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# STAR SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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Abstract. A graph $G$ is called edge-magic if there is a bijective function $f\colon V(G)\cup E(G)\to\{1,2,\dots,|V(G)|+|E(G)|\}$ such that for every edge $xy\in E(G), f(x)+f(xy)+f(y)=c$ is a constant, called the valence of $f$. A graph $G$ is said to be super edge-magic if $f(V(G))=\{1,2,\dots,|V(G)|\}$. Let $G$ be a graph with $p$ vertices with $V(G)=\{v_1,v_2,\dots,v_p\}$ and let $S_m$ be the star with $m$ leaves. If in $G$, every vertex $v_i$ is identified to the center vertex of $S_{m_i}$, for some $m_i\geq 0, 1\leq i\leq n$, where $S_0=K_1$, then the graph obtained is denoted by $G_{(m_1,m_2,\dots,m_p)}$. Let $M(G)=\{(m_1,m_2,\dots,m_p)|G_{(m_1,m_2,\dots,m_p)}$ is a super edge-magic graph $\}$. The star super edge-magic deficiency $S\mu^*(G)$ is defined as

$$S\mu^*(G)=\begin{cases}\min_{(m_1,,m_2,\dots,m_p)}(m_1+m_2+\cdots+m_p), & \text{if } M(G)\neq\emptyset,\\ +\infty, & \text{if } M(G)=\emptyset.\end{cases}$$

In this paper we determine the star super edge-magic deficiency of certain classes of graphs.

## 1. Introduction

In 1970, Kotzig and Rosa [12] introduced the concept of edge-magic labeling using a different name: magic valuations. Meanwhile, the super edge-magic labeling was introduced by Enomoto et al. [6]. In [12], Kotzig and Rosa proved that for every graph $G$ there exists an edge-magic graph $H$ such that $H\cong G\cup nK_1$ for some non-negative integer $n$. This fact motivates the emergence of the concept of the edge-magic deficiency of a graph.

The *edge-magic deficiency* $\mu(G)$ of a graph $G$ is the minimum non-negative integer $n$ such that $G\cup nK_1$ has an edge-magic labeling. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno et al. [8] defined a similar concept for the super edge-magic labeling.

The *super edge-magic deficiency* $\mu_s(G)$ of a graph $G$ is the minimum non-negative integer $n$ such that $G\cup nK_1$ has a super edge-magic labeling or $+\infty$ if there exists no such $n$. Figueroa-Centeno et al. [8] provided the exact values for the super edge-magic deficiencies of several classes of graphs, such as, cycles, some classes of forests and complete bipartite graphs $K_{m,n}$. Ahmad et al. [3] provided the exact values for super edge-magic deficiencies of graphs,

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$(n,1)-$ kite graphs, $(n,3)-$kite graphs, $K_2 \cup C_n$ when $n \equiv 1 \pmod 4$. They also provided the upper bound of the super edge-magic deficiency of $K_2 \cup C_n$ when $n \equiv 3 \pmod 4$. Nadeem et al. [13] provided the upper bound for the super edge-magic deficiencies of kite graphs. Ahmad et al. [3] provided the upper bound for the super edge-magic deficiencies of ladder graphs. Acharya and Hegde introduced the concept of strongly indexable graph that is equivalent to the concept of super edge-magic graph [1]. For further details, see [10].

We observe some drawbacks of the super edge-magic deficiency of a graph.

- For several graphs, $\mu_s(G) = \infty$.
- To find $\mu_s(G)$, we construct a disconnected graph with large number of components (consisting of isolated vertices) having a super edge-magic labeling.
- The distribution of non-utilized numbers to the isolated vertices is very trivial.

Motivated by the concept of super edge-magic deficiency, we introduce a new deficiency for a graph without some of the above drawbacks, namely the star super edge-magic deficiency, $S\mu^*(G)$. We prove that $S\mu^*(G)$ is finite for several classes of graphs for which $\mu_s(G) = \infty$.

In this paper, we provide the exact values for the star super edge-magic deficiencies of several classes of graphs such as, cycles, $nK_2$ forests , $nP_2$ graphs, $(n,3)$-kite graphs, and $(n,2)$-kite graphs. We give an upper bound for the star super edge-magic deficiencies of kite graphs, ladder graphs, Mongolian tent graphs $Mt_n$ when $n$ is odd and triangular chain graphs $TC_n$ when $n$ is odd.

Figueroa-Centeno et al. [7] showed the following connection between the super edge-magic labeling and a special vertex labeling. This result characterizes super edge-magic graphs.

