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THE ENDOMORPHISMS MONOIDS OF HELM GRAPH AND ITS GENERALIZATION

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Abstract. Let G be a graph. Then G is said to be End-regular if the set of all endomorphisms of G forms a regular monoid. In this paper, we discuss the End-regularity of Helm graphs and our generalization. We also prove that the generalized Helm graph is End-orthodox if and only if it is End-regular. Moreover, we investigate the End-regularity of the join of two generalized Helm graphs.

1. Introduction

There are many relations between graph theory and algebraic structures. For instance, the notion of End-regular graphs relates to both the semi-group theory of algebra and graph theory.

The motivation of this paper comes from an open problem, posed by Knauer and Wilkeit (see [12]), which states which graphs are End-regular. It is very hard to determine and characterize all End-regular graphs, so most researchers deal with some types of well-known graphs like End-regular bipartite, End-orthodox bipartite graphs (see [19, 1] for more details) and End-regular split graphs as considered in [10].

For nonbipartite and nonsplit graphs, there are some researches on End-regularity, for instance, End-regularity complement of a path [5], End-regularity of n-prism graphs [17], unicyclic graphs [11], generalized bicycle graphs [15], and also End-regularity of book graphs [13, 16]. Pipattanajinda, Knauer, and Arworn [14] defined a generalized wheel graph and obtained some conditions that imply a generalized wheel graph be End-regular.

Let us remind some basic definitions, which are necessary. An element x of a semi-group S is said to be *regular* if there exists an element $y \in S$ such that $xyx = x$. Also the element y is called a *pseudo-inverse* of x. A regular element x of the semi-group S is called *completely regular* if $xy = yx$ for some pseudo-inverse y of x. A semi-group S is *regular* (*completely regular*)

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if every element of S is regular (completely regular). An element e of semi-group S is said to be *idempotent* if $e^2 = e$. If the set of all idempotents of a semi-group S is a subsemi-group of S, then S is called *idempotent-closed*. A semi-group S is called *orthodox* if S is regular and idempotent-closed.

In this paper, G stands as a finite simple graph (with no loops and multiple edges). The vertex set and edge set of graph G are denoted by $V(G)$ and $E(G)$, respectively. If u and v are two vertices such that u adjacent to v, then we denote it shortly by $u \sim v$ and say that v is a *neighbor* of u. The number of neighbors of the vertex u is called the *degree* of u, denoted by $\deg(u)$. The set of all neighbors of vertex u in the graph G is denoted by $N_G(u)$. The graph G is *complete* if all of its vertices are adjacent. The complete graph of order n is shown by K_n. A subset K of vertices of the graph G is called a *clique* if the induced subgraph over K is a complete graph. A subset S of $V(G)$ is called an *independent set* if there is no edge between any two vertices in S.

Moreover, the graph G is said to be *split* if we can partition $V(G)$ into two subsets K and S such that the induced subgraph over K is a clique and the induced subgraph over S is an independent set. A *path* is a finite sequence of edges that joins a sequence of distinct vertices. A path of length r is denoted by P_r. A *cycle* C_n can be obtained from a path P_n in which the first and the last vertices coincide.

If a path P_r attaches to a vertex u, then it is called a *pendant* path of length r. It is clear that if $r = 1$, then a pendant edge (or pendant vertex) appears.

The *join* of two graphs G and H, shown by $G + H$, is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{(u, v), u \in V(G), v \in V(H)\}.$$

Note that in the above definition, we suppose that the vertex sets of G and H are disjoint. The *wheel graph* W_n is $C_n + K_1$.

Let G and H be two graphs. Then a map f from G to H is a *homomorphism* if $u \sim v$ implies that $f(u) \sim f(v)$. The set of all homomorphisms from G to H is denoted by $\mathrm{Hom}(G, H)$. Moreover f is an *isomorphism* if f is a bijective homomorphism and its set inverse f^{-1} is a homomorphism. A homomorphism from G to itself is called an *endomorphism*. The set of all endomorphisms of G is denoted by $\mathrm{End}(G)$. We know that $\mathrm{End}(G)$ is a semi-group. Also an isomorphism of G onto itself is called an *automorphism*, and the set of all automorphisms of a graph G is denoted by $\mathrm{Aut}(G)$. A graph G is *rigid* if $\mathrm{End}(G) = 1$, and G is *unretractive* if $\mathrm{Aut}(G) = \mathrm{End}(G)$. It is important to note that since our graphs are finite, injective endomorphisms are automorphisms.

A graph G is called *End-regular* (*End-orthodox, End-completely-regular, End-idempotent-closed*) if the semi-group $\mathrm{End}(G)$ is regular (orthodox, completely-regular, idempotent-closed).

Let $f \in \mathrm{Hom}(G, H)$. For $u, v \in V(G)$, let $\{f(u), f(v)\} \in E(H)$. We set $U = f^{-1}(f(u))$ and $V = f^{-1}(f(v))$. Then f is a *half-strong* homomorphism if there exists at least one edge between U and V. If for each $u \in U$, there exists at least one element $v \in V$ such that $u \sim v$ or for each $v \in V$, there exists at least one element $u \in U$ such that $u \sim v$, then f is called a *locally strong homomorphism*. We call f is a *quasi-strong* homomorphism if there exists $u' \in U$ such that for any $v \in V$, we have $u' \sim v$ and there exists $v' \in V$ such that for any $u \in U$, we have $u \sim v'$. Also f is a *strong homomorphism* if all elements of U and V are adjacent. The set of all half-strong endomorphisms of G, (locally-strong endomorphisms of G, quasi-strong endomorphisms of G, strong endomorphisms of G) is denoted by $\mathrm{HEnd}(G)$ ($\mathrm{LEnd}(G)$, $\mathrm{QEnd}(G)$, $\mathrm{SEnd}(G)$). Also the set of idempotent endomorphisms of a graph G is shown by $\mathrm{Idp}(G)$.

Let $f \in End(G)$. Then we denote I_f as the endomorphic image of G under f that is a subgraph of G with $V(I_f) = f(V(G))$ and $f(a) \sim f(b)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $c \sim d$ in G. Finally, for fixed $f \in \mathrm{End}(G)$, we define a relation ρ_f by $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$ for every $a, b \in V(G)$. One can see that ρ_f is an equivalence relation and for every $a \in V(G)$, the related equivalence class of a is denoted by $[a]_{\rho_f}$.

Helm graphs are known graphs, which are not bipartite or split. The paper is organized as follows. In section 2, we discuss when these graphs are End-regular. We also define a generalization of Helm graphs and determine when these generalized Helm graphs are End-regular, End-orthodox, and completely-regular. Moreover, in section 3, we prove that the join of two End-regular generalized Helm graphs is End-regular. In the last section, we attempt to compute the endotype and endospectrum of Helm graphs.

2. End-regularity of generalized Helm Graphs

Now, we recall the following theorem from [8], which determines sufficient and necessary conditions for an endomorphism $f \in \mathrm{End}(G)$ to be a regular.

Theorem 2.1 (Li [8]). *Let G be a graph and let $f \in \mathrm{End}(G)$. Then f is regular if and only if there exist idempotents $g, h \in \mathrm{End}(G)$ such that $I_g = I_f$ and $\rho_h = \rho_f$.*

The following two lemmas from [10, 11] study regular endomorphisms.

