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ON THE EQUATION $\sum_{i=1}^{n} \frac{1}{x_i} = 1$ IN DISTINCT ODD OR EVEN NUMBERS

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Abstract. In this paper we combine theoretical results and computer search to obtain information about the solutions of the equation

$$\sum_{i=1}^{n} \frac{1}{x_i} = 1.$$

We calculate (a) $m_o(n) = \min\{\max\{x_i \mid 1 \le i \le n\}\}$ and (b) $m_e(n) = \min\{\max\{x_i \mid 1 \le i \le n\}\}$, where the minimum is taken over all sets $\{x_i\}$ satisfying the above equation in distinct odd integers when $13 \le n \le 41$ (for case a) and in distinct even integers when $3 \le n \le 29$ (for case b). We compute the number of solutions of the above equation for:

- $n = 13, 15$ when $x_i \in \{3^\alpha \cdot 5^\beta \cdot 7^\gamma\}$;
- $n \le 17$ when $x_i \in \{3^\alpha \cdot 5^\beta \cdot 11^\gamma\}$;
- $n \le 23$ when $x_i \in \{3^\alpha \cdot 5^\beta \cdot 13^\gamma\}$.

We also compute $\max x_i$ for $\{x_i\}$ satisfying the mentioned equation in distinct even integers. Finally, we compute $\liminf(x_n/x_{n-k})$ for fixed k.

1. Introduction

In the book "Unsolved Problems in Number Theory" [7], Section D11 discusses the topic of Egyptian fractions. These are fractions that can be expressed as a finite sum $\sum_{i=1}^{n} 1/x_i$ of reciprocals of distinct positive integers. Special attention has been given to the equation

$$\sum_{i=1}^{n} \frac{1}{x_i} = 1, \tag{1.1}$$

where $x_1 < \cdots < x_n$. The number of solutions of Equation (1.1) for $1 \le n \le 8$ over distinct positive integers are given in [13]. Furthermore, in [4, 11, 1], the authors computed the number of solutions of Equation (1.1) for $n = 9, 11$ over distinct odd positive integers.

A question proposed by Erdős and Graham about these fractions that appears in [7] is to determine the value of $m(n)$, the min max x_i, where the minimum is taken over all sets $\{x_i\}$ satisfying Equation (1.1). Section D11

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in [7] contains a table showing $m(k)$ for $3 \leq n \leq 28$. It is observed that $m(n)$ is nondecreasing for these values.

In [2, 15], the authors proved that any rational number with an odd denominator can be written as the sum of distinct odd unit fractions. In [10], G. Martin computed the asymptotic behavior of $m(n)$ for Equation (1.1). More precisely, he proved

$$m(n) = \min \max_{1 \leq i \leq n} x_i = \frac{n}{1 - e^{-1}} + O_1\left(\frac{n \log\log 3n}{\log 3n}\right). \tag{1.2}$$

In this paper we consider the following problem: to determine the values of

$$m_o(n) = \min \max_{1 \leq i \leq n} x_i$$

and

$$m_e(n) = \min \max_{1 \leq i \leq n} x_i,$$

where x_i are distinct odd(even) positive integers satisfying Equation (1.1).

In fact, for each odd n, $13 \leq n \leq 41$, we found the value of $m_o(n)$ and all the sequences where $m_o(n)$ is the minimum value for the corresponding n. The minimum n for which $m_o(n)$ exists is $n = 9$; the values $m_o(9) = 231$ and $m_o(11) = 105$, were independently found by P. Shiu [11] and N. Burshtein [3].

In [1], Arce et. al. computed $m_o(13) = 115$. In this paper we compute $m_o(n)$ for $15 \leq n \leq 41$. For the case $m_e(n)$ we compute $m_e(n)$ for $3 \leq n \leq 29$. Our results imply that $m_o(n)$ and $m_e(n)$ are nondecreasing for $11 \leq n \leq 41$, $3 \leq n \leq 29$, respectively. Note that Martin's result in [10] implies that

$$m_e(n) = \frac{n}{1 - e^{-2}} + \text{(error term)}.$$

In [5], Chen-Elsholtz-Jiang proved a criterion for existence of solutions of Equation (1.1) over subsets of the type

$$S(p_1, \ldots, p_r) = \{p_1^{\alpha_1} \cdots p_r^{\alpha_r} \mid \alpha_1, \ldots, \alpha_r \in \mathbb{N}_0\},$$

where $\mathbb{N}_0$ are the nonnegative integers. Let $N_n(p_1, \ldots, p_r)$ be the set of the solutions of Equation (1.1) with $x_i \in S(p_1, \ldots, p_r)$ where any solution contains at least one x_j that is divisible by p_i for $i = 1, ..., r$. Note that in general $|N_n(p_1, \ldots, p_r)|$ is not equal to $T_n(p_1, \ldots, p_r)$ of [5], as $|N_n(p_1, \ldots, p_r)| \leq T_n(p_1, \ldots, p_r)$ and $|N_n(p_1, p_2, p_3)| = T_n(p_1, p_2, p_3)$. It is not difficult to prove that $|N_n(p_1, p_2)| = 0$, where p_1, p_2 are odd primes; see [5] for details. Hence the simplest case is $N_n(p_1, p_2, p_3)$. In [3], Burshtein proved that $N_{11}(3, 5, 7) = 17$. Using the information in [1], we have $N_{11}(p_1, p_2, p_3) = 0$ except in the case of $p_1 = 3, p_2 = 5$, and $p_3 = 7$. Motivated by the results of [5, 3, 1], we compute

- $N_{13}(3, 5, 7) = 2034$ and $N_{15}(3, 5, 7) = 374349$.
- $N_n(3, 5, 11) = 0$ for $n \leq 15$ and $N_{17}(3, 5, 11) = 11$.
- $N_n(3, 5, 13) = 0$ for $n \leq 21$ and $N_{23}(3, 5, 13) = 63$.