**Lemma 1.1** ([7]). *A $(p,q)$ graph $G$ is super edge-magic if and only if there exists a bijective function $f\colon V(G) \longrightarrow \{1,2,\dots,p\}$ such that the set $S = \{f(u)+f(v)\colon uv \in E(G)\}$ consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic labeling of $G$ with valence $k = p+q+s$, where $s = \min S$ and $S = \{k-(p+1), k-(p+2), \dots, k-(p+q)\}$.*

## 2. Main Results

**Definition 2.1.** *Let $G$ be a graph with $p$ vertices with vertex set $V(G) = \{v_1, v_2, \dots, v_p\}$. In $G$, every vertex $v_i$ is identified to the center vertex of $S_{m_i}$, for some $m_i \geq 0, 1 \leq i \leq n$, where $S_0 = K_1$; this graph is denoted by $G_{(m_1,m_2,,m_p)}$. Let $M(G) = \{(m_1, m_2, ..., m_p) | G_{(m_1,m_2,...,m_p)}$ is a super edge-magic graph $\}$. The star super edge-magic deficiency $S\mu^*(G)$ is defined as*

$$S\mu^*(G) = \begin{cases} \min_{(m_1,m_2,\ldots,m_p)}(m_1 + m_2 + \cdots + m_p), & \text{if } M(G) \neq \emptyset, \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases}$$

**Remark.** If $G$ is super edge-magic, then $S\mu^*(G) = 0$.

In the next theorem, we show the exact value for the star super edge-magic deficiency for the forest $nK_2$.

**Theorem 2.2.** *The star super edge-magic deficiency of the forest $nK_2$ is given by*

$$S\mu^*(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

*Proof.* The vertex set and edge set of the forest $nK_2$ are $V(nK_2) = \{x_i \colon 1 \leq i \leq n\} \cup \{y_i \colon 1 \leq i \leq n\}$ and $E(nK_2) = \{x_iy_i \colon 1 \leq i \leq n\}$, respectively. Kotzig and Rosa [12] showed that the forest $nK_2$ is super edge-magic if and only if $n$ is odd. Therefore, $S\mu^*(nK_2) = 0$ when $n$ is odd and $S\mu^*(nK_2) \geq 1$ when $n$ is even. When n is even, we define the graph $G = (nK_2)_{(m_{x_1},m_{x_2},\ldots,m_{x_n},m_{y_1},m_{y_2},\ldots,m_{y_n})}$, where

$$m_i = \begin{cases} 1, & \text{if } i = y_{\frac{n}{2}+1}, \\ 0, & \text{otherwise}. \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{x_i \colon 1 \leq i \leq n\} \cup \{y_i \colon 1 \leq i \leq n\} \cup \{s\}$ and $E(G) = \{x_iy_i \colon 1 \leq i \leq n\} \cup \{y_{(n/2)+1}s\}$, respectively. Consider the vertex labeling $f \colon V(G) \to \{1, 2, \ldots, 2n+1\}$ such that

- $f(x_i) = i, \quad 1 \leq i \leq n,$
- $f(y_i) = \begin{cases} \frac{3n}{2} + i + 1, & \text{if } 1 \leq i \leq \frac{n}{2}, \\ \frac{n}{2} + i, & \text{if } \frac{n}{2} + 1 \leq i \leq n, \end{cases}$
- $f(s) = \frac{3n+2}{2}$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(3n+4)/2, (3n+6)/2, \ldots, (5n+4)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(9n/2)+4$ and consequently, $S\mu^*(nK_2) \leq 1$. Therefore, we conclude that $S\mu^*(nK_2) = 1$, when n is even. □

In the next theorem, we show the exact value for the star super edge-magic deficiency of the forest $nP_2$ where $P_2$ is a path of length 2.

**Theorem 2.3.** *The star super edge-magic deficiency of the forest $nP_2$ is given by*

$$S\mu^*(nP_2) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

*Proof.* The vertex set and edge set of $nP_2$ are $V(nP_2) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}$, and $E(nP_2) = \{x_i y_i : 1 \leq i \leq n\} \cup \{y_i z_i : 1 \leq i \leq n\}$, respectively. Chen [5] proved that the graph $nP_2$ is super edge-magic if and only if $n$ is odd. Therefore, $S\mu^*(nP_2) = 0$ when $n$ is odd and $S^*(nP_2) \geq 1$ when $n$ is even. When n is even, we define the graph $G = (nP_2)_{(m_{x_1}, m_{x_2}, \ldots, m_{x_n}, m_{y_1}, m_{y_2}, \ldots, m_{y_n}, m_{z_1}, m_{z_2}, \ldots, m_{z_n})}$, where

$$m_i = \begin{cases} 1, & \text{if } \; i = y_{\frac{n}{2}+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of G are $V(G) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\} \cup \{s\}$, and $E(G) = \{x_i y_i : 1 \leq i \leq n\} \cup \{y_i z_i : 1 \leq i \leq n\} \cup \{y_{\frac{n}{2}+1} s\}$.

Consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, 2n+1\}$ such that

- $f(x_i) = i, 1 \leq i \leq n$,
- $f(y_i) = \frac{5n}{2} + 1 + i, 1 \leq i \leq \frac{n}{2}$,
- $f(y_i) = \frac{3n}{2} + i, \frac{n}{2} + 1 \leq i \leq n$,
- $f(z_i) = n + i, 1 \leq i \leq n$,
- $f(s) = \frac{5n+2}{2}$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(5n+4)/2, (5n+6)/2, \ldots, (9n+4)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(15n+8)/2$. This shows that $S\mu^*(nP_2) \leq 1$. Therefore, $S\mu^*(nP_2) = 1$. □

In the next theorem, we find the exact value for the star super edge-magic deficiency of the disconnected graph, $K_2 \cup C_n$ .

**Theorem 2.4.** *The star super edge-magic deficiency of the disconnected graph $K_2 \cup C_n$ is given by*

$$S\mu^*(K_2 \cup C_n) = \begin{cases} 0, & \textit{if } n \textit{ is even}, \\ 1, & \textit{if } n \textit{ is odd}. \end{cases}$$

*Proof.* Let $G = K_2 \cup C_n$. The vertex set and edge set of $G$ are $V(G) = \{v_i : 1 \leq i \leq n\} \cup \{u, w\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1, uw\}$. Kim and Park [11] proved that $K_2 \cup C_n$ is super edge-magic if and only if $n$ is even. Hence $S\mu^*(G) = 0$ for $n$ even and $S\mu^*(G) \geq 1$ for $n$ odd. When n is odd, define $G^* = (G)_{(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_u, m_w)}$, where

$$m_i = \begin{cases} 1, & \text{if } i = v_{n-2}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G^*$ are $V(G^*) = \{v_i : 1 \leq i \leq n\} \cup \{u, w\} \cup \{s\}$ and $E(G^*) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1, uw\} \cup \{v_{n-2} s\}$. We label the vertices of $G^*$ in the following manner,

- $f(u) = \frac{n+1}{2}, f(w) = n + 3$,

- $f(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd and } i \leq n-2, \\ \frac{n+3}{2}, & \text{if } i = n, \\ \frac{n+3+i}{2}, & \text{if } i \text{ is even.} \end{cases}$
- $f(s) = n + 2$.

The set of all edge sums generated by the above formula forms a set of n+1 consecutive integers $(n+5)/2, (n+7)/2, \ldots, (3n+7)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(5n+15)/2$. This shows that $S\mu^*(G) \leq 1$. Therefore, $S\mu^*(G) = 1$. □

In the next theorem, we determine the exact value for the star super edge-magic deficiency of $C_n$ .

**Theorem 2.5.** *The star super edge-magic deficiency of the cycle $C_n$ is given by*

$$S\mu^*(C_n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* The vertex set and edge set of $C_n$ are $V(C_n) = \{x_i : 1 \leq i \leq n\}$ and $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$, respectively. Enomoto et al. [6] proved that $C_n$ is super edge-magic if and only if $n$ is odd. Hence $S\mu^*(C_n) = 0$ if $n$ is odd and $S\mu^*(C_n) \geq 1$ if $n$ is even. Thus, assume that $n$ is even.

*Case 1*: : $n \equiv 0 \pmod 4$.

Now we define the graph $G \equiv (C_n)_{(m_{x_1}, m_{x_2}, \ldots, m_{x_n})}$ , where

$$m_i = \begin{cases} 1, & \text{if } i = x_1, \\ 1, & \text{if } i = x_{\frac{n}{2}+2}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{x_i : 1 \leq i \leq n\} \cup \{s_1, s_2\}$ and $E(G) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1, x_1 s_1\} \cup \{x_{(n/2)+2} s_2\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(x_i) = \begin{cases} \frac{n}{2}, & \text{if } i = 1, \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i > 1, \\ \frac{n+i}{2}, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n}{2}, \\ \frac{n+i+2}{2}, & \text{if } i \text{ is even and } \frac{n}{2}+1 \leq i \leq n, \end{cases}$
- $f(s_1) = n + 2$,
- $f(s_2) = 3(\frac{n}{4} - 1) + 4$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(n+4)/2, (n+6)/2, \ldots, (3n+6)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(5n+12)/2$ and consequently, $S\mu^*(C_n) \leq 2$ when $n \equiv 0 \pmod 4$.

*Case 2*: $n \equiv 2 \pmod 4$.