Lemma 2.2 (Li and Chen [10]). *Let G be a graph and let $f \in \mathrm{End}(G)$. If f is regular, then $f \in \mathrm{HEnd}(G)$.*

Lemma 2.3 (Ma, Wong, and Zhou [11]). *Let f be a regular endomorphism of a graph G with a pseudo-inverse g. Then $g(x) \in f^{-1}(x)$ for any $x \in f(V(G))$.*

The following theorem states a sufficient and necessary condition for the End-regularity of split graphs.

Theorem 2.4 (Li and Chen [10]). *Let G be a connected split graph with $V = K \cup S$ and $|K| = n$. Then G is End-regular if and only if there exists $r \in \{1, 2, \ldots, n\}$ such that $\deg(x) = r$ for any $x \in S$, or there exists a vertex $a \in S$ with $\deg(a) = n$ and there exists $r \in \{1, 2, \ldots, n-1\}$ such that $\deg(x) = r$ for any $x \in S - \{a\}$ if $S - \{a\} \neq \emptyset$.*

Now, we are going to investigate the End-regularity of Helm graphs.

We recall that the *wheel* graph W_n consisting of a cycle on n vertices such as $u_1, u_2, \ldots, u_n$ and a vertex such as c that is adjacent to each of $u_1, u_2, \ldots, u_n$.

Definition 2.5. *The Helm graph H_n is a graph obtained from the wheel graph W_n by adjoining pendant vertices $v_1, v_2, \ldots, v_n$, respectively. Examples of Helm graphs are presented in Figs. 1 and 2 for $n = 3$ and $n = 4$, respectively.*

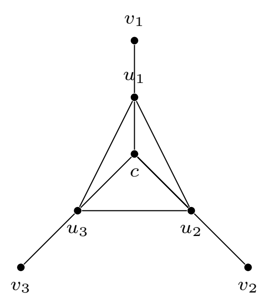

Fig. 1. H_3

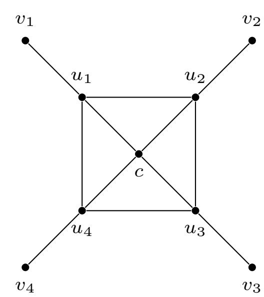

Fig. 2. H_4

We recall that the *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G such that no two adjacent vertices share the same color. Note that if there is a homomorphism from a graph G to a graph H, then $\chi(G) \leq \chi(H)$.

We use the fact that odd cycles are unretractive (see [7, Corollary 7.2.2]). This amounts to the fact there is no retraction from an odd cycle onto a proper subset. This fact is due to the result that odd cycles are 3-chromatic while proper subsets are 2-colorable. Similarly, the chromatic number of the wheel graph W_n for odd n is 4 and the chromatic number of proper subsets is at most 3. Hence the wheel graph is unretractive, too. We use that fact in the next lemma. To determine the End-regularity of H_n, we state the following two lemmas.

Lemma 2.6. *Let $f \in \operatorname{End}(H_n)$, where n is an odd number and $n \geq 5$. Then $f(c) = c$ and $f(u_i) \in V(C_n)$ for every $u_i \in C_n$.*

Proof. Suppose that $f(c) = v_i$. This case is impossible. Indeed, each triangle containing c must be sent on a triangle containing v_i, but there is no such triangle. Suppose that $f(c) = u_i$. There are n triangles containing c and two triangles containing u_i, since $n > 3$. Thus the wheel graph W_n is sent on a proper subset of itself. This is impossible, indeed, since n is odd, the chromatic number of C_n is 3. Hence the chromatic number of the wheel graph W_n is 4 and the chromatic number of proper subsets is at most 3. Hence $f(c) = c$. Next, since v_j is a pendant vertex, no u_i can be sent on v_j. Finally, since $f(c) = c$ and u_i is joined to c, u_i cannot be sent on c. Hence $f(V(C_n)) \subseteq V(C_n)$ as claimed. □

One can observe that the condition $n \geq 5$ in the above lemma is necessary. Because, if we consider the case when $n = 3$, then it is possible to define the endomorphism $f \in \mathrm{End}(H_3)$ such that $f(u_1) = u_2$, $f(u_2) = f(v_1) = f(v_3) = c$, $f(u_3) = f(v_2) = u_1$, and $f(c) = u_3$, which does not hold the property of Lemma 2.6.

As a consequence of Lemma 2.6, we can state the following lemma, which proves that the image of odd cycle C_{2k+1} in H_{2k+1} under every endomorphism is again the odd cycle C_{2k+1}.
It is not difficult to see that we can give other proofs for Lemma 2.6 directly and without the use of coloring arguments from graph theory.

Lemma 2.7. *If $k \geq 2$ and $f \in \mathrm{End}(H_{2k+1})$, then $f(C_{2k+1}) = C_{2k+1}$.*

Proof. It follows from the property that C_{2n+1} is unretractive. □

Now, we have enough tools to investigate the End-regularity of Helm graphs.

Theorem 2.8. *The Helm graph H_n is End-regular if and only if n is an odd number.*

Proof. Suppose that $n = 2k$, for $k \geq 2$. Then we define the map f by the following rule:

$$f(x) = \begin{cases} c, & x = c, v_1, \\ u_3, & x = v_2, \\ u_2, & x = v_{2i-1}, i \neq 1, x = u_{2i}, \\ u_1, & x = v_{2i}, i \neq 1, x = u_{2i-1}. \end{cases}$$

It is easy to check that $f \in \mathrm{End}(H_{2k})$, but $f \notin \mathrm{HEnd}(H_{2k})$. As in the above definition, $f(v_2)$ is adjacent to $f(v_1)$, but there exists no vertex in $f^{-1}(f(v_1))$ such that it is adjacent to a vertex in $f^{-1}(f(v_2))$. Thus H_{2k} is not End-regular, by Lemma 2.2.

Assume that $n = 2k+1$ for $k \geq 1$. If $k = 1$, then it is easy to see that H_3 is a split graph made of the clique $K = \{u_1, u_2, u_3, c\}$ and the independent set $S = \{v_1, v_2, v_3\}$. Since $\deg(x) = 1$ for all $x \in S$, then H_3 is End-regular by Theorem 2.4. Now, suppose that $k \geq 2$; then $n = 2k + 1 \geq 5$. Let $f \in \mathrm{End}(H_{2k+1})$; then we define endomorphisms g and h as follows:

$$g(x)=\begin{cases} x, & x=u_i, x=c,\\ x, & x=v_i, v_i\in \operatorname{Im} f,\\ c, & x=v_i, v_i\notin \operatorname{Im} f,\end{cases} \qquad h(x)=\begin{cases} x, & x=u_i, x=c,\\ u_j, & x=v_i, f(v_i)=f(u_j),\\ c, & x=v_i, f(v_i)=c,\\ v_i, & x=v_i, f(v_i)\neq f(u_j), c.\end{cases}$$

It is clear that g and h are idempotent endomorphisms in H_{2k+1} and $I_f=I_g$ is a wheel graph W_n maybe with some pendant vertices. We show that $\rho_f=\rho_h$. Let $f(x)=f(y)$, where $x,y\in V(H_{2k+1})$. Since $f(C_{2k+1})=C_{2k+1}$ by Lemma 2.7, there are the following two cases:

Case 1: $x=c$ and $y=v_i$ for some $i\in\{1,2,\ldots,2k+1\}$.