We also prove $N_n(p_1,\ldots,p_r) \geq 1$ over $S(p_1,p_2,\ldots,p_r)$ for $n \geq n_o$ odd and some $p_1,\ldots,p_r$ odd primes.

In this paper we study solutions of Equation (1.1) satisfying some restrictions. We prove the existence of solutions of Equation (1.1) where the last k terms are close to each other in a multiplicative sense. The optimization problem of finding $m(n)$ was solved in [8, 14]. In this paper we solve the optimization problem of finding $\max\{x_i\}$ over positive even numbers of the Equation (1.1).

The results in Section 2 are mainly computational, but in order to significantly reduce the computational time we had to develop some elementary constraints. We used some divisibility properties to obtain these constraints and they appear as theorem and corollaries in Section 2. The Tables in Section 2 summarize our results for $m_o(n), m_e(n)$. A complete list of sequences can be found in our page http://ccom.uprrp.edu/∼rarce/dmath.html. In Section 3, we present the calculation of $m_o(27)$ and $m_o(29)$ to show that in some cases the constraints reduce the solution space enough that they can be completed by hand. In Section 4, we study the solutions of Equation (1.1) over $S(p_1,\ldots,p_r)$. In section 4 we also combine computational results with an identity to prove the solvability of Equation (1.1). In Section 5, we construct solutions of Equation (1.1) satisfying certain properties. Our results of this section imply that $\liminf x_n/x_{n-k} = 1$ for fixed k. In the last section, we answer the optimization problem for $\max\{x_i\}$ over positive even numbers.

Our calculations have two goals:

- Prove that the optimization problem of computing $m_o(n)$ and $m_e(n)$ can be simplified significantly by using elementary arguments of divisibility, as well as prove that the divisibility constraints imposed by the Equation (1.1) allow us to compute $m_o(n)$ and $m_e(n)$ by hand.
- Provide more data about the behavior of $m_o(n)$ and $m_e(n)$.

2. Computation of $m_o(n)$ and $m_e(n)$

In this section we study the divisibility properties of the solutions of Equation (1.1). Using these properties, we were able to compute $m_o(n)$ and $m_e(n)$ for $13 \leq n \leq 41$ and $3 \leq n \leq 29$, respectively.

Theorem 2.1. *Let p be an odd prime number. Let M be a positive integer and let $x_1, x_2, \ldots, x_n$ be a sequence of distinct odd positive integers such that*

$$1 = \sum_{i=1}^{n} \frac{1}{x_i}$$

and $x_i < M$, for all i.

(1) *If p^t divides exactly k terms $x_1,\ldots,x_k$, and $p^{t+1} \nmid x_i$, for all i, let $a_i = x_i/p^t$ for $i = 1,...,k$, and let $A = a_1 \cdot a_2 \cdots a_k$. Then p divides $\sum_{i=1}^{k} A/a_i$*

(2) *If* $p^t \mid x_j$*, for some* j*, and* $p^{t+1} \nmid x_i$*, for all* i*, then* p^t *divides another term of the sequence* $\{x_i\}$*.*
(3) *If* $p^s \mid x_j$*, for some* j*, and* $p^{s+1} > M$*, then* p^s *divides at least three terms of the sequence* $\{x_i\}$*.*

Proof. (1) For $j > k$ we have that $p^t/x_j = c_j/d_j$ where $p \mid c_j$ and $p \nmid d_j$. From Equation (1.1) we have

$$p^t = \sum_{i=1}^{n} \frac{p^t}{x_i} = \sum_{i=1}^{k} \frac{1}{a_i} + \sum_{j=k+1}^{n} \frac{c_j}{d_j}.$$

Let $D = d_{k+1} \cdot d_{k+2} \cdots d_n$, so $p \nmid D$. From

$$ADp^t = D\left(\sum_{i=1}^{k} \frac{A}{a_i}\right) + A\left(\sum_{j=k+1}^{n} c_j \left(\frac{D}{d_j}\right)\right)$$

it follows that p divides $\sum_{i=1}^{k} A/a_i$

(2) This follows from part 1.

(3) If $x_1 = a_1 p^s$, $x_2 = a_2 p^s$ and $p^s \nmid x_j$ for $j > 2$, then $a_1 + a_2 \equiv 0 \bmod p$ by part 1. Since $a_1 + a_2$ is even, then $a_1 + a_2 \geq 2p$ so $a_1 > p$ (or $a_2 > p$) and $x_1 > p^{s+1} > M$.