See Figure 1 for the labeling of $(C_6)_{(1,0,1,0,0,0)}$.

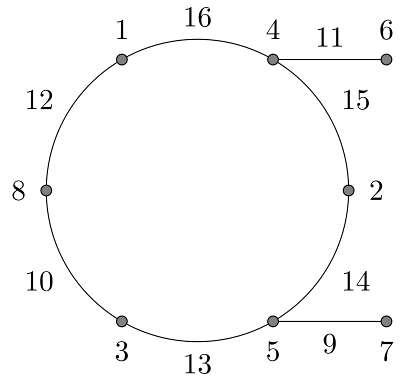


FIGURE 1. Labeling of $(C_6)_{(1,0,1,0,0,0)}$

Now consider $n > 6$. Define the graph $G \equiv (C_n)_{((m_{x_1}, m_{x_2}, \ldots, m_{x_n})}$, where

$$m_i = \begin{cases} 1, & \text{if } i = x_{\frac{n-8}{2}}, x_{\frac{n-2}{2}}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{x_i : 1 \le i \le n\} \cup \{s_1, s_2\}$ and $E(G) = \{x_i x_{i+1} : 1 \le i \le n\} \cup \{x_n x_1, x_{(n-8)/2} s_1\} \cup \{x_{(n-2)/2} s_2\}$, respectively. Consider the following labeling $f$ of $G$.

- $f(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd and } 1 \le i \le n, \\ \frac{n+2+i}{2}, & \text{if } i \text{ is even and } 1 \le i \le \frac{n-4}{2}, \\ \frac{n+6+i}{2}, & \text{if } i \text{ is even and } \frac{n-4}{2} + 1 \le i \le n-1, \\ \frac{n+2}{2}, & \text{if } i = n. \end{cases}$
- $f(s_1) = 3(\frac{n-2}{4} - 1) + 6$,
- $f(s_2) = 3(\frac{n-2}{4} - 1) + 5$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(n+4)/2, (n+6)/2, \ldots, (3n+6)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(5n+12)/2$ and consequently, $S\mu^*(C_n) \le 2$ when $n \equiv 2 \pmod 4$. In both the cases $S\mu^*(C_n) \le 2$. Kim and Park [11], proved that $(n, 1)$-kite is super edge-magic if and only if $n$ is odd. That is $(C_n)_{(1,0,0,..,0)}$ is not super edge-magic if $n$ is even. Therefore, $S\mu^*(C_n) = 2$, for $n$ even. □

In the next theorem, we prove an upper bound for the star super edge-magic deficiency of Fan graphs.

**Theorem 2.6.** *The star super edge-magic deficiency of the fan graph $F_n$ is given by*

$$S\mu^*(F_n) \le \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd,} \\ \frac{n-2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* The vertex set and edge set of $F_n$ are $V(F_n) = \{x_i : 1 \leq i \leq n\} \cup \{c\}$, and $E(F_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{cx_i : 1 \leq i \leq n\}$, respectively.

*Case 1*: $n$ is odd.

We define the graph $G = (F_n)_{(m_{x_1}, m_{x_2}, \ldots, m_{x_n}, m_c)}$, where

$$m_i = \begin{cases} \frac{n-1}{2}, & \text{if } i = c, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{x_i : 1 \leq i \leq n\} \cup \{c\} \cup \{s_i : 1 \leq i \leq (n-1)/2\}$. $E(G) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{cx_i : 1 \leq i \leq n\} \cup \{cs_i : 1 \leq i \leq \frac{n-1}{2}\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{n+i+1}{2}, & \text{if } i \text{ is even,} \end{cases}$
- $f(c) = \frac{3n+1}{2}$,
- $f(s_i) = n + i, 1 \leq i \leq \frac{n-1}{2}$.

The set of all edge sums generated by the above formula forms a set of consecutive integers $\{(n+5)/2, (n+7)/2, \ldots, (6n)/2\}$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(9n+3)/2$. This shows that $S\mu^*(F_n) \leq (n-1)/2$.

*Case 2*: $n$ is even.

We define the graph $G \cong (F_n)_{(m_1, m_2, \ldots, m_n, m_c)}$, where

$$m_i = \begin{cases} \frac{n-2}{2}, & \text{if } i = c, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{x_i : 1 \leq i \leq n\} \cup \{c\} \cup \{s_i : 1 \leq i \leq \frac{n-2}{2}\}$, $E(G) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{cx_i : 1 \leq i \leq n\} \cup \{cs_i : 1 \leq i \leq \frac{n-2}{2}\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{n+i}{2}, & \text{if } i \text{ is even,} \end{cases}$
- $f(c) = \frac{3n}{2}$,
- $f(s_i) = n + i, 1 \leq i \leq \frac{n-2}{2}$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(n+4)/2, (n+6)/2, \ldots, (6n-2)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $9n/2$. This shows that $S\mu^*(F_n) \leq (n-2)/2$.