In this case, we have $h(x)=c=h(v_i)=h(y)$.

Case 2: $x=u_j$ and $y=v_i$ for some $i,j\in\{1,2,\ldots,2k+1\}$, $i\neq j$.

We have $h(x)=h(u_j)=u_j=h(v_i)=h(y)$.

Thus $h(x)=h(y)$ in both cases and it implies that $\rho_f\subseteq\rho_h$. According to the definition, $h(x)=h(y)$ entails that $x=v_i$ and $y=v_j$ for some $i\neq j$. If $h(x)=h(y)=c$, then $f(x)=f(y)=c$ or if $h(x)=h(y)=u_k$, then $f(x)=f(u_k)=f(y)$; thus $\rho_f=\rho_h$ and $\operatorname{End}(H_{2k+1})$ is regular for all $k\geq 2$ by Theorem 2.1. □

In 2016, Indriati et al. [6] defined a type of generalized Helm graph. Now, we are going to give another generalization of Helm graphs, which is also a generalization of [6]. Before we define that, let us recall that the *Cartesian product* $G\Box H$ of the graphs G and H is a graph such that the vertex set of $G\Box H$ is the Cartesian product $V(G)\times V(H)$; and two vertices (x,y) and (x',y') are adjacent in $G\Box H$ if and only if either $x=x'$ and y is adjacent to y' in H, or $y=y'$ and x is adjacent to x' in G. Also, a *G-layer* G_x ($x\in V(H)$) of the Cartesian product $G\Box H$ is the subgraph induced by the set of vertices $\{(u,x): u\in V(G)\}$. An H-layer is defined analogously (see [2, p. 40]).

Definition 2.9. *Let $n\geq 3$, let $m\geq 1$, and let $r\geq 1$. Consider the Cartesian product $C_n\Box P_{(m+1)r}$ that joins a vertex c to every vertex of C_n-layer C_{n_1} and removes edges between all of vertices of C_n-layer C_{n_x} for $x\neq kr$ and $1<k<m$. This graph is called a generalized Helm graph and denoted by $H_n(m,r)$. In other words, $H_n(m,r)$ is m cycles of length n such that each vertex of each cycle joins to the corresponding vertex of the later cycle with a path of length r, also a pendant path P_r joins to each vertex of the last cycle, and a vertex such as c joins to vertices of the first cycle with n P_rs.*

As an example of a generalized Helm graph, $H_5(3,2)$ is shown in Fig. 3. We observe that the generalized Helm graph $H_n(1,1)$ coincides with the ordinary Helm graph H_n. With the above definition, we determine the vertices of the generalized Helm graph precisely by the following method.

Let $n\geq 3$, let $m\geq 1$, let $r\geq 1$, and let $1\leq j\leq m$. We denote C_n^j as a cycle of length n for every $1<j<m$, and G_j as a graph isomorphic to a cycle C_n^j whose each vertex has a pendant path of length r, and G_0 denotes

a graph with a vertex c, in which it has n pendant paths of length r. We label vertices of G_0 and G_j as follows:

$$V(G_0) = \{c, v_1^1, v_1^2, \ldots, v_1^r, v_2^1, v_2^2, \ldots, v_2^r, \ldots, v_n^1, v_n^2, \ldots, v_n^r\}$$

such that for each i, v_i^1 is adjacent to c and $v_i^1 \sim v_i^2 \sim v_i^3 \sim \cdots \sim v_i^r$.

$$V(C_n^j) = \{v_1^{jr}, v_2^{jr}, v_3^{jr}, \ldots, v_n^{jr}\},$$

such that $v_i^{jr} \sim v_{i+1}^{jr}$, and indices are in module n.

$$V(G_j) = V(C_n^j) \cup \{v_i^{(j-1)r+1}, v_i^{(j-1)r+2}, \ldots, v_i^{(j-1)r+r}\}$$

such that $v_i^{(j-1)r+l} \sim v_i^{(j-1)r+l+1}$ for every $0 \le l \le r-1$.

So, $V(H_n(m,r)) = \bigcup_{i=0}^m V(G_i)$ and $E(H_n(m,r)) = \bigcup_{i=1}^m E(G_i)$.

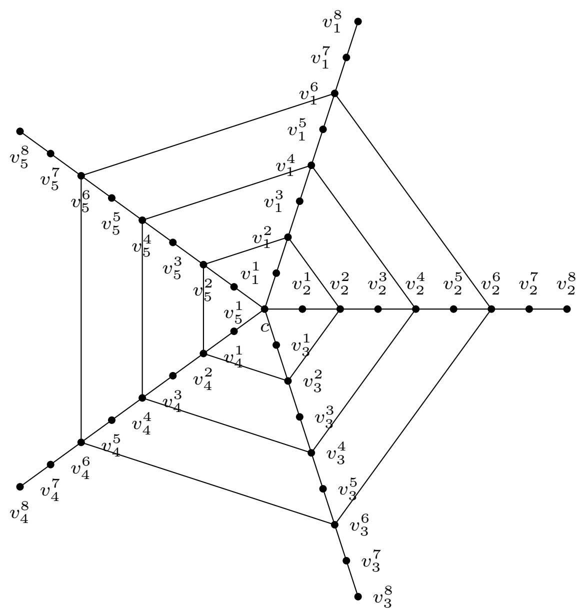

Fig. 3. $H_5(3,2)$

Theorem 2.10. *If $m \ge 2$ or $r \ge 2$, then $H_n(m,r)$ is not End-regular.*

Proof. Suppose that $r \ge 2$. It is not difficult to see that the following endomorphism is in $\mathrm{End}(H_n)$:

$$f(x) = \begin{cases} v_{i+1}^j, & x = v_i^j, 1 \le j \le mr \text{ and } 1 \le i \le n-1, \\ v_1^j, & x = v_n^j, \text{ and } 1 \le j \le mr, \\ v_1^{mr}, & x = v_1^{mr+1}, \\ v_1^{mr+1}, & x = v_1^{mr+2}, \\ v_i^{mr}, & x = v_i^{mr+j}, \ j \text{ is odd } 1 \le j \le r \text{ and } x \ne v_1^{mr+1}, \\ v_i^{mr-1}, & x = v_i^{mr+j}, \ j \text{ is even, } 2 \le j \le r \text{ and } x \ne v_1^{mr+2}. \end{cases}$$