□

The following corollaries are for specific values of M:

Corollary 2.2. *Let* p *and* $x_1, x_2, \ldots, x_n$ *be as in Theorem 2.1 and let* $M = 240$*. Then:*

(1) *If* $p > 37$*,* $p \nmid x_i$ *for all* i*;*
(2) *If* $p \geq 7$*,* $p^2 \nmid x_i$*, for all* i*;*
(3) $81 \nmid x_i$ *and* $29 \nmid x_i$*, for all* i*.*

Proof. (1) Assume $p \mid x_1$. Since $p^2 > M$ and $7p > M$, we have that p divides exactly three terms, say $x_1 = p$, $x_2 = 3p$ and $x_3 = 5p$, so $a_1 = 1, a_2 = 3$, and $a_3 = 5$. By Theorem 2.1, $\sum A/a_i = 23 \equiv 0 \bmod p$; since $p > 37$, this is a contradiction.

(2) Assume p^2 divides some x_i. Then $p^2 < M$ so $p = 7, 11, 13$. Since $p^3 > M$ and $7p^2 > M$, p^2 divides exactly three terms and, as before, p divides 23 but $p < 23$, which is a contradiction.

(3) The case 81 follows from Theorem 2.1. For $p = 29$ we have that p divides three or four terms. In the first case, $x_1 = a_1 p$, $x_2 = a_2 p$, $x_3 = a_3 p$, and $a_1, a_2, a_3 \in \{1, 3, 5, 7\}$, since $9p > M$. If $\{a_1, a_2, a_3\} = \{1, 3, 5\}$,$\{1, 3, 7\}$, $\{1, 5, 7\}$, or $\{3, 5, 7\}$, we respectively have that $\sum A/a_i = 23, 31, 47$, or 71, so p does not divide $\sum A/a_i$. If p divides 4 terms, then $a_1 = 1, a_2 = 3, a_3 = 5$ and $a_4 = 7$, and in this case $\sum A/a_i = 176 \not\equiv 0 \bmod p$. □

Corollary 2.3. *Let* p *and* $x_1, x_2, \ldots, x_n$ *be as in Theorem 2.1. Let* $M = 200$*, then* $31 \nmid x_i$*, and* $19 \nmid x_i$ *for all* i*.*

We used an exhaustive computer search over the integers that comply with Theorem 2.1 and Corollaries 2.2, 2.3 to find all the sequences $x_1 < x_2 < \cdots < x_n = m_o(n)$ satisfying Equation (1.1) with x_i odd positive integers. We also did an exhaustive computer search over the distinct even integers that comply with similar results to the ones in this section (we omit the proof because it follows along the same line of the proofs of the results of this section) to find all the sequences $x_1 < x_2 < \cdots < x_n = m_e(n)$ satisfying Equation (1.1) with x_i even positive integers. In the computation of $m_o(n)$ and $m_e(n)$, the results of this section limited the amount of possible values of the x_i's. Tables 1 and 2 summarize our results.

The column $N_e(n)$ in Table 1 shows the number of sequences where the value $m_e(n)$ was obtained. In Table 2 the column $N_o(n)$ shows the number of sequences where the value $m_o(n)$ was obtained.

The restrictions imposed on the valid values for x_i, $1 \le i \le n$, by using Theorem 2.1 and Corollaries 2.2, 2.3 significantly reduced the solution search space. The search space reduction translated to speedups of up to $70\times$ for the computation of $m_e(n)$ where $14 \le n \le 18$, as seen in Figure 1. This allowed us to obtain results for $n \le 29$ in $m_o(n)$ and $n \le 41$ in $m_e(n)$, respectively.

Table 1. $m_e(n)$ for $3 \le n \le 29$

n	$m_e(n)$	$N_e(n)$	n	$m_e(n)$	$N_e(n)$
4	12	1	17	66	4
5	24	1	18	66	2
6	30	1	19	72	1
7	30	1	20	84	4
8	36	1	21	84	2
9	40	1	22	90	7
10	48	2	23	90	2
11	48	1	24	96	9
12	48	1	25	96	2
13	56	7	26	104	32
14	56	2	27	104	5
15	56	1	28	104	2
16	66	15	29	120	1

Conjectures.

- $m_e(n)$ is a nondecreasing function for $n \ge 3$.
- $m_o(n)$ is a nondecreasing function for $n \ge 11$.

3. Calculations of $m_o(27)$ and $m_o(29)$

In this section we show how to use Theorem 2.1 and Corollaries 2.2 and 2.3 to obtain $m_o(27)$ and $m_o(29)$. Other cases can be obtained similarly. We

TABLE 2. $m_o(n)$ for $11 \le n \le 41$

n	$m_o(n)$	$N_o(n)$	n	$m_o(n)$	$N_o(n)$
13	115	3	29	187	4
15	117	9	31	195	1
17	117	3	33	209	11
19	135	5	35	209	1
21	143	6	37	217	1
23	175	6	39	221	1
25	187	106	41	231	2
27	187	21			

