Contributions to Discrete Mathematics

Volume 13, Number 1, Pages 79–119
ISSN 1715-0868

ON THE ENUMERATION OF A CLASS OF TOROIDAL GRAPHS

DIPENDU MAITY AND ASHISH KUMAR UPADHYAY

Abstract. We present enumerations of a class of toroidal graphs which give rise to semiequivelar maps. There are eleven different types of semiequivelar maps on the torus: $\{3^6\}$, $\{4^4\}$, $\{6^3\}$, $\{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3^4, 6\}$, $\{4, 8^2\}$, $\{3, 12^2\}$, $\{4, 6{,}12\}$, $\{3, 4, 6, 4\}$. We know the classification of the maps of type $\{3^6\}$, $\{4^4\}$, $\{6^3\}$. In this article, we attempt to classify maps of type $\{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3^4, 6\}$, $\{4, 8^2\}$, $\{3, 12^2\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$.

1. Introduction

A *map* M is an embedding of a graph G on a surface S such that the closure of components of $S \setminus G$, called the *faces* of M, are closed 2-cells, that is, each face is homeomorphic to a 2-disk. A map M is said to be a *polyhedral map* (see Brehm and Schulte [4]) if the intersection of any two distinct faces are either empty, a common vertex, or a common edge. An *a-cycle* C_a is a finite connected 2-regular graph with a vertices, and the *face sequence* of a vertex v in a map is a finite sequence $(a^p, b^q, \cdots, m^r)$ of powers of positive integers $a, b, \cdots, m \geq 3$ and $p, q, \cdots, r \geq 1$ in *cyclic* order such that through the vertex v, p number of C_a's, q number of C_b's, $\cdots$, r number of C_m's are incident. A map K is said to be *semiequivelar* if the face sequence of each vertex is same, see [8]. Two maps of fixed type on the torus are called *isomorphic* if there exists a homeomorphism of the torus which sends vertices to vertices, edges to edges, faces to faces, and preserves incidents. That is, if we consider two polyhedral complexes K_1 and K_2, then an isomorphism is a map $f : K_1 \to K_2$ such that $f|_{V(K_1)} : V(K_1) \to V(K_2)$ is a bijection and $f(\sigma)$ is a cell in K_2 if and only if σ is a cell in K_1. There are eleven types of semiequivelar maps on the torus: $\{3^6\}$, $\{4^4\}$, $\{6^3\}$, $\{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3^4, 6\}$, $\{4, 8^2\}$, $\{3, 12^2\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$.

In this article, we classify the remaining semiequivelar maps on the torus up to isomorphism, completing their classification. In this context, Altshuler [1] has shown construction and enumeration of maps of types $\{3^6\}$ and $\{6^3\}$. Kurth [5] has given an enumeration of semiequivelar maps of types

Received by the editors February 21, 2015, and in revised form May 29, 2017.
2000 *Mathematics Subject Classification.* 52B70, 05C30, 05C38.
Key words and phrases. Toroidal Graphs, Semiequivelar Maps, Cycles.

$\{3^6\}$, $\{4^4\}$, $\{6^3\}$. Negami [6] has studied uniqueness and faithfulness of embeddings for a class of toroidal graphs. Brehm and Kühnel [3] have presented a classification of semiequivelar maps of types $\{3^6\}$, $\{4^4\}$, $\{6^3\}$. Tiwari and Upadhyay [7] have classified semiequivelar maps of types $\{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3^4, 6\}$, $\{4, 8^2\}$, $\{3, 12^2\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$ with up to twenty vertices. In this article, we devise a way of enumerating all semiequivelar maps of types $\{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3^4, 6\}$, $\{4, 8^2\}$, $\{3, 12^2\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$ on the torus and explicitly determine the maps with a small number of vertices. Therefore, we have the following theorem.

Theorem 1.1. *The semiequivelar maps with n vertices of types $\{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3, 12^2\}$, $\{3^4, 6\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$, $\{4, 8^2\}$ can be classified up to isomorphism on the torus. In Tables 1–8, we have given nonisomorphic objects with few vertices.*

More precisely, let $X = \{3^3, 4^2\}$, $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3, 12^2\}$, $\{3^4, 6\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$, or $\{4, 8^2\}$ be a semiequivelar type on the torus. We present an algorithmic approach of calculating different maps for the type X for different numbers of vertices in the subsequent sections.

2. Definitions

We now define some operations on graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of the same graph $G = (V, E)$. Then the *union* $G_1 \cup G_2$ is a graph $G_3 = (V_3, E_3)$ where $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$. Similarly, the *intersection* $G_1 \cap G_2$ is a graph $G_4 = (V_4, E_4)$ where $V_4 = V_1 \cap V_2$ and $E_4 = E_1 \cap E_2$. For more on graph theory see [2].