Let f be regular. Then there exists a pseudo-inverse g such that $fgf = f$. So $g(v_1^{mr+1}) = v_1^{mr+2}$ by Lemma 2.3. Also, $g(v_1^{mr}) \subseteq \{v_1^{mr+j}, v_n^{mr}\}$ such that j is odd and $1 \leq j \leq r$ by Lemma 2.3. On the other hand, g is an endomorphism and $v_1^{mr+1} \sim v_1^{mr}$ results in $g(v_1^{mr+1}) \sim g(v_1^{mr})$. Consequently just $g(v_1^{mr}) = v_1^{mr+1}$ or $g(v_1^{mr}) = v_1^{mr+3}$ can be accepted. Similarly, $g(v_n^{mr}) \subseteq \{v_n^{mr+j}, v_{n-1}^{mr}\}$ for odd j with $1 \leq j \leq r$. We have $v_n^{mr} \sim v_1^{mr}$ and so $g(v_n^{mr}) \sim g(v_1^{mr}) = v_1^{mr+1}$ or $g(v_n^{mr}) \sim g(v_1^{mr}) = v_1^{mr+3}$. No element is found for $g(v_n^{mr})$ and this is a contradiction. Thus, f is not regular. Now, let $r = 1$ and let $m \geq 2$. Define $f(v_i^{m+1}) = v_{i+1}^{m}$, $f(v_i^{m}) = v_{i+1}^{m-1}$, and $f(x) = x$ for $x \neq v_i^{m+1}, v_i^{m}$; then we can see that $f \in \mathrm{End}(H_n(m,1))$. Moreover $f \notin \mathrm{HEnd}(H_n(m,1))$. Because $f(v_i^{m+1}) \sim f(v_{i+1}^{m+1})$, but

$$v_i^{m+1} = f^{-1}(f(v_i^{m+1})) \nsim f^{-1}(f(v_{i+1}^{m+1})) = v_{i+1}^{m+1},$$

so, $H_n(m,1)$ is not End-regular by Lemma 2.2. □

Corollary 2.11. *The generalized Helm graph $H_n(m,r)$ is End-regular if and only if $m = r = 1$ and n is odd.*

Proof. It follows from Theorems 2.8 and 2.10 directly. □

As we mentioned earlier in the first section, if the graph G is End-orthodox or End-completely-regular, then G is End-regular. So, to check that when $H_n(m,r)$ is End-orthodox or End-completely-regular, we just need to consider H_{2k+1}.

Theorem 2.12. *The graph H_{2k+1} is End-idempotent-closed for any positive integer k.*

Proof. Let $k \geq 2$ and let $f \in \mathrm{Idp}(H_{2k+1})$. Also let $f(u_i) = u_j$ for some $i \neq j$. We know $f^2(u_i) = f(u_i)$, so $f(u_j) = u_j$, which is a contradiction with the definition of homomorphisms. Thus $f(u_i) = u_i$. In addition, if $f \in \mathrm{Idp}(H_3)$, then $f(c) \neq u_i$. Otherwise, since f is an idempotent $f(u_i) = u_i$, which is again a contradiction with the definition of homomorphism. Hence if $f \in \mathrm{Idp}(H_{2k+1})$, for any positive integer k, we have $f(u_i) = u_i$ for all $i = 1, 2, \ldots, 2k+1$ and $f(c) = c$. So here each vertex images to itself. Thus we have $f(v_i) \in \{c, v_i, u_{i+1}, u_{i-1}\}$. For each of cases of $f(v_i)$, it is not difficult to check that $(fog)^2(v_i) = fog(v_i)$ for $f, g \in \mathrm{Idp}(H_{2k+1})$, for instance, we check one case and left another one to the reader. Let $f(v_i) = u_{i+1}$, let $g(v_i) = c$, let $fg^2(v_i) = fgfg(v_i) = fgf(c) = fg(c) = f(c) = c$, and let $fg(v_i) = f(c) = c$. Then $fg^2 = fg$. Now $fg \in \mathrm{Idp}(H_{2k+1})$ for every $f, g \in \mathrm{Idp}(H_{2k+1})$. □

Corollary 2.13. *The graph $H_n(m,r)$ is End-orthodox if and only if it is End-regular.*

Proof. It is a direct consequence of Corollary 2.11 and Theorem 2.12. □

We recall that a function f is called *square injective* if $f^2(a) = f^2(b)$ implies that $f(a) = f(b)$. Now, we give the following result, which states a necessary and sufficient condition for the endomorphism f to be completely-regular (see [18, p. 14 Theorem 1.4.7] for more details).

Theorem 2.14 (Wanichsombat [18])**.** *Let G be a finite graph and let $f \in \mathrm{End}(G)$. Then f is completely-regular if and only if for all $a, b \in V(G)$, $f(a) \neq f(b)$ implies $f^2(a) \neq f^2(b)$, that is, f is a square injective.*

Theorem 2.15. *The graph $H_n(m,r)$ is not End-completely-regular for every $n \geq 3$, $m \geq 1$, and $r \geq 1$.*

Proof. If $m \geq 2$ or $r \geq 2$, then $H_n(m,r)$ is not End-regular and so is not End-completely-regular as well. Moreover, $H_n(1,1) = H_n$ is not End-regular for all even number n. Thus, we only need to consider the case H_{2k+1}, where $k \geq 1$. For H_{2k+1}, we define $f \in \mathrm{End}(H_n(m,r))$ by the rules $f(u_i) = u_{i+1}$, $f(v_1) = v_2$, and $f(c) = f(v_i) = c$ for $i \neq 1$. We see that $f(c) \neq f(v_1)$, but $f^2(c) = f^2(v_1)$, and so H_{2k+1} is not End-completely-regular by Theorem 2.14. Hence $H_n(m,r)$ is not End-completely-regular as required. □

One of the interesting topics in this area is to characterize all graphs whose set of idempotent endomorphisms is a monoid. We recall that this property is called *End-idempotent-closed.* Here, we state some theorems on this property for the generalized Helm graph $H_n(m,r)$.

Theorem 2.16. *The Helm graph H_{2k} is not End-idempotent-closed.*

Proof. Suppose that f and g have the following rules:

$$f(x) = \begin{cases} u_1, & x = u_{2i+1}, \\ u_2, & x = u_{2i}, i \neq k, \\ u_{2k}, & x = u_{2k}, x = v_3, \\ c, & x = c, x = v_i, i \neq 3, \end{cases} \qquad g(x) = \begin{cases} u_3, & x = u_{2i+1}, \\ u_2, & x = u_{2i}, \\ v_3, & x = v_3, \\ c, & x = c, x = v_i, i \neq 3. \end{cases}$$

We show that f and g are idempotent endomorphisms. Indeed $u_{2i+1} \sim u_{2i}$ results in $f(u_{2i+1}) = u_1 \sim f(u_{2i}) \in \{u_2, u_{2k}\}$. Also $v_i \sim u_i$ for $i \neq 3$ results in $f(v_i) = c \sim f(u_i) \in \{u_1, u_2, u_{2k}\}$ and $f(v_3) = u_{2k} \sim f(u_3) = u_1$. Eventually, $c \sim u_i$ and $f(c) = c \sim f(u_i) \in \{u_1, u_2, u_{2k}\}$. So, f is an endomorphism. Also, $f^2(u_i) = f(u_i)$, $f^2(c) = f(c)$, and $f^2(v_i) = f(v_i)$; then $f \in \mathrm{Idp}(H_{2k})$. Similarly it can be shown that $g \in \mathrm{Idp}(H_{2k})$. Now,

$$(fg)^2(v_3) = fgfg(v_3) = fgf(v_3) = fg(u_{2k}) = f(u_2) = u_2$$

and $fg(v_3) = f(v_3) = u_{2k}$. Since $k > 1$, we have $fg \notin \mathrm{Idp}(H_{2k})$. Hence H_{2k} is not End-idempotent-closed. □