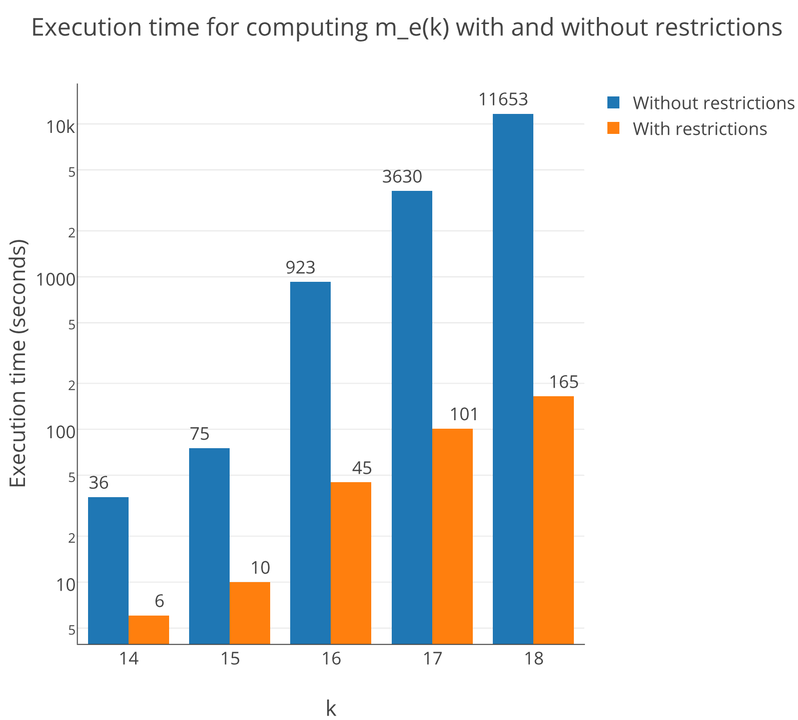

FIGURE 1. Execution time for the computation of $m_e(n)$ with and without the restrictions imposed by Theorem 2.1 and Corollaries 2.2, 2.3. The results were obtained on a workstation with one Intel(R) Xeon(R) CPU 5138 @ 2.13GHz, with 4MB of cache and 32GB of RAM.

are going to prove that

$$m_o(27) = \min \max_{1 \le i \le 27} \{x_i\} = 187$$

and

$$m_o(29) = \min \max_{1\le i\le 29} \{x_i\} = 187.$$

Using the results of Section 1, if $m_o(n) < 200$, then the x_i's are in the following set: $\{3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 27, 33, 35, 39, 45, 51, 55, 63, 65, 69, 75, 77, 85, 91, 99, 105, 115, 117, 119, 135, 143, 153, 161, 165, 175, 187, 189, 195\}$. The following is a solution of Equation (1.1) when $n = 27$:

$$\begin{aligned} &[5, 7, 9, 11, 13, 15, 23, 27, 39, 45, 51, 63, 65, 69, 75, 77, 85, 91, 99, \\ &\qquad 105, 115, 119, 135, 143, 153, 175, 187]. \end{aligned} \tag{3.1}$$

Therefore $m_o(27) \le 187$. Substituting 11 and 105 in (3.1) by 21, 35, 55, 165, we obtain a solution for $n = 29$:

$$\frac{1}{11} + \frac{1}{105} = \frac{1}{21} + \frac{1}{35} + \frac{1}{55} + \frac{1}{165}.$$

Hence, $m_o(29) \le 187$.

Suppose $m_o(n) < 187$, where $n \in \{27, 29\}$. This implies that $m_o(n) \le 175$. Using Theorem 2.1, if 17 appears in a minimal solution then one of the following has to appear:

(1) 17, $153 = 3^2 \cdot 17$, $187 = 11 \cdot 17$, or
(2) $51 = 3 \cdot 17$, $85 = 5 \cdot 17$, $119 = 9 \cdot 17$, $187 = 11 \cdot 17$.

Therefore, since $m_0(n) \le 175$, 17 cannot appear in a minimal solution. Hence there are 29 possible values: 3, 5, 7, 9, 11, 13, 15, 21, 23, 27, 33, 35, 39, 45, 55, 63, 65, 69, 75, 77, 91, 99, 105, 115, 117, 135, 143, 165, 175. This implies that $m_o(29) = 187$ since

$$\frac{1}{3} + \cdots + \frac{1}{175} \ne 1.$$

If 13 appears in a minimal solution, then one of the following will appear:

(1) 13, $39 = 3 \cdot 13$, $117 = 9 \cdot 13$
(2) 13, $39 = 3 \cdot 13$, $91 = 7 \cdot 13$, $117 = 9 \cdot 13$, $143 = 11 \cdot 13$
(3) $65 = 5 \cdot 13$, $91 = 7 \cdot 13$, $117 = 9 \cdot 13$
(4) $39 = 3 \cdot 13$, $65 = 5 \cdot 13$, $117 = 9 \cdot 13$, $143 = 11 \cdot 13$

Among the 29 possible values we count 13, $3 \cdot 13$, …, $11 \cdot 13$ (six appearances of 13), but the number of times that 13 can appear without getting a contradiction are 4 or 5. If 13 appears four times, then we have $29 - 2 = 27$ possible values. Hence, the only possible solution including 39, 65, 117, and 143 is: [3, 5, 7, 9, 11, 15, 21, 23, 27, 33, 35, 39, 45, 55, 63, 65, 69, 75, 77, 99, 105, 115, 117, 135, 143, 165, 175], but its sum is not equal to 1. If 13 appears 5 times, then we have $29 - 1 = 28$ possible values, as we do not have to consider 65. Hence, in the list of 28 values we choose 22 from 23 possible values (distinct from 13, 39, 91, 117, 143) and sum them. Those sums are not equal to 1, and therefore $m_o(27) = 187$.

The following is an example of the solutions we found for $m_o(27)$: [3, 5, 13, 21, 23, 27, 35, 39, 51, 55, 63, 65, 69, 75, 77, 85, 91, 99, 105, 115, 119, 135, 143, 153, 165, 175, 187].

