $K_{r+1,k}$ is $S_k$-decomposable. By Lemma 3.6, there exists a decomposition

$\mathscr{D}$ of $G$ into $t-1$ copies of $P_{k+1}$, $k$ copies of $S_k$ with centers at distinct vertices in $A_1$, and $r+1$ copies of $S_{k-r}$ with centers at distinct vertices in $B'$. For $i \in \{k, k+1, \ldots, k+r\}$, let $Z_i$ be the $(k-r)$-star with center $b_i$ in $\mathscr{D}$ and let $Z_i' = H[A' \cup b_i]$. Trivially $Z_i \cup Z_i' = S_k$. Thus $C_{n,n-1}$ is $(P_{k+1}, S_k)$-decomposable.

□

**Lemma 3.8.** *If $k$ is an integer with $k \geq 3$, then $C_{k+1,k}$ is $(P_{k+1}, S_k)$-decomposable.*

*Proof.* The proof is divided into two parts according to the parity of $k$.
*Case 1*: $k$ is odd.

Note that $C_{k+1,k}$ can be decomposed into $C_{k,k-1}$ and two copies of $S_k$. Since $k-1$ is even and $k \leq 2k-3$ for $k \geq 3$, Proposition 2.3 implies that $C_{k,k-1}$ is $P_{k+1}$-decomposable. Hence $C_{k+1,k}$ is $(P_{k+1}, S_k)$-decomposable.
*Case 2*: $k$ is even.

Let $A_1 = \{a_0, a_1, \ldots, a_{k/2-1}\}$ and $A_2 = \{a_{k/2}, a_{k/2+1}, \ldots, a_k\}$. For $i \in \{1,2\}$, define $G_i = C_{k+1,k}[A_i \cup B]$. Clearly $C_{k+1,k} = G_1 \cup G_2$. We will show that $G_1$ is $P_{k+1}$-decomposable and $G_2$ is $S_k$-decomposable.

Let $P = b_1 a_0 b_2 a_1 \ldots b_{k/2} a_{k/2-1} b_{k/2+1}$. Note that $P$ is a $(k+1)$-path containing all of the edges incident with the vertices in $A_1$, having labels 1 and 2. Furthermore, $P_{+2i}$ is a $(k+1)$-path containing all of the edges incident with the vertices in $A_1$, having labels $2i+1$ and $2i+2$ for $i = 0, 1, 2, \ldots, k/2-1$. Hence $P \cup P_{+2} \cup P_{+4} \cup \cdots \cup P_{+(k-2)}$ is a subgraph of $C_{k+1,k}$ consisting of all edges incident with the vertices in $A_1$, that is, $\bigcup_{i=0}^{k/2-1} P_{+2i} = G_1$. Therefore, $G_1$ is $P_{k+1}$-decomposable.

Let $Q_i = G_2[\{a_i\} \cup B]$ for $i \in \{k/2, k/2+1, \ldots, k\}$. It is easy to check that $Q_i = S_k$ and $\bigcup_{i=k/2}^{k} Q_i = G_2$. Hence $G_2$ is $S_k$-decomposable. This completes the proof.

□

Now, we are ready for the main result. It is obtained by combining Lemmas 3.1–3.3, 3.7, and 3.8.

**Theorem 3.9.** *Let $k$ and $n$ be positive integers and let $I$ be a 1-factor of $K_{n,n}$. The graph $K_{n,n} - I$ is $(P_{k+1}, S_k)$-decomposable if and only if $k \leq n-1$ and $n(n-1) \equiv 0 \pmod{k}$.*

## References


1. A. Abueida and M. Daven, *Multidesigns for graph-pairs of order 4 and 5*, Graphs Combin. **19** (2003), 433–447.
2. ———, *Multidecompositons of the complete graph*, Ars Combin. **72** (2004), 17–22.
3. ———, *Multidecompositions of several graph products*, Graphs Combin. **29** (2013), 315–326.
4. A. Abueida and C. Lian, *On the decompositions of complete graphs into cycles and stars on the same number of edges*, Discuss. Math. Graph Theory **34** (2014), 113–125.
5. A. Abueida and T. O'Neil, *Multidecomposition of $\lambda k_m$ into small cycles and claws*, Bull. Inst. Combin. Appl. **49** (2007), 32–40.