□

**Open Problem.** *Verify whether equality holds in the above inequality.*

The following theorem gives an upper bound for the star super edge-magic deficiency of Wheel graph.

**Theorem 2.7.** *For $n$ odd, the star super edge-magic deficiency of the Wheel graph $W_n$ is given by $S\mu^*(W_n) \le (n-1)/2$.*

*Proof.* The vertex set and edge set of $W_n$ are $V(W_n) = \{x_i\colon 1 \le i \le n\} \cup \{c\}$, $E(W_n) = \{x_i x_{i+1}\colon 1 \le i \le n-1\} \cup \{cx_i\colon 1 \le i \le n\} \cup \{x_n x_1\}$, respectively. Define the graph $G \cong (W_n)_{(m_{x_1}, m_{x_2}, \ldots, m_{x_n}, m_c)}$, where

$$m_i = \begin{cases} \frac{n-1}{2}, & \text{if } i = c, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{x_i\colon 1 \le i \le n\} \cup \{c\} \cup \{s_i\colon 1 \le i \le (n-1)/2\}$, $E(G) = \{x_i x_{i+1}\colon 1 \le i \le n-1\} \cup \{cx_i\colon 1 \le i \le n\}$ $\cup \{cs_i\colon 1 \le i \le (n-1)/2\} \cup \{x_n x_1\}$ respectively. We label the vertices of $G$ in the following manner,

- $f(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{n+i+1}{2}, & \text{if } i \text{ is even,} \end{cases}$
- $f(c) = \frac{3n+1}{2}$,
- $f(s_i) = n + i$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(n+3)/2, (n+5)/2, \ldots, 6n/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(9n+3)/2$. This shows that $S\mu^*(W_n) \le (n-1)/2$ if $n$ is odd. □

In the next theorem, we show an upper bound for the super edge-magic deficiency of $(n,t)-$kite graph for odd $n$ and for even $t$.

**Theorem 2.8.** *Let $G$ be the $(n,t)$-kite graph. If $n$ is odd and $t$ is even, then $S\mu^*(G) \le t/2$.*

*Proof.* *Case 1*: $G = (3,t)$-kite, $t$ is even.

The vertex set and edge set of $G$ is $V(G) = \{v_i\colon 1 \le i \le 3\} \cup \{u_i\colon 1 \le i \le t\}$ and $E(G) = \{v_i v_{i+1}\colon 1 \le i \le 2\} \cup \{v_3 v_1, v_1 u_t\} \cup \{u_i u_{i+1} : 1 \le i \le t-1\}$ respectively. Define $G^* = (G)_{(m_{v_1}, m_{v_2}, m_{v_3}, m_{u_1}, m_{u_2}, \ldots, m_{u_t})}$, where

$$m_i = \begin{cases} \frac{t}{2}, & \text{if } i = v_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $v_1 s_1, v_1 s_2, v_1 s_{t/2}$ be the edges of the star attached at $v_1$. The labeling of $G^*$ is

- $f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{t+i+2}{2}, & \text{if } i \text{ is even,} \end{cases}$
- $f(v_i) = \begin{cases} \frac{t+i+1}{2}, & \text{if } i = 1, \\ t+2, & \text{if } i = 2, \\ t+3, & \text{if } i = 3, \end{cases}$
- $f(s_i) = t + 3 + i, \quad 1 \le i \le \frac{t}{2}$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(t+6)/2, (t+8)/2, \ldots, (4t+10)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(7t+18)/2$. This shows that $S\mu^*((3,t)\text{-kite}) \leq t/2$.

*Case 2*: $G = (n,t)$-kite, $n > 3$ and $t$ is even.

The vertex set and edge set of $(n,t)$-kite are $V(G) = \{v_i \colon 1 \leq i \leq n\} \cup \{u_i \colon 1 \leq i \leq t\}$, $E(G) = \{v_i v_{i+1} \colon 1 \leq i \leq n-1\} \cup \{v_n v_1, v_1 u_t\} \cup \{u_i u_{i+1} \colon 1 \leq i \leq t-1\}$, respectively. Now, we define

$$G^* = ((n,t)\text{-kite})_{(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_{u_1}, m_{u_2}, \ldots, m_{u_t}))}$$

where

$$m_i = \begin{cases} \frac{t}{2}, & \text{if } i = v_{n-2}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G^*$ are $V(G^*) = \{v_i \colon 1 \leq i \leq n\} \cup \{u_i \colon 1 \leq i \leq t\} \cup \{s_i \colon 1 \leq i \leq \frac{t}{2}\}$, $E(G^*) = \{v_i v_{i+1} \colon 1 \leq i \leq n-1\} \cup \{v_{n-2} s_i \colon 1 \leq i \leq \frac{t}{2}\} \cup \{v_n v_1, v_1 u_t\} \cup \{u_i u_{i+1} \colon 1 \leq i \leq t-1\}$, respectively. We label the vertices of $G^*$ in the following manner,