We denote a cycle u_1-u_2-$\cdots$-u_k-u_1 by $C(u_1, u_2, \cdots, u_k)$ and a path w_1-w_2-$\cdots$-w_x by $P(w_1, w_2, \cdots, w_x)$. Let $Q_1 = P(u_1, \cdots, u_k)$ be a path. We call a path $Q_2 = P(v_1, \cdots, v_r)$ to be a path extended from Q_1 if $V(Q_1) \subset V(Q_2), E(Q_1) \subset E(Q_2)$, i.e., Q_1 is a subpath of Q_2. We say that the Q_2 is an *extended path* of the path Q_1.

We say that a cycle is *contractible* if it bounds a 2-disk. If the cycle does not bound any 2-disk on the torus then we say that the cycle is *noncontractible.*

3. Examples

Example 3.1. *Let M be a semiequivelar map of type $\{3^3, 4^2\}$ with n vertices on the torus. The map M has a $T(r, s, k)$ representation (defined later in Section 4) for some $r, s, k \in \mathbb{N} \cup \{0\}$. Let the number of vertices $n = 14$. By Lemma 4.9, $n = rs = 14$ where $2 \mid s$. Hence, $s = 2$, $r = 7$, and $k = 2, 3, 4$ by Lemma 4.9. So, $T(r, s, k) = T(7, 2, 2)$, $T(7, 2, 3)$, and $T(7, 2, 4)$, see Figure 1, 2, and 3, respectively. In $T(7, 2, 2)$, $C_{1,1} = C(u_1, u_2, \cdots, u_7)$ is a cycle of type A_1 (see the definition of type A_1 in Section 4),*

$$C_{1,2} = C(u_1, u_8, u_3, u_{10}, u_5, u_{12}, u_7, u_{14}, u_2, u_9, u_4, u_{11}, u_6, u_{13})$$

and

$$C_{1,3} = C(u_1, u_8, u_4, u_{11}, u_7, u_{14}, u_3, u_{10}, u_6, u_{13}, u_2, u_9, u_5, u_{12})$$

are two cycles of type A_2 *(see definition of type* A_2 *in Section 4), and* $C_{1,4} = C(u_3, u_{10}, u_5, u_4)$ *is a cycle of type* A_4 *(see definition of type* A_4 *in Section 4). In* $T(7,2,4)$, $C_{2,1} = C(v_1, v_2, \cdots, v_7)$ *is of type* A_1,

$$C_{2,2} = C(v_1, v_8, v_5, v_{12}, v_2, v_9, v_6, v_{13}, v_3, v_{10}, v_7, v_{14}, v_4, v_{11})$$

and

$$C_{2,3} = C(v_1, v_8, v_6, v_{13}, v_4, v_{11}, v_2, v_9, v_7, v_{14}, v_5, v_{12}, v_3, v_{10})$$

are of type A_2*, and* $C_{2,4} = C(v_5, v_{12}, v_3, v_4)$ *is of type* A_4*. In* $T(7,2,3)$, $C_{3,1} = C(w_1, w_2, \cdots, w_7)$ *is of type* A_1,

$$C_{3,2} = C(w_1, w_8, w_4, w_{11}, w_7, w_{14}, w_3, w_{10}, w_6, w_{13}, w_2, w_9, w_5, w_{12})$$

and

$$C_{3,3} = C(w_1, w_8, w_5, w_{12}, w_2, w_9, w_6, w_{13}, w_3, w_{10}, w_7, w_{14}, w_4, w_{11})$$

are of type A_2*, and* $C_{3,4} = C(w_4,\ w_5, w_6, w_7, w_{11})$ *is of type* A_4.