Theorem 2.17. *If $r \geq 2$, then $H_n(m,r)$ is not End-idempotent-closed.*

Proof. It is not difficult to see that f and g defined below are two idempotent endomorphisms from $H_n(m,r)$.

$$f(x) = \begin{cases} v_1^{mr+1}, & x = v_1^{mr+j}, \ j \text{ odd}, \\ v_1^{mr+2}, & x = v_1^{mr+j}, \ j \text{ even}, \\ v_2^{mr-1}, & x = v_2^{mr+j}, \ j \text{ odd}, \\ v_2^{mr}, & x = v_2^{mr+j}, \ j \text{ even}, \\ x, & \text{otherwise}, \end{cases}$$

$$g(x) = \begin{cases} v_2^{mr}, & x = v_1^{mr+j}, \ j \text{ odd}, \\ v_2^{mr+1}, & x = v_1^{mr+j}, \ j \text{ even}, \\ v_2^{mr+1}, & x = v_2^{mr+j}, \ j \text{ odd}, \\ v_2^{mr}, & x = v_2^{mr+j}, \ j \text{ even}, \\ x, & \text{otherwise}. \end{cases}$$

Note that in the above endomorphisms, we assumed that $j \geq 1$. Since

$$\begin{aligned} (gf)^2(v_1^{mr+2}) &= gfg(f(v_1^{mr+2})) \\ &= gf(g(v_1^{mr+2})) \\ &= g(f(v_2^{mr+1})) \\ &= g(v_2^{mr-1}) \\ &= v_2^{mr-1} \end{aligned}$$

and $gf(v_1^{mr+2}) = g(v_1^{mr+2}) = v_2^{mr+1}$, we see that $(gf)^2(v_1^{mr+2}) \neq gf(v_1^{mr+2})$. So $H_n(m, r)$ is not End-idempotent-closed for all $r \geq 2$. □

In spite of the fact that we believe $H_n(m, 1)$ is End-idempotent-closed for all $m \geq 2$, we are not yet able to prove it. So, let us state the following conjecture.

Conjecture. *$H_n(m, r)$ is End-idempotent-closed if and only if $r = 1$ and $m \geq 2$ or $r = m = 1$ and n is odd.*

3. End-regularity of the join of two generalized Helm graphs

In this section, we prove that the join of two generalized Helm graphs is End-regular if and only if that is as the form $H_m + H_n$ such that m and n are odd. We start with the following three lemmas.

Lemma 3.1 (Li [9]). *Let G and H be two graphs. If $G + H$ is End-regular, then both G and H are End-regular.*

Lemma 3.2 (Hou and Luo [4]). *Let G and H be two End-regular graphs. If for every $f \in \text{End}(G + H)$, we have $f(G) \subseteq G$ and $f(H) \subseteq H$, then $G + H$ is End-regular.*

Recall that the join of n graphs G_1, G_2, …, G_n, denoted by $G_1 + G_2 + \cdots + G_n$, is a graph with

$$V(G_1 + G_2 + \cdots + G_n) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$$

and

$$E(G_1 + G_2 + \cdots + G_n) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n) \cup \{\{x, y\} | x \in V(G_i), y \in V(G_j)\}$$

(where $i \neq j$). Now, we have the following lemma for End-regularity join n-split graphs.

Lemma 3.3 (Hou, Feng, and Gu [3])**.** *Let G_i, $i = \{1, 2, \ldots, n\}$, be split graphs and let each G_i have a complete subgraph K_i and an independent set S_i. Then $G_1 + G_2 + \cdots + G_n$ is End-regular if and only if*

(i) *G_i is End-regular for every $1 \leq i \leq n$,*
(ii) *$q_i - d_i = q_j - d_j$ for every i and j, where $q_i = |V(K_i)|$, $d_i = \deg_{G_i}(x_i)$, and $x_i \in S_i$,*
(iii) *$V(K_i) \neq N_{G_i}(x) \cup N_{G_i}(x')$ for every i and every $x, x' \in S_i$.*

Now, we investigate the End-regularity of the join of two generalized Helm graphs. The following lemma will be used in the proof of the main theorem of this section.

Lemma 3.4. *Let $m, n \geq 5$ be odd and let $f \in \mathrm{End}(H_n + H_m)$, where H_n and H_m be Helm graphs with vertices $\{c, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and $\{c', u'_1, u'_2, \ldots, u'_m, v'_1, v'_2, \ldots, v'_m\}$ (as in the definition of Helm graphs), respectively. Then*

(i) *$f(C_n) = C_n$ and $f(C'_m) = C'_m$ or $f(C_n) = C'_m$ and $f(C'_m) = C_n$,*
(ii) *$f(c) = c$ and $f(c') = c'$ or $f(c) = c'$ and $f(c') = c$.*

Proof. (*i*) If $f(H_n) \subseteq H_n$, then $f(C_n) = C_n$ by Lemma 2.7, and since $C_m \sim C_n$ and $f(C_n) \sim f(C_m)$, we have $f(C_m) = C_m$. Similarly, if $f(H_n) \subseteq H_m$, then $f(C_n) = C'_m$ and $f(C'_m) = C_n$. Assume that $f(H_n) \nsubseteq H_n$, that $f(H_n) \nsubseteq H_m$, that $f(C_n) \neq C_n$, and that $f(C_n) \neq C_m$. We know that $f(C_n)$ is a cycle and that $|f(C_n)| \geq 5$. So $f(C_n)$ has a path of length 5, and since $c \sim C_n$, there is one vertex that is adjacent to all of vertices of this path. All vertices of $f(C_m)$ are adjacent to all vertices $f(C_n)$; so we should find another path with the above property. If $v_i, v'_i \in f(C_n)$, then there is no path with the said features in $f(C_m)$. So $v_i, v'_i \notin f(C_n)$. Let $f(C_n)$ contain at least two vertices in each of H_n and H_m. Then again there are no two paths with the said features such that they are adjacent together. Suppose that $|f(C_n)| \geq 4$ in H_n and that $|f(C_n)| = 1$ in H_m. Without loss of generality, we can assume the following assumptions. For $i \geq 1$,

$$\{u_i, u_{i+1}, u_{i+2}, u_{i+3}, c'\} \subseteq f(C_n)$$

and

$$\{u'_i, u'_{i+1}, u'_{i+2}, u'_{i+3}, c\} \subseteq f(C_m)$$

such that c and c' are adjacent to $f(C_n)$ and $f(C_m)$, respectively. Since $f(v_{i+4}) \sim f(u_{i+4}) = c'$, $f(v_{i+4}) \sim f(u'_{i+4}) = c$, and $f(v_{i+4}) \sim f(u_{i+3}) = u_{i+3}$, there is no vertex for $f(v_{i+4})$ and this is a contradiction. Hence $f(C_n) = C_n$ or $f(C_n) = C'_m$. By the same method as above, we can show that

$f(C'_m) = C'_m$ or $f(C'_m) = C_n$. If $f(C_n) = C_n$, since $f(C_m) \sim f(C_n)$, then $f(C_m) = C_m$. Similarly, if $f(C_n) = C_m$, then $f(C_m) = C_n$, as required.