4. Solutions of $\sum_{i=1}^{n} 1/x_i = 1$ with Restrictions

In [5], the authors considered egyptian fractions with restrictions. They considered solutions of Equation (1.1) over

$$S(p_1, \dots, p_r) = \{p_1^{\alpha_1} \cdots p_r^{\alpha_r} \mid \alpha_i \in \mathbb{N}_0,\ i = 1, \dots, r\},$$

where $\mathbb{N}_0$ is the set of nonnegative integers. In this section we consider the solutions of Equation (1.1) over the set $S(p_1, \dots, p_r)$, where $p_1, \dots, p_r$ are odd primes. In some sense the simplest case is $S(p_1, p_2, p_3)$, since Equation (1.1) does not have solution over $S(p_1, p_2)$ with p_1 and p_2 odd primes (see Theorem 2.3 in [5]). Let $N_n(p_1, \dots, p_r)$ be the set of solutions $(x_1, \dots, x_n)$ of Equation (1.1) with $x_i \in S(p_1, \dots, p_r)$, where any solution contains at least one x_j that is divisible by p_i for $i = 1, ..., r$. Note that in general $|N_n(p_1, \dots, p_r)|$ is not equal to the value $T_n(p_1, \dots, p_r)$ of [5]. Using Theorem 2.3 in [5], we have that $|N_n(p_1, p_2, p_3)| \geq 1$ for $(p_1, p_2, p_3) \in \{(3, 5, 7), (3, 5, 11), (3, 5, 13)\}$, n sufficiently large, and $|N_n(p_1, p_2, p_3)| = 0$ otherwise.

The algorithm to compute the solutions with restrictions employs the same backtracking strategy that was used to find the solutions for m_e and m_o. The main difference is that we begin the algorithm with L, a precomputed list of integers $x_{\alpha_1 \cdots \alpha_r}$ in increasing order, where $x_{\alpha_1 \cdots \alpha_r} = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $\alpha_i \in \mathbb{N}_0$. During backtracking, our algorithm can only choose among integers in L, which significantly speeds the computation. The largest integer $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ used from L during the computation of $|N_{15}(3, 5, 7)|$ was 19297377225.

Theorem 4.1. *Let $|N_n(p_1, \dots, p_r)|$ be the number of solutions of Equation (1.1) over $S(p_1, p_2, p_3)$. Then:*

(1) $|N_{13}(3, 5, 7)| = 2034$;
(2) $|N_{15}(3, 5, 7)| = 374349$;
(3) $|N_{17}(3, 5, 11)| = 11$ *and* $N_n(3, 5, 11) = 0$ *for* $n \leq 15$;
(4) $|N_{23}(3, 5, 13)| = 63$ *and* $N_n(3, 5, 13) = 0$ *for* $n \leq 21$.

Remark: $N_{13}(3, 5, 7)$ can be obtained using an argument similar to the one in [3]. We decided not to proceed in that way since the result can be obtained quickly via computer. For the $N_{15}(3, 5, 7)$ case it would be very difficult to apply this method since there are too many subcases.

We now illustrate the process that is going to be used in the proof of Theorem 4.3. In [1] we introduced the following identity

$$\frac{1}{abc} = \frac{1}{ab(a+b+c)} + \frac{1}{ac(a+b+c)} + \frac{1}{bc(a+b+c)} \tag{4.1}$$

to construct new solutions of Equation (1.1). Using

$$\frac{1}{105} = \frac{1}{3 \cdot 5 \cdot 7} = \frac{1}{15 \cdot 15} + \frac{1}{21 \cdot 15} + \frac{1}{35 \cdot 15}, \tag{4.2}$$

the following solution (given in [12]) for Equation (1.1) in 11-variables over $S(3,5,17)$,

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{35} + \frac{1}{63} + \frac{1}{105} + \frac{1}{135},$$

can be lifted to a new solution in 13-variables:

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{35} + \frac{1}{63} + \frac{1}{135} + \frac{1}{225} + \frac{1}{315} + \frac{1}{5 \cdot 105}.$$

This process can be repeated to obtain a solution for any odd n greater that 9. In [5], the authors proved $N_n(3,5,7) \geq c_1 \cdot 62^{k/2}$ for a computable $c_1 > 0$ and any odd number $n = 2k+1 \geq 11$. We now prove $|N_n(p_1,\ldots,p_r)| \geq 1$ for some prime numbers $p_1,\ldots,p_r$ and $n \geq n_0$ odd, where n_0 is an odd positive integer.

Theorem 4.3.

(1) *Let* $11 \leq p \leq 37$ *be prime. Then* $|N_n(3,5,7,p)| \geq 1$ *if and only if* $n \geq 11$ *is odd.*
(2) $|N_n(3,5,11)| \geq 1$ *if and only if* $n \geq 17$ *is odd.*
(3) $|N_n(3,5,13)| \geq 1$ *if and only if* $n \geq 23$ *is odd.*

Proof. For the first part of the theorem we give a complete proof for the case $N_{13}(3,5,7,11)$. For the other cases, we give a solution and the number that we need to substitute in order to get a new solution.