- $f(v_i) = \begin{cases} \frac{t+i+1}{2}, & \text{if } i \text{ is odd and } 1 \leq i \leq n-1, \\ \frac{n+2t+i+1}{2}, & \text{if } i \text{ is even,} \\ \frac{n+2t+1}{2}, & \text{if } i = n, \end{cases}$
- $f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{n+t+i-1}{2}, & \text{if } i \text{ is even,} \end{cases}$
- $f(s_i) = n + t + i, \quad 1 \leq i \leq \frac{t}{2}$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(n+t+3)/2, (n+t+5)/2, \ldots, (3n+4t+1)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(5n+7t+3)/2$. This shows that $S\mu^*((n,t)\text{-kite}) \leq t/2$. □

**Corollary 2.9.** *The star super edge-magic deficiency of the* $(n,2)$*-kite is*

$$S\mu^*((n,2)\text{-kite}) = \begin{cases} 0, & \textit{if } n \textit{ is even,} \\ 1, & \textit{if } n \textit{ is odd.} \end{cases}$$

*Proof.* The $(n,2)$-kite is not super edge-magic if and only if $n$ is even [14]. Hence $S\mu^*((n,2)\text{-kite}) = 0$ if $n$ is even and $S\mu^*((n,2)\text{-kite}) \geq 1$ if $n$ is odd. Thus assume that $n$ is even. By Theorem 2.9, $S\mu^*((n,2)\text{-kite}) \leq 1$. Therefore, $S\mu^*((n,2)\text{-kite}) = 1$. □

In the next theorem we find the exact value for the star super edge-magic deficiency of the $(n,3)$-kite.

**Theorem 2.10.** *The star super edge-magic deficiency of the* $(n,3)$*-kite is given by*

$$S\mu^*((n,3)\text{-kite}) = \begin{cases} 0, & \textit{if } n \textit{ is odd,} \\ 1, & \textit{if } n \textit{ is even.} \end{cases}$$

*Proof.* The vertex set and edge set of the $(n,3)$-kite are $V((n,3)\text{-kite}) = \{v_i \colon 1 \le i \le n\} \cup \{u_i \colon 1 \le i \le 3\}$, $E((n,3)\text{-kite}) = \{v_i v_{i+1} \colon 1 \le i \le n-1\} \cup \{v_n v_1, v_1 u_3\} \cup \{u_i u_{i+1} \colon 1 \le i \le 2\}$, respectively. Kim and Park [11] proved that the $(n,3)$-kite is super edge-magic if and only if $n$ is odd. Hence $S\mu^*((n,3)\text{-kite}) = 0$ if $n$ is odd and $S\mu^*((n,3)\text{-kite}) \ge 1$ if $n$ is even. Thus assume that $n$ is even.

*Case 1*: $n \equiv 0 \pmod 4$.

We define $G = ((n,3)\text{-kite})_{(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_{u_1}, m_{u_2}, m_{u_3})}$ where

$$m_i = \begin{cases} 1, & \text{if } i = v_{\frac{n-2}{2}}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{v_i \colon 1 \le i \le n\} \cup \{u_i \colon 1 \le i \le 3\} \cup \{s_1\}$, $E(G) = \{v_i v_{i+1} \colon 1 \le i \le n-1\} \cup \{v_{(n-2)/2} s_1\} \cup \{v_n v_1, v_1 u_3\} \cup \{u_i u_{i+1} \colon 1 \le i \le 2\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(v_i) = \begin{cases} \frac{i}{2} + 3, & \text{if } 1 \le i \le n-1 \text{ and } i \text{ is even,} \\ \frac{n+7+i}{2}, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is odd,} \\ \frac{n+9+i}{2}, & \text{if } \frac{n}{2} + 1 \le i \le n-1 \text{ and i is odd,} \\ 3, & \text{if } i = n, \end{cases}$
- $f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i = 1, 3, \\ \frac{n}{2} + 3, & \text{if } i = 2, \end{cases}$
- $f(s_1) = \frac{3n+16}{4}$.

The set of all edge sums generated by the above formula forms a consecutive integer sequence $(n+8)/2, (n+10)/2, \ldots, (3n+14)/2$. Therefore, $f$ can be extended to a super edge-magic labeling with valence $(5n+24)/2$. This shows that $S\mu^*((n,3)\text{-kite}) = 1$.