(ii) We know that c and c' are the only vertices that are adjacent to all of vertices C_n and C'_m. On the other hand, $f(c)$ is adjacent to C_n and C'_m. So by part (i), we have $f(c) = c$ or $f(c) = c'$. Similarly $f(c') = c'$ or $f(c') = c$. □

Theorem 3.5. *The join of two generalized Helm graphs is End-regular if and only if both of them are End-regular.*

Proof. If the join of two generalized Helm graphs is End-regular, then each of them must be End-regular by Lemma 3.1. Now considering the notation of Lemma 3.4, suppose that $H_n(m_1, r_1)$ and $H_m(m'_1, r'_1)$ are two End-regular generalized Helm graphs. Thus $H_n(m_1, r_1) = H_n$ and $H_m(m'_1, r'_1) = H_m$ such that m and n are odd numbers by Corollary 2.11. Let

$$f \in \mathrm{End}(H_n + H_m).$$

Then we consider the following cases:

Case 1: $m \neq n$ and $m, n \neq 3$.

Without loss of generality, let $n < m$. So C_n cannot image to C_m and $f(C_n) \neq C'_m$. Then $f(C_n) = C_n$ and $f(C'_m) = C'_m$ by Lemma 3.4. Now, if $f(c) = c$, then $f(c') = c'$, $f(H_n) \subseteq H_n$, and $f(H_m) \subseteq H_m$. So, $H_n + H_m$ is End-regular by Lemma 3.2. If $f(c) = c'$, then $f(c') = c$, and, for each i, we have $f(v_i) = c'$ and $f(v'_i) = c$. We define $g \in \mathrm{Idp}(H_n + H_m)$ by the rules $g(v_i) = c$, $g(v'_i) = c'$, and if $x \neq v_i, v'_i$, then $g(x) = x$. Then it is easy to check that $I_f = I_g$ and $\rho_f = \rho_g$, which implies that f is regular by Lemma 2.1.

Case 2: $m \neq n$ and one of them is 3.

Let $n = 3$. Clearly $n < m$ and $f(C_n) \neq C'_m$. If $f(C_n) = C_n$, then similar to case 1, the result follows. Now, let $f(C_n) \neq C_n$. We define $g, h \in \mathrm{Idpt}(H_n + H_m)$ as follows:

$$g(x) = \begin{cases} v_i, & x = v_i,\ v_i \in \mathrm{Im}\, f, \\ c, & x = v_i,\ v_i \notin \mathrm{Im}\, f, \\ v'_i, & x = v'_i,\ v'_i \in \mathrm{Im}\, f, \\ c', & x = v'_i,\ v'_i \notin \mathrm{Im}\, f, \\ x, & x \neq v_i, v'_i, \end{cases} \qquad h(x) = \begin{cases} u_j, & x = v_i,\ f(v_i) = f(u_j), \\ c, & x = v_i,\ f(v_i) = c, \\ u'_j, & x = v'_i,\ f(v'_i) = f(u'_j), \\ c', & x = v'_i,\ f(v'_i) = c', \\ x, & \text{otherwise.} \end{cases}$$

We can see that $I_f = I_g$ and $\rho_f = \rho_h$, and consequently f is regular by Lemma 2.1.

Case 3: $m = n = 3$.

We know that H_3 is a split graph. If we check three conditions of Lemma 3.3, then get that $H_3 + H_3$ is End-regular.

Case 4: $m = n \neq 3$, $f(C_n) = C_n$, and $f(c) = c$.

In this case, $f(C'_m) = C'_m$ and $f(c') = c'$, so $f(H_n) \subseteq H_n$ and $f(H_m) \subseteq H_m$. Then $H_n + H_m$ is End-regular by Lemma 3.2.

Case 5: $m = n \neq 3$, $f(C_n) = C_n$, and $f(c) = c'$.

We know that $f(C'_m) = C'_m$, $f(c') = c$, $f(v_i) = c'$, and $f(v'_i) = c$ for every i. Define $g \in \mathrm{Idp}(H_n + H_m)$ by the rules $g(v_i) = c$, $g(v'_i) = c'$, and $g(x) = x$ for $x \neq v_i, v'_i$. Therefore, $I_f = I_g$ and $\rho_f = \rho_g$. So by Lemma 2.1, f is regular.

Case 6: $m = n \neq 3$, $f(C_n) = C'_m$, and $f(c) = c$.

It is very similar to the proof of case 5.

Case 7: $m = n \neq 3$, $f(C_n) = C'_m$, and $f(c) = c'$.

In this case, $f(C'_m) = C_n$ and $f(c') = c$. We define $g \in \mathrm{Idp}(H_n + H_m)$ by the rules $g(x) = c$ if $x = v_i$ and $v_i \notin \mathrm{Im}(f)$, $g(x) = c'$ if $x = v'_i$ and $v'_i \notin \mathrm{Im}(f)$, and $g(x) = x$ if $x \neq v_i, v'_i$. We see that $I_f = I_g$. Now see the following endomorphism:

$$h(x) = \begin{cases} x, & x \in C_n,\ x \in C_m,\ x = c \text{ and } x = c', \\ u_j, & x = v_i \text{ and } f(v_i) = f(u_j), \\ u'_j, & x = v'_i \text{ and } f(v'_i) = f(u'_j), \\ c, & x = v_i \text{ and } f(v_i) = c', \\ c', & x = v'_i \text{ and } f(v'_i) = c, \\ v_i, & x = v_i,\ f(v_i) \neq f(u_j), \text{ and } f(v_i) \neq c', \\ v'_i, & x = v'_i,\ f(v'_i) \neq f(u'_j), \text{ and } f(v_i) \neq c. \end{cases}$$

It is not difficult to check that $h \in \mathrm{Idp}(H_n + H_m)$ and $\rho_f = \rho_h$, so f is regular by Lemma 2.1.

□

The following theorem gives a necessary and sufficient condition on the orthodoxy join of two graphs.

Theorem 3.6 (Hou and Luo [4]). *Let G_1 and G_2 be two graphs. Then $G_1 + G_2$ is End-orthodox if and only if $G_1 + G_2$ is End-regular and both of G_1 and G_2 are End-orthodox.*

Corollary 3.7. *The join of two generalized Helm graphs is End-orthodox if and only if that is as the form $H_{2k+1} + H_{2k'+1}$.*

Proof. This is a consequent of Corollary 2.13 and Theorems 3.5 and 3.6. □

4. Endospectrum of Helm graphs

First, we observe that the following sequence occurs for every graph G:

$$\mathrm{End}\, G \supseteq \mathrm{HEnd}\, G \supseteq \mathrm{LEnd}\, G \supseteq \mathrm{QEnd}\, G \supseteq \mathrm{SEnd}\, G \supseteq \mathrm{Aut}\, G.$$

Recall that the *endospectrum* of G (endomorphism spectrum of G) is the following 6-tuples:

$$\mathrm{Endospec}(G) = (|\,\mathrm{End}\, G|, |\,\mathrm{HEnd}\, G|, |\,\mathrm{LEnd}\, G|, |\,\mathrm{QEnd}\, G|, |\,\mathrm{SEnd}\, G|, |\,\mathrm{Aut}\, G|).$$

We may also associate the 5-tuple $(s_1, s_2, s_3, s_3, s_4, s_5)$ to the Endospec(G) with the rule that $s_i \in \{0, 1\}$ for $1 \le i \le 5$, where

$$s_i = \begin{cases} 0, & i\text{-entry} = i+1\text{-entry in Endospec}(G), \\ 1, & i\text{-entry} \neq i+1\text{-entry in Endospec}(G). \end{cases}$$

For instance, $s_1 = 1$ means that $|\operatorname{End}(G)| \neq |\operatorname{HEnd}(G)|$, $s_2 = 0$ means that $|\operatorname{HEnd}(G)| = |\operatorname{LEnd}(G)|$, and so on. We call the integer $\sum_{i=1}^{5} s_i 2^{i-1}$ the *endotype* or *endomorphism type* of G and denote it by Endotype(G). The endotype is a number between 0 and 31. There are no graphs of endotypes 1 and 17. A rigid graph is a graph with $|\operatorname{End}(G)| = 1$. The endotype of this graph is 0 (see [7] for more details). One of the interesting problems is to determine the endotype and endospectrum of a graph. Although finding these values for any graph is not only easy but sometimes impossible, they are computed for some graphs. For instance, for trees, it was obtained in [7].