To prove that $N_{13}(3,5,7,11) \geq 1$, we use the solution $[3,5,7,11,15,21,27,33,35,45,2079]$ and $2029 = 3 \cdot 231$. We then have that

$$\frac{1}{231} = \frac{1}{1 \cdot 11 \cdot 21} = \frac{1}{363} + \frac{1}{693} + \frac{1}{33 \cdot 231}.$$

Hence $[3,5,7,11,15,21,27,33,35,45,441,693,33 \cdot 231]$ is a new solution of Equation (1.1). We can repeat this process for any odd n greater than 13.

- To prove that $N_{13}(3,5,7,13) \geq 1$, we use the following solution: $[3,5,7,9,15,21,35,39,45,49,637]$ and $637 = 1 \cdot 13 \cdot 49$.
- To prove that $N_{13}(3,5,7,17) \geq 1$, we use the following solution: $[3,5,7,9,15,17,21,35,153,357,595]$ and $595 = 1 \cdot 5 \cdot 119$.
- To prove that $N_{13}(3,5,7,19) \geq 1$, we use the following solution: $[3,5,7,9,15,19,21,35,95,285,315]$ and $315 = 3 \cdot 105$.
- To prove that $N_{13}(3,5,7,23) \geq 1$, we use the following solution: $[3,5,7,9,15,21,23,35,69,115,315]$ and $315 = 3 \cdot 105$.
- To prove that $N_{13}(3,5,7,29) \geq 1$, we use the following solution: $[3,5,7,9,15,21,25,29,45,725,3045]$ and $3045 = 29 \cdot 105$.
- To prove that $N_{13}(3,5,7,31) \geq 1$, we use the following solution: $[3,5,7,9,15,21,27,31,35,1953,29295]$ and $29295 = 279 \cdot 105$.
- To prove that $N_{13}(3,5,7,37) \geq 1$, we use the following solution: $[3,5,7,9,15,21,25,37,45,111,6475]$ and $6475 = 5(5 \cdot 7 \cdot 37)$.

For the second part of the theorem, we find the following solution for Equation (1.1) in 17-variables over $S(3,5,11)$:

$$\begin{aligned}1 =&\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{25}+\frac{1}{27}+\frac{1}{33}+\frac{1}{45}+\frac{1}{55}+\frac{1}{75}+\frac{1}{81}+\frac{1}{99}\\ &+\frac{1}{135}+\frac{1}{297}+\frac{1}{405}+\frac{1}{825}.\end{aligned}$$

Using

$$\frac{1}{165}=\frac{1}{3\cdot 5\cdot 11}=\frac{1}{1\cdot 11\cdot 15}=\frac{1}{11\cdot 27}+\frac{1}{15\cdot 27}+\frac{1}{165\cdot 27}$$

we then obtain the following solution for $n = 19$:

$$\begin{aligned}1 &= \frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{25}+\frac{1}{27}+\frac{1}{33}+\frac{1}{45}+\frac{1}{55}+\frac{1}{75}+\frac{1}{81}+\frac{1}{99}\\ &+\frac{1}{135}+\frac{1}{297}+\frac{1}{405}+\frac{1}{5(3\cdot 5\cdot 11)}\\ &=\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{25}+\frac{1}{27}+\frac{1}{33}+\frac{1}{45}+\frac{1}{55}+\frac{1}{75}+\frac{1}{81}+\frac{1}{99}\\ &+\frac{1}{135}+\frac{1}{297}+\frac{1}{405}+\frac{1}{1485}+\frac{1}{2025}+\frac{1}{135\cdot 165}.\end{aligned}$$

We can continue this process to obtain $N_n(3,5,11) \geq 1$ for all odd $n \geq 17$. Using direct computation, we do not find any solution when $n \leq 15$.

For the third part of the theorem, we find the following solution for Equation (1.1) in 23-variables over $S(3,5,13)$:

$$[3, 5, 9, 13, 15, 25, 27, 39, 45, 65, 75, 81, 117, 125, 135, 195, 225, 243, 325, 351,\\ 675, 1125, 15795].$$

Using

$$\frac{1}{195}=\frac{1}{3\cdot 5\cdot 13}=\frac{1}{1\cdot 5\cdot 39}=\frac{1}{5\cdot 45}+\frac{1}{39\cdot 45}+\frac{1}{195\cdot 45}$$

and $3^4 \cdot 195 = 15795$, we obtain a solution for equation (1.1) in 25-variables over $S(3,5,13)$. We can repeat the process to obtain a solution for $n \geq 25$. Using the computer we do not find any solution for $n \leq 21$. $\square$

In [5], the authors proved that the set of natural numbers n such that the equation (1.1) has at least one solution over $S(p_1,\ldots,p_r)$ is a union of finitely many arithmetic progressions. Our calculations suggest the following: Let n_0 be the smallest natural number such that $N_{n_0}(p_1,\ldots,p_r) \geq 1$. Then $N_n(p_1,\ldots,p_r) \geq 1$ for $n = 2k+1 \geq n_0$.

5. On the calculation of $\liminf \frac{x_n}{x_{n-k}}$

In this section we prove the existence of solutions of Equation (1.1) satisfying certain properties. Applying the obtained results, we prove that $\liminf x_n/x_{n-k} = 1$ for fixed k.