*Case 2*: $n \equiv 2 \pmod 4$.

The labeling of $((6,3)\text{-kite})_{(0,0,0,0,0,1)}$ is given in Figure 2. Now consider $(n,3)$-kite, $n > 6$. We define

$$G = ((n,3)\text{-kite})_{(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_{u_1}, m_{u_2}, m_{u_3})}$$

where

$$m_i = \begin{cases} 1, & \text{if } i = v_{\frac{n-8}{2}}, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{v_i \colon 1 \le i \le n\} \cup \{u_i \colon 1 \le i \le 3\} \cup \{s_1\}$, $E(G) = \{v_i v_{i+1} \colon 1 \le i \le n-1\} \cup \{v_{(n-8)/2} s_1\} \cup \{v_n v_1, v_1 u_3\} \cup \{u_i u_{i+1} \colon 1 \le i \le 2\}$ respectively.

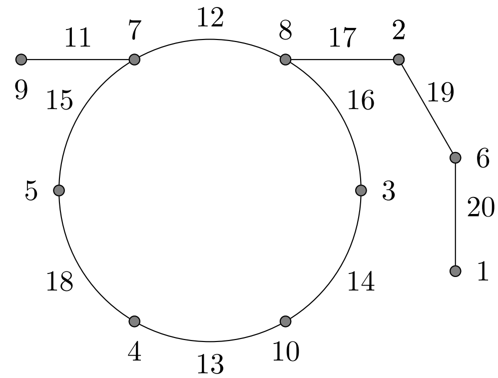


FIGURE 2. Labeling of $((6,3)\text{-kite})_{(0,0,0,0,0,1)}$

We label the vertices of $G$ in the following manner,

- $f(v_i) = \begin{cases} \frac{n+i+9}{2}, & \text{if } 1 \le i \le \frac{n}{2}-1 \text{ and } i \text{ is odd,} \\ \frac{n+i+11}{2}, & \text{if } \frac{n}{2} \le i \le \frac{n}{2}+1 \text{ and } i \text{ is odd,} \\ \frac{i+5}{2}, & \text{if } \frac{n}{2}+2 \le i \le n-1 \text{ and } i \text{ is odd,} \\ \frac{i+4}{2}, & \text{if } 1 \le i \le \frac{n}{2}+1 \text{ and } i \text{ is even,} \\ \frac{n+10+i}{2}, & \text{if } \frac{n}{2}+2 \le i \le n-1 \text{and } i \text{ is even,} \\ \frac{n+8}{2}, & \text{if } i = n, \end{cases}$
- $f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i = 1,3, \\ \frac{n}{2}+3, & \text{if } i = 2, \end{cases}$
- $f(s_1) = \frac{3n+18}{4}$.

The set of all edge sums generated by the above formula forms a consecutive integers $\{(n+8)/2, (n+10)/2, \dots, (3n+14)/2\}$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(5n+24)/2$. This shows that $S\mu^*((n,3)\text{-kite}) = 1$.

□

In the next theorem we show that $S\mu^*(L_n) \le 1$.

**Theorem 2.11.** *For $n$ even, the star super edge-magic deficiency of the ladder graph $L_n$ is $S\mu^*(L_n) \le 1$.*

*Proof.* The vertex set and edge set of $L_n$ are $V(L_n) = \{v_i, u_i : 1 \le i \le n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_iu_i : 1 \le i \le n\}$, respectively. Now we define $G = (L_n)_{(m_{v_1}, m_{v_2}, \dots, m_{v_n}, m_{u_1}, m_{u_2}, \dots, m_{u_n})}$ where

$$m_i = \begin{cases} 1, & \text{if } i = u_2, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{v_i, u_i\colon 1 \le i \le n\} \cup \{s_1\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1}\colon 1 \le i \le n-1\} \cup \{v_iu_i\colon 1 \le i \le n\} \cup \{u_2s_1\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(v_i) = \begin{cases} i, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is odd,} \\ \frac{4i-n}{2}, & \text{if } \frac{n}{2}+1 \le i \le n \text{ and } i \text{ is odd,} \\ \frac{4i+n+2}{2}, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is even,} \\ n+1+i, & \text{if } \frac{n}{2}+1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$
- $f(u_i) = \begin{cases} i, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is even,} \\ \frac{4i-n}{2}, & \text{if } \frac{n}{2}+1 \le i \le n \text{ and } i \text{ is even,} \\ \frac{4i+n+2}{2}, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is odd,} \\ n+1+i, & \text{if } \frac{n}{2}+1 \le i \le n \text{ and } i \text{ is odd,} \end{cases}$
- $f(s_1) = \frac{n+2}{2}$.