Now, we may ask the same problem for the generalized Helm graphs.

Problem 4.1. *What are the endotype and endospectrum of the generalized Helm graphs?*

We attempt to determine the endotype and endospectrum of Helm graph H_n, for both cases when n is even or odd. We have been successful to find it when n is odd, but for even n, we determined the endotype and we were able to compute only three entries of Endospec(H_{2k}) and the remaining three entries are left unknown.

Theorem 4.2. *If k is a positive integer, then*

$$|\operatorname{LEnd}(H_{2k+1})| = |\operatorname{Aut}(H_{2k+1})| = 2(2k+1).$$

Proof. First, we suppose $k \geq 2$ and show that every local strong endomorphism H_{2k+1} is an automorphism in H_{2k+1}. Let $f \in \operatorname{LEnd}(H_{2k+1})$; then by Lemma 2.7, it is enough to prove that $f(v_i) \neq c$ and $f(v_i) \neq u_j$ for all $i, j \in \{1, 2, \ldots, 2k+1\}$. We recall that $f^{-1}(u_i)$ is a singleton set for each $i \in \{1, 2, \ldots, 2k+1\}$, by Lemma 2.7. Without loss of generality, we show that $f(v_1) \neq c$ and that $f(v_1) \neq u_j$ for each $j \in \{1, 2, \ldots, 2k+1\}$. On the contrary, let $f(v_1) = c$ and put $f^{-1}(u_m) = \{u_2\}$. Then $f(v_1) \sim f(u_2)$, but v_1 is not adjacent to any vertex in $f^{-1}(u_m)$, a contradiction. Thus $f(v_1) \neq c$. Let $f(v_1) = u_j$ for some $j \in \{1, 2, \ldots, 2k+1\}$. We know that $u_{j+1} \sim f(v_1) = u_j \sim u_{j-1}$, but $f^{-1}(u_{j+1}) \neq \{u_1\}$ or $f^{-1}(u_{j-1}) \neq \{u_1\}$. This contradicts with that f is a local strong endomorphism of H_{2k+1}. Hence $f(v_1) \neq u_j$ for each $j \in \{1, 2, \ldots, 2k+1\}$.

Now, we compute the order of Aut(H_{2k+1}). Let $f \in \operatorname{Aut}(H_{2k+1})$. Then for $f(u_1)$ and $f(u_2)$, we have $2k+1$ and 2 choices, respectively. By Lemma 2.7 for $f(u_i)$, $i \in \{3, 4, \ldots, 2k+1\}$, we have just one choice. Since $f(c) = c$ and f is an automorphism for all $j = 1, 2, \ldots, 2k+1$, we have $v_j \in \operatorname{Im} f$.

Hence for all v_j, there is only one choice and consequently

$$|\operatorname{Aut}(H_{2k+1})| = 2(2k+1).$$

For $k = 1$ and $f \in \operatorname{LEnd}(H_3)$, it is enough to show that $f(C_{2k+1}) = C_{2k+1}$. Then by the same proof as in above, $\operatorname{LEnd}(H_3) = \operatorname{Aut}(H_3)$ and the result follows. We claim that $f(c) = c$. On the contrary, let $f(c) = u_j$ for some $j \in \{1, 2, 3\}$. Thus $f(u_k) = c$ for some $k \in \{1, 2, 3\}$. Hence, $f(v_k) = u_m$ for some $m \in \{1, 2, 3\}$, also $f^{-1}(u_m) = \{v_k, u_l\}$ and $f^{-1}(u_j) = \{c\}$. This is a contradiction since f is a local strong endomorphism. □

The following lemma from [7] will help us to prove the next theorem.

Lemma 4.3. *Let G be a graph such that $N(x) \subsetneq N(x')$ for some $x, x' \in G$, with $\{x, x'\} \notin E(G)$. Then* $\operatorname{HEnd}(G) \neq \operatorname{LEnd}(G)$.

Theorem 4.4. *For all positive integers n, we have* $\operatorname{HEnd}(H_n) \neq \operatorname{LEnd}(H_n)$.

Proof. Put $x = v_1$ and $x' = c$ in Lemma 4.3 and the proof follows. □

Theorem 4.5. *For every k, we have*

$$|\operatorname{HEnd}(H_{2k+1})| = |\operatorname{End}(H_{2k+1})| = 2^{4k+3}(2k+1).$$

Proof. It is clear that $\operatorname{HEnd}(H_{2k+1}) = \operatorname{End}(H_{2k+1})$, by Theorem 2.8 and Lemma 2.2. To prove the order, assume that $f \in \operatorname{End}(H_{2k+1})$. Then $f(u_1)$ and $f(u_2)$ have $2k+1$ and 2 choices, respectively. For $f(u_i)$ when $i \in \{3, 4, \ldots, 2k+1\}$ and $f(c)$, there is only one choice by Lemma 2.7. So, $|f(C_{2k+1} \cup \{c\})| = 2(2k+1)$. If $f(u_j) = u_k$ for some $j \in \{1, 2, \ldots, 2k+1\}$, then $f(v_j)$ can be one of four vertices u_{k+1},u_{k-1}, c, and v_k. Thus for every j, we have four possibilities for $f(v_j)$ and it implies that

$$|\operatorname{End}(H_{2k+1})| = 2(2k+1)4^{2k+1} = 2^{4k+3}(2k+1).$$

□

Corollary 4.6. *For all positive integers $k \geq 2$, we have* $\operatorname{Endospec}(H_{2k+1})$ *is equal to*

$$\Big(2^{4k+3}(2k+1), 2^{4k+3}(2k+1), 2(2k+1), 2(2k+1), 2(2k+1), 2(2k+1)\Big)$$

and $\operatorname{Endotype}(H_{2k+1})$ *is* 2.

Proof. It is deduced from Theorems 4.2 and 4.5. □

Theorem 4.7. *It follows that* $\operatorname{Endospec}(H_3) = (2^3 \times 3^5, 2^3 \times 3^5, 24, 24, 24, 24)$.