Lemma 5.1. *For each odd positive integer q there exists an increasing sequence $x_1, x_2, \ldots, x_n$ of odd positive integers such that $q|x_n$ and*

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}.$$

Proof. Let $y_1, y_2, \ldots, y_m$ be an increasing sequence of odd positive integers such that $1 = \sum_{i=1}^{m} 1/y_i$. By using one of the identities

$$\frac{1}{x} = \frac{1}{(3x+1)/2} + \frac{1}{3x} + \frac{1}{3x(3x+1)/2} \qquad \text{if } x \equiv 3 \bmod 4$$

or

$$\frac{1}{x} = \frac{1}{(3x+3)/2} + \frac{1}{3x} + \frac{1}{x(3x+3)/2} \qquad \text{if } x \equiv 1 \bmod 4$$

we can replace $1/y_m$ by one of these sums of three distinct odd positive integers and obtain a new increasing sequence with $m+2$ terms of odd positive integers such that

$$1 = \sum_{i=1}^{m+2} \frac{1}{y_i},$$

but now with a different y_m. Thus, we may assume that the sequence $y_1, y_2, \ldots, y_m$ also has the property that $q < y_m$. In [2, 15], it was proved that any rational number with odd denominator is a sum of a finite number of distinct terms from the sequence $1/3, 1/5, 1/7, \cdots$, so the fraction q/y_m has an expansion of the form,

$$\frac{q}{y_m} = \sum_{j=1}^{s} \frac{1}{z_j},$$

where the z_j are distinct odd positive integers. From here it follows that

$$\frac{1}{y_m} = \sum_{j=1}^{s} \frac{1}{(qz_j)}.$$

Notice that each term qz_j is greater than y_m, so after substituting this expression for $1/y_m$ into the equation $1 = \sum_{i=1}^{m} 1/y_i$, we obtain a sequence that satisfies the statement of the lemma. $\square$

Lemma 5.2. *Let k be a positive even integer and $n_1 < n_2 < \cdots < n_k$ be a sequence of odd positive integers such that $n_1 > k+2$. Let $d = n_1 \cdot n_2 \cdots n_k$ and $q = d - \sum_{i=1}^{k} d/n_i$. Then q is an odd integer, $d < qn_1$, and*

$$\frac{1}{q} = \frac{1}{d} + \frac{1}{qn_1} + \frac{1}{qn_2} + \cdots + \frac{1}{qn_k}. \tag{5.1}$$

Proof. Since k is an even integer and the n_i are odd, the integer q is odd. Moreover,

$$\frac{1}{q} - \sum_{i=1}^{k} \frac{1}{qn_i} = \frac{1}{q}\left(1 - \sum_{i=1}^{k} \frac{1}{n_i}\right) = \frac{1}{q} \cdot \frac{q}{d} = \frac{1}{d}.$$

To show that $d < qn_1$ notice that

$$\sum_{j=1}^{k} \frac{n_1 d}{n_j} < kd,$$

so

$$qn_1 = n_1 d - \sum_{j=1}^{k} \frac{n_1 d}{n_j} > (n_1 - k)d > d.$$

This proves the lemma. $\square$

Theorem 5.3. *Let k, $n_1 < n_2 < \cdots < n_k$, and q be as in Lemma 5.2. Then there exists a sequence $x_1 < x_2 < \cdots < x_n$ of odd positive integers such that*

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

and whose last k terms are

$$x_n = aqn_k,\ x_{n-1} = aqn_{k-1},\ \cdots,\ x_{n-(k-1)} = aqn_1.$$

Proof. By Lemma 5.1there exists a sequence $z_1 < z_2 < \cdots < z_m$ such that $q|z_m$ and

$$1 = \frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_m}.$$

Therefore $z_m = aq$ for some odd integer a. We now multiply both sides of (5.1) by $1/a$ and then substitute $1/z_m = 1/qa$ by this new value to give the theorem. $\square$

The following is a consequence of Theorem 5.3:

Corollary 5.4. *Let $k \geq 1$ be a natural number. Then*

$$\liminf \frac{x_n}{x_{n-k}} = 1.$$

Proof. We are going to prove the case when $k = 2$. By Theorem 5.3, there exists a solution of Equation (1.1) for each pair $(n_1, n_2) = (2n+1, 2n+3)$, where n is a natural number greater than 1. This implies the corollary. $\square$

6. On $\max x_i$ satisfying $\sum_{i=1}^{n} 1/x_i = 1$ over the even numbers

In this section we compute $\max x_i$ for sets of x_i satisfying Equation (1.1) over distinct even numbers.

Let $\mathbb{E}$ be the set of positive even numbers. In [16], Sylvester introduced the sequence $a_1 = 2, a_2 = 3, a_3 = 7, \ldots, a_{n+1} = a_1 a_2 \cdots a_n + 1$, which satisfies the equation

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{1}{a_1 a_2 \cdots a_n} = 1.$$

From [8, 6, 14], it is known that

$$\max \left\{ x_i \in \mathbb{N} \;\middle|\; \sum_{i=1}^{n+1} \frac{1}{x_i} = 1 \right\} = a_1 \cdots a_n, \tag{6.1}$$

and if $x_1 \le x_2 \le \cdots \le x_n$ are natural numbers that satisfy

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} < 1,$$

then

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \le \frac{1}{a_1} + \cdots + \frac{1}{a_n}. \tag{6.2}$$

In fact, Equation (6.2) implies Equation (6.1).