The set of all edge sums generated by the above formula forms a set of consecutive integers $\{(n+6)/2, (n+8)/2, \ldots, (7n+2)/2\}$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(11n+6)/2$. This shows that $S\mu^*(L_n) \le 1$, when $n$ is even. □

**Open Problem.** *Prove that the same bound holds when n is odd.*

The following theorem gives an upper bound for the star super edge-magic deficiency of the Mongolian tent graph.

**Theorem 2.12.** *The star super edge-magic deficiency of the Mongolian tent graph $Mt_n$, for $n$ odd, is bounded by $S\mu^*(Mt_n) \le (n-3)/2$.*

*Proof.* The vertex set and edge set of $Mt_n$ are $V(Mt_n) = \{v_i, u_i\colon 1 \le i \le n\} \cup \{u\}$ and $E(Mt_n) = \{v_iv_{i+1}, u_iu_{i+1}\colon 1 \le i \le n-1\} \cup \{uu_i, u_iv_i\colon 1 \le i \le n\}$, respectively. Let $n$ be any odd non-negative integer. According to Lemma 1.1, it is sufficient to prove that there is a vertex labeling with the property that the edge sums under this labeling are consecutive integers. Define $G = (Mt_n)_{(m_{u_1}, m_{u_2}, \ldots, m_{u_n}, m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_u)}$, where

$$m_i = \begin{cases} \frac{n-3}{2}, & \text{if } i = u, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ is $V(G) = \{v_i, u_i\colon 1 \le i \le n\} \cup \{u\} \cup \{s_i\colon 1 \le i \le (n-3)/2\}$. and $E(G) = \{u_iu_{i+1}, v_iv_{i+1}\colon 1 \le i \le n-1\} \cup \{v_iu_i, uu_i\colon 1 \le i \le n\} \cup \{us_i\colon 1 \le i \le (n-3)/2\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is odd,} \\ \frac{n+i+1}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$
- $f(u_i) = \begin{cases} \frac{3n+i}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is odd,} \\ \frac{2n+i}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$
- $f(u) = \frac{5n-1}{2}$.

The vertices $s_i$ under the labeling $f$ are labeled by $f(s_i) = 2n + i, for 1 \leq i \leq (n-3)/2$. The edge sums form a consecutive integer sequence $(n+5)/2, (n+7)/2, \ldots, (10n-4)/2$.Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(15n-3)/2$. This shows that $S\mu^*(Mt_n) \leq (n-3)/2$. □

**Open Problem.** *Prove that the above upper bound is $(n-2)/2$ when $n$ is even.*

The following theorem gives an upper bound for the star super edge-magic deficiency of the triangular chain graph.

**Theorem 2.13.** *For $n$ odd, the star super edge-magic deficiency of the triangular chain graph, $TC_n$, is $S\mu^*(TC_n) \leq \lfloor n/2 \rfloor$.*

*Proof.* The vertex set and edge set of $TC_n$ are $V(TC_n) = \{v_i : 1 \leq i \leq 2n\} \cup \{u_i : 1 \leq i \leq n\}$, $E(TC_n) = \{v_i v_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_i v_{2i-1}, u_i v_{2i} : 1 \leq i \leq n\}$, respectively. We define $G = (TC_n)_{(m_{u_1}, m_{u_2}, \ldots, m_{u_n}, m_{v_1}, m_{v_2}, \ldots, m_{v_{2n}})}$, where,

$$m_i = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } i = u_n, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{v_i : 1 \leq i \leq 2n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq \lfloor n/2 \rfloor\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_i v_{2i-1}, u_i v_{2i} : 1 \leq i \leq n\} \cup \{u_n s_i : 1 \leq i \leq \lfloor n/2 \rfloor\}$ respectively. We label the vertices of $G$ in the following manner,

- $f(v_{2i-1}) = i$ and $f(v_{2i}) = n + i$, for $1 \leq i \leq n$,
- $f(u_i) = \begin{cases} 3n + i, & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ 2n + i, & \text{if } \lfloor \frac{n}{2} + 1 \rfloor \leq i \leq n. \end{cases}$

The vertices $s_i$ are labeled by $f(s_i) = 2n + i$, for $1 \leq i \leq \lfloor n/2 \rfloor$. The set of all edge sums generated by the above formula forms a consecutive integer sequence $(2n+4)/2, (2n+6)/2, \ldots, (11n-1)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $8n+1+2\lfloor n/2 \rfloor$. Hence $G$ admits a super edge-magic labeling. This shows that $S\mu^*(TC_n) \leq \lfloor n/2 \rfloor$. □

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