6. F. Beggas, M. Haddad, and H. Kheddouci, *Decomposition of complete multigraphs into stars and cycles*, Discuss. Math. Graph Theory **35** (2015), 629–639.
7. A. Bouchet and J. L. Fouquet, *Trois types de décomposition d'un graphe en chaînes*, Ann Discrete Math. **17** (1983), 131–141.
8. D. E. Bryant, S. El-Zanati, C. V. Eyden, and D. G. Hoffman, *Star decompositions of cubes*, Graphs Combin. **17** (2001), 55–59.
9. K. Heinrich, *Path-decompositions*, Matematiche (Catania) **47** (1992), 241–258.
10. K. Heinrich, J. Liu, and M. Yu, $P_4$*-decomposition of regular graphs*, J. Graph Theory **31** (1999), 135–143.
11. M. S. Jacobson, M. Truszczyński, and Z. Tuza, *Decompositions of regular bipartite graphs*, Discrete Math. **89** (1991), 17–27.
12. S. Jeevadoss and A. Muthusamy, *Decomposition of complete bipartite graphs into paths and cycles*, Discrete Math. **331** (2014), 98–108.
13. ———, *Decomposition of complete bipartite multigraphs into paths and cycles having k edges*, Discuss. Math. Graph Theory **35** (2015), 715–731.
14. A. Kotzig, *From the theory of finite regular graphs of degree three and four*, Časopis Pěst Mat **82** (1957), 76–92.
15. C. S. Kumar, *On* $p_4$*-decomposition of graphs*, Taiwanese J. Math. **7** (2003), 657–664.
16. H.-C. Lee, *Multidecompositions of complete bipartite graphs into cycles and stars*, Ars Combin. **108** (2013), 355–364.
17. ———, *Decomposition of the complete bipartite multigraph into cycles and stars*, Discrete Math. **338** (2015), 1362–1369.
18. H.-C. Lee, M.-J. Lee, and C. Lin, *Isomorphic path decompositions of* $\lambda k_{n,n,n}$ $(\lambda k^*_{n,n,n})$ *for odd n*, Taiwanese J. Math **13** (2009), 393–402.
19. H.-C. Lee and J.-J. Lin, *Decomposition of the complete bipartite graph with a 1-factor removed into cycles and stars*, Discrete Math. **313** (2013), 2354–2358.
20. C. Lin, J.-J. Lin, and T.-W. Shyu, *Isomorphic star decomposition of multicrowns and the power of cycles*, Ars Combin. **53** (1999), 249–256.
21. J.-J. Lin, *Decompositions of multicrowns into cycles and stars*, Taiwanese J. Math. **19** (2015), 1261–1270.
22. C. A. Parker, *Complete bipartite graph path decompositions*, Ph.D. thesis, Auburn University, Auburn, Alabama, 1998.
23. H. M. Priyadharsini and A. Muthusamy, $(G_m, H_m)$*-multifactorization of* $\lambda k_m$, J. Combin. Math. Combin. Comput. **69** (2009), 145–150.
24. ———, *Bull. inst. combin. appl.*, $(G_m, H_m)$-multidecomposition of $K_{m,m}(\lambda)$ **66** (2012), 42–48.
25. T.-W. Shyu, *Path decompositions of* $\lambda k_{n,n}$, Ars Combin. **85** (2007), 211–219.
26. ———, *Decomposition of complete graphs into paths and stars*, Discrete Math. **310** (2010), 2164–2169.
27. ———, *Decompositions of complete graphs into paths and cycles*, Ars Combin. **97** (2010), 257–270.
28. ———, *Decomposition of complete graphs into paths of length three and triangles*, Ars Combin. **107** (2012), 209–224.
29. ———, *Decomposition of complete bipartite graphs into paths and stars with same number of edges*, Discrete Math. **313** (2013), 865–871.
30. ———, *Decomposition of complete graphs into cycles and stars*, Graphs Combin. **29** (2013), 301–313.
31. T.-W. Shyu and C. Lin, *Isomorphic path decomposition of crowns*, Ars Combin. **67** (2003), 97–103.
32. M. Tarsi, *Decomposition of complete multigraphs into stars*, Discrete Math. **26** (1979), 273–278.
33. ———, *Decomposition of complete multigraph into simple paths: nonbalanced handcuffed designs*, J. Combin. Theory Ser. A **34** (1983), 60–70.

34. S. Tazawa, *Decomposition of a complete multipartite graph into isomorphic claws*, SIAM J. Algebraic Discrete Methods **6** (1985), 413–417.
35. M. Truszczyński, *Note on the decomposition of* $\lambda k_{m,n}(\lambda k^*_{m,n})$ *into paths*, Discrete Math. **55** (1985), 89–96.
36. K. Ushio, S. Tazawa, and S. Yamamoto, *On claw-decomposition of complete multipartite graphs*, Hiroshima Math. J. **8** (1978), 207–210.
37. S. Yamamoto, H. Ikeda, S. Shigeede, K. Ushio, , and N. Hamada, *On claw decomposition of complete graphs and complete bigraphs*, Hiroshima Math. J. **5** (1975), 33–42.

Department of Finance, Ling Tung University, Taichung, Taiwan
*E-mail address*: jjlin@teamail.ltu.edu.tw

Department of Information Technology, Ling Tung University, Taichung, Taiwan
*E-mail address*: birdy@teamail.ltu.edu.tw