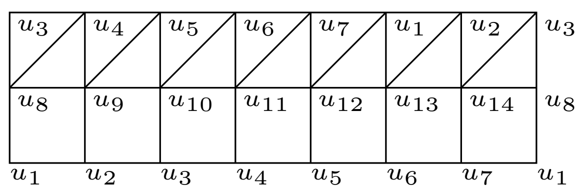

FIGURE 1. $T(7, 2, 2) : O_1$

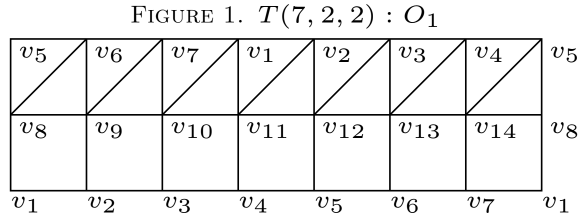

FIGURE 2. $T(7, 2, 4) : O_2$

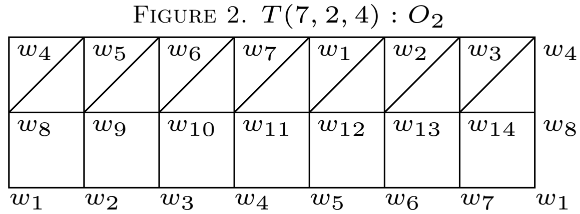

FIGURE 3. $T(7, 2, 3) : O_3$

In Section 4, by Lemma 4.8, the cycles of type A_1 have the same length and the cycles of type A_2 have at most two different lengths in M. So, $O_3 \ncong O_1$ since length$(C_{3,4}) \neq$ length$(C_{1,4})$ and $O_3 \ncong O_2$ since length$(C_{3,4}) \neq$ length$(C_{2,4})$. Thus, $O_3 \ncong O_i$ for $i = 1, 2$. Now, length$(C_{1,1})$ = length$(C_{2,1})$, {length$(C_{1,2})$, length$(C_{1,3})$} = {length$(C_{2,2})$, length$(C_{2,3})$}, and length$(C_{1,4})$ = length$(C_{2,4})$. We cut $T(7,2,4)$ along the path $P(v_5, v_{12}, v_3)$ and identify along the path $P(v_1, v_8, v_5)$. This gives the presentation of $T(7,2,4)$ in Figure 4. Figure 4 has a $T(7,2,2)$ representation. So, $O_1 \cong O_2$ (see the proof of Lemma 4.10 for the isomorphism between O_1 and O_2). Thus, $O_1 \cong O_2$ and $O_3 \ncong O_1, O_2$. Therefore, there are two semiequivelar maps of type $\{3^3, 4^2\}$ with 14 vertices on the torus up to isomorphism.

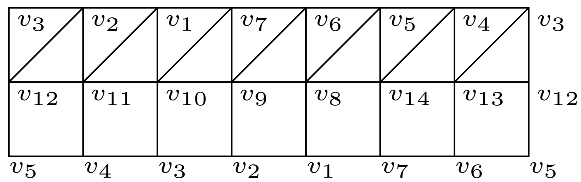

FIGURE 4. $T(7,2,2) : O_2$

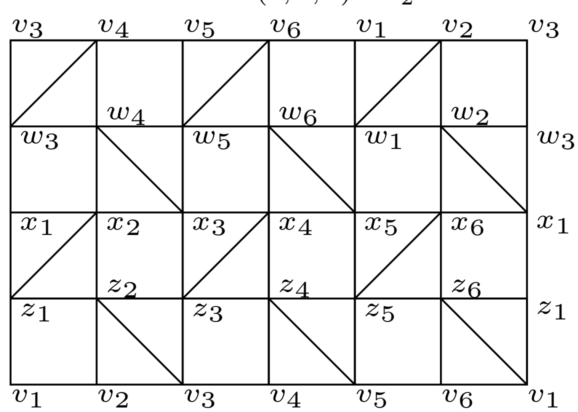

FIGURE 5. $T(6,4,2)$

Example 3.2. *Let M be a semiequivelar map of type $\{3^2, 4, 3, 4\}$ with 16 vertices on the torus. As above, by Lemma 5.5, there are three representations of M, namely, $T(8,2,4)$, $T(4,4,0)$, and $T(4,4,2)$, see Figures 9, 6, and 7, respectively. In $T(8,2,4)$, $C_{1,1} = C(u_1, u_2, \cdots, u_8)$ and $C_{1,2} = C(u_1, u_9, u_5, u_{13})$ are two cycles of type B_1 (see the definition of type B_1 in Section 5). In $T(4,4,0)$, $C_{2,1} = C(v_1, v_2, v_3, v_4)$ and $C_{2,2} = C(v_1, v_5, v_9, v_{13})$ are two cycles of type B_1. In $T(4,4,2)$, $C_{3,1} = C(w_1, w_2, w_3, w_4)$ and $C_{3,2} = C(w_1, w_5, w_9, w_{13}, w_3, w_7, w_{11}, w_{15})$ are two cycles of type B_1. In Section 5, the cycles of type B_1 have at most two different lengths. So, $O_5 \ncong O_6$ since $\{\text{length}(C_{1,1}), \text{length}(C_{1,2})\} \neq \{\text{length}(C_{2,1}), \text{length}(C_{2,2})\}$. Now, $\{\text{length}(C_{1,1}), \text{length}(C_{1,2})\} = \{\text{length}(C_{3,1}), \text{length}(C_{3,2})\}$. We identify boundaries of O_7 and cut along the cycle $C_{3,2} = C(w_1, w_5, w_9, w_{13}, w_3, w_7, w_{11}, w_{15})$ and next along $C_{3,1}$. Thus, we get a $T(8,2,4)$ representation in Figure 8. So, $O_5 \cong O_7$ (see the proof of Lemma 5.6 for the isomorphism). Therefore, we have two maps of type $\{3^2, 4, 3, 4\}$ with 16 vertices on the torus up to isomorphism.*