Proof. Since H_3 is End-regular, so $|\operatorname{End}(H_3)| = |\operatorname{HEnd}(H_3)|$ by Lemma 2.2. If $f \in \operatorname{End}(G)$, then $f(u_i) \in \{c, u_1, u_2, u_3\}$. So there exists 4 choices for $f(u_1)$. Since $W_3 \cong K_4$, for $f(u_2)$, $f(u_3)$ and $f(c)$ there exist 3, 2 and 1 choice respectively. For every $1 \leq i \leq 3$, $f(v_i) \in W_3 - \{f(u_i)\}$. So

$$|\operatorname{HEnd}(H_3)| = |\operatorname{End}(H_3)| = 4 \times 3 \times 2 \times 1 \times 3 \times 3 \times 1 = 2^3 \times 3^4.$$

.

Now, we claim that $|\operatorname{LEnd}(H_3)| = |\operatorname{Aut}(H_3)| = 24$. Let $f \in \operatorname{LEnd}(H_3)$, it is enough to show that $f(C_{2k+1}) = C_{2k+1}$. Then by the same proof as in Theorem 4.2, $\operatorname{LEnd}(H_3) = \operatorname{Aut}(H_3)$ and the result follows. We claim that $f(c) = c$. On the contrary, let $f(c) = u_j$ for some $j \in \{1,2,3\}$. Thus $f(u_k) = c$ for some $k \in \{1,2,3\}$. Hence, $f(v_k) = u_m$ for some $m \in \{1,2,3\}$, also $f^{-1}(u_m) = \{v_k, u_l\}$ and $f^{-1}(u_j) = \{c\}$. This is a contradiction since f is a local strong endomorphism. Now, for the computation of automorphisms of H_3, similarly as for endomorphisms of H_3, there exist 4 choices for $V(W_3)$, but for every $f(v_i)$ is just one choice, therefore

$$|\operatorname{LEnd}(H_3)| = |\operatorname{Aut}(H_3)| = 4 \times 3 \times 2 \times 1 \times 1 \times 1 \times 1 = 24.$$

□

Corollary 4.8. *It follows that* $\operatorname{Endotype}(H_3) = 2$*.*

Proof. It is a consequent of Theorem 4.7. □

Now, we state the following theorem.

Theorem 4.9. *It follows that* $|\operatorname{QEnd}(H_{2k})| = |\operatorname{Aut}(H_{2k})| = 4k$*.*

Proof. Let $f \in \operatorname{QEnd}(H_{2k})$ and let $x, y \in V(H_{2k})$. If $x \sim y$, then $f(x) \neq f(y)$. Otherwise, we consider the following four cases:

CASE 1: $x = v_i$ and $y = v_j$, $i \neq j$.

Without loss of generality, suppose that $f(v_i) = f(v_j)$ for some $i, j \in \{1, 2, \ldots, 2k\}$. We know that $\{f(u_i), f(v_i)\} \in E(H_{2k})$ and that $v_i, v_j \in f^{-1}(f(v_i))$. Therefore there exists one member in $f^{-1}(f(u_i))$ that is adjacent to v_i and v_j, but this is impossible, since v_i and v_j do not have a common neighbor. Thus $f(x) \neq f(y)$.

CASE 2: $x = v_i$ and $y = c$.

Let $f(v_i) = f(c)$; then we know $f(c) \sim f(u_{i+1})$. Since $f \in \operatorname{QEnd}(H_{2k})$, there exists one vertex in $f^{-1}(f(u_{i+1}))$ that is adjacent to both of c and v_i. On the other hand, the only common neighbor of c and v_i is u_i, which means $f(u_i) = f(u_{i+1})$, a contradiction. So $f(x) \neq f(y)$.

CASE 3: $x = u_i$ and $y = u_j$, $i \neq j$.

If $|i - j| = 1$, then $u_i \sim u_j$ and $f(u_i) \neq f(u_j)$. Let $|i - j| = 2$ and let $f(u_i) = f(u_j)$. Then common neighbors of u_i and u_j are c and u_{i+1}. We know that $\{f(u_i), f(v_i)\} \in E(H_{2k})$ and that $\{f(u_i), f(v_j)\} \in E(H_{2k})$. Since $f \in \operatorname{QEnd}(H_{2k})$, thus $f^{-1}(f(v_i))$ and $f^{-1}(f(v_j))$ have at least one common neighbor of u_i and u_j. By case 2, $c \notin f^{-1}(f(v_i))$ and $c \notin f^{-1}(f(v_j))$. So, $u_{i+1} \in f^{-1}(f(v_i))$ and $u_{i+1} \in f^{-1}(f(v_j))$. Then

$$f(v_i) = f(u_{i+1}) = f(v_j),$$

a contradiction to case 1. Therefore $f(u_i) \neq f(u_j)$. Let $|i-j| \geqslant 3$ and let $f(u_i) = f(u_j)$. Since $\{f(u_i), f(v_i)\} \in E(H_{2k})$, there exists one vertex in $f^{-1}(f(v_i))$ that is a common neighbor of u_i and u_j. On the other hand, c is only a common neighbor of u_i and u_j, that is, $f(v_i) = f(c)$, which is a contradiction to case 2. Therefore $f(x) \neq f(y)$.

CASE 4: $x = u_i$ and $y = v_j$, $i \neq j$.

Let $f(u_i) = f(v_j)$. We know that $\{f(u_i), f(c)\} \in E(H_{2k})$ and that $f \in \mathrm{QEnd}(H_{2k})$. Thus $Y = f^{-1}(f(c))$ has a common neighbor of u_i and v_j. If $j \neq i+1$ and $j \neq i-1$, then u_i and v_j have no common neighbor. Then without loss of generality, suppose that $j = i+1$. Then $u_i \in Y$, which means $f(c) = f(u_i)$, a contradiction. Therefore $f(x) \neq f(y)$.

Now, we compute the order of $\mathrm{Aut}(H_{2k})$. If $f \in \mathrm{Aut}(H_{2k})$, then clearly $f(C_{2k}) = C_{2k}$ and $\mathrm{Im}(f) = H_{2k}$. So we have $2k$ choices for $f(u_1)$, two choices for $f(u_2)$, and one choice for other vertices. So, we have $4k$ automorphisms from H_{2k}. □

Theorem 4.10. *It follows that* $\mathrm{Endotype}(H_{2k}) = 7$.

Proof. By the given endomorphism in Theorem 2.8, $\mathrm{End}(H_{2k}) \neq \mathrm{HEnd}(H_{2k})$, and by Theorem 4.4, we have $\mathrm{HEnd}(H_{2k}) \neq \mathrm{LEnd}(H_{2k})$. Now, we define the following endomorphism:

$$g(x) = \begin{cases} u_3, & x = u_i \text{ and } i \text{ is odd}, \\ u_4, & x = u_i \text{ and } i \text{ is even}, \\ v_3, & x = v_i \text{ and } i \text{ is odd}, \\ v_4, & x = v_i \text{ and } i \text{ is even}, \\ c, & x = c. \end{cases}$$

We see that $g \in \mathrm{LEnd}(H_{2k})$, but $g \notin \mathrm{QEnd}(H_{2k})$. Thus, $\mathrm{LEnd}(H_{2k}) \neq \mathrm{QEnd}(H_{2k})$, and Theorem 4.9 implies $\mathrm{QEnd}(H_{2k}) = \mathrm{Aut}(H_{2k})$. □

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