Theorem 6.1. *Let $a_1, \cdots, a_n$ be the sequence defined above. Then*

$$\max\left\{ x_{n+1} \;\middle|\; \sum_{i=1}^{n+1} \frac{1}{x_i} = 1, x_1 < \cdots < x_{n+1}, x_i \in \mathbb{E} \right\} = 2a_1 \cdots a_{n-1}.$$

Proof. Observe that

$$S_e = \frac{1}{2} + \frac{1}{2a_1} + \cdots + \frac{1}{2a_{n-1}} + \frac{1}{2a_1 \cdots a_{n-1}} = 1.$$

Let $c_1 = 2$ and $c_i = 2a_{i-1}$ for $2 \le i \le n$. Then

$$\frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n} + \frac{1}{2a_1 \cdots a_{n-1}} = 1.$$

We claim that

$$\max\left\{ x_i \in \mathbb{E} \;\middle|\; \sum_{i=1}^{n+1} \frac{1}{x_i} = 1 \right\} = 2a_1 \cdots a_{n-1}. \tag{6.3}$$

Let $x_1, x_2, \cdots, x_{n+1} \in \mathbb{E}$ be a sequence such that

$$\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}} = 1 = \frac{1}{2} + \frac{1}{2a_1} + \cdots + \frac{1}{2a_1 \cdots a_{n-1}},$$

where $x_1 < x_2 < \cdots < x_{n+1}$. Then

$$\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}} - \frac{1}{2} = \frac{1}{2a_1} + \cdots + \frac{1}{2a_1 \cdots a_{n-1}}$$

so

$$\frac{1}{(x_1/2)} + \cdots + \frac{1}{(x_{n+1}/2)} - 1 = \frac{1}{a_1} + \cdots + \frac{1}{a_1 \cdots a_{n-1}} = 1.$$

This implies that

$$\frac{1}{(x_1/2)} + \cdots + \frac{1}{(x_{n+1}/2)} = 2.$$

If $x_1 = 2$, then

$$\frac{1}{(x_2/2)} + \cdots + \frac{1}{(x_{n+1}/2)} = 1.$$

Therefore $x_{n+1}/2 \le a_1 \cdots a_{n-1}$ and hence $x_{n+1} \le 2a_1 \cdots a_{n-1}$.

Suppose $x_1 \ge 4$ and let $x_i' = x_i/2$. Suppose there exists a subset A of $B = \{x_1', \ldots, x_{n+1}'\}$ such that

$$\sum_{i \in A} \frac{1}{x_i'} = 1.$$

Then

$$\sum_{i\in A}\frac{1}{x'_i}=1$$

and

$$\sum_{i\in A^c}\frac{1}{x'_i}=1,$$

where $A^c = B - A$. Note that $x'_i \leq a_1\cdots a_m$, where $m = \max\{|A|, |A^c|\}$. Hence $x_i \leq 2a_1\cdots a_m$,. In particular, $x_{n+1} \leq 2a_1\cdots a_{n-1}$, so we can assume that $\sum_{i\in A} 1/x'_i \neq 1$ for any subset A of $B = \{x'_1, \ldots, x'_{n+1}\}$.

Suppose we take out some collection of x'_i from $B = \{x'_1, \ldots, x'_{n+1}\}$ such that the sum of the reciprocal of the elements left in B is less than 1, but when we add x'_{n+1} to the set, we obtain a sum greater than 1. Call this set A and observe that

$$\sum_{x'_i\in A}\frac{1}{x'_i}<1.$$

Let $|A| = k$ and use Equation (6.2) to obtain

$$\sum_{x'_i\in A}\frac{1}{x'_i}\leq\frac{1}{a_1}+\cdots+\frac{1}{a_k}.$$

We then have that

$$\sum_{x'_i\in A}\frac{1}{x'_i}+\frac{K}{\prod_{x'_i\in A}x'_i}=1=\frac{1}{a_1}+\cdots+\frac{1}{a_k}+\frac{1}{a_1\cdots a_k},$$

for some natural number K. Then

$$\frac{1}{x'_{n+1}}>\frac{K}{\prod_{x'_i\in A}x'_i}$$

and hence $Kx'_{n+1} < \prod_{x'_i\in A} x'_i$. Furthermore, we have that

$$\frac{K}{\prod_{x'_i\in A}x'_i}\geq\frac{1}{a_1\ldots a_{n-1}}$$

and hence $\prod_{x'_i\in A} x'_i \leq Ka_1\cdots a_{n-1}$. This implies that $x'_{n+1} < a_1a_2\cdots a_{n-1}$.

Suppose now for any subset A of B satisfying

$$\sum_{x'_i\in A}\frac{1}{x'_i}<1,$$

we have that

$$\sum_{x'_i\in A}\frac{1}{x'_i}+\frac{1}{x'_{n+1}}<1.$$

We choose A_0 with maximal sum, i.e, for any $A \subseteq B$ satisfying

$$\sum_{x'_i\in A}\frac{1}{x'_i}<1,$$

we have

$$\sum_{x'_i \in A} \frac{1}{x'_i} \leq \sum_{x'_i \in A_0} \frac{1}{x'_i}.$$

We can substitute values x'_i of A_0 by y'_i such that

$$\sum_{y'_i \in A_0} \frac{1}{y'_i} < 1$$

and

$$\sum_{y'_i \in A_0} \frac{1}{y'_i} + \frac{1}{x'_{n+1}} \geq 1.$$

Now we can apply the method of the previous case to obtain the inequality $x'_{n+1} \leq a_1 \cdots a_{n-1}$. $\square$

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