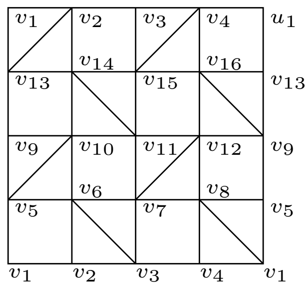

FIGURE 6. $T(4,4,0) : O_6$

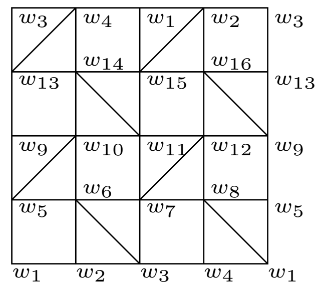

FIGURE 7. $T(4,4,2) : O_7$

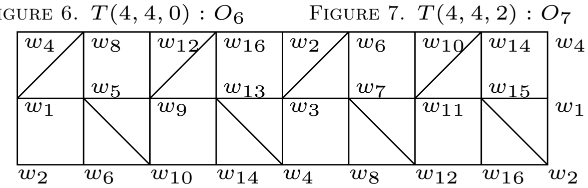

FIGURE 8. $T(8,2,4) : O_4$

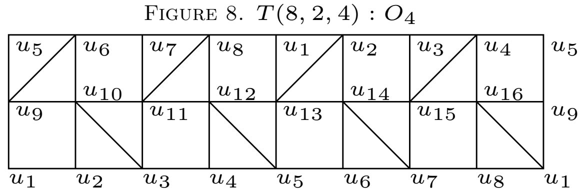

FIGURE 9. $T(8,2,4) : O_5$

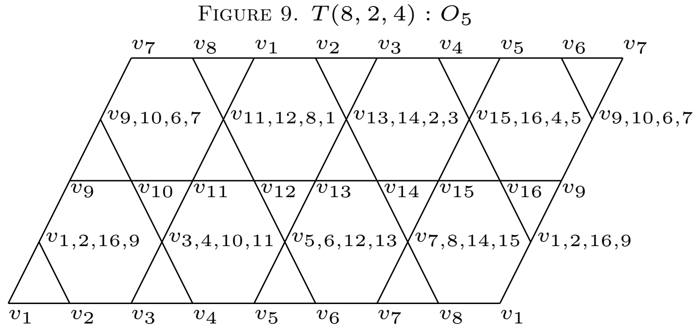

FIGURE 10. $T(8,2,6)$

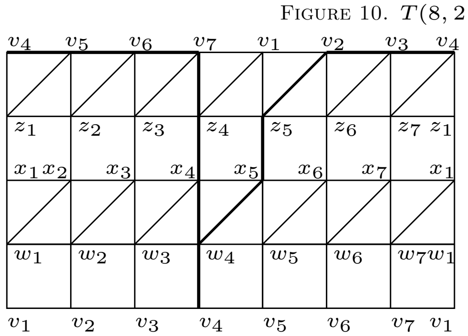

FIGURE 11. $T(7,4,3)$

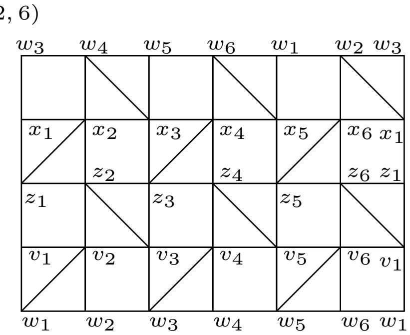

FIGURE 12. R

4. MAPS OF TYPE $\{3^3, 4^2\}$

Let M be a map of type $\{3^3, 4^2\}$ on the torus. Through each vertex in M there are three distinct types of paths as follows.

Definition. *Let* $P_1 := P(\cdots, u_{i-1}, u_i, u_{i+1}, \cdots)$ *be a path in the edge graph of* M. *We say* P_1 *is of type* A_1 *if all the triangles incident with an inner (degree two in* P_1*) vertex* u_i *lie on one side and all quadrangles incident with* u_i *lie on the other side of the subpath* $P' = P(u_{i-1}, u_i, u_{i+1})$ *(as in Figure 13) at* u_i. *Since the link of vertex* u_i *is a cycle and the path* P' *is a cord of cycle* $lk(u_i)$*, so, the path* P' *divides the region into two parts. If* u_t *is a boundary vertex (degree one in* P_1*) of* P_1 *then there is an extended path of* P_1 *where* u_t *is an inner vertex.*

Observe that a link of a vertex in M contains some vertices which are adjacent to the vertex. To identify the nonadjacent vertices in the link, we use bold letters. That is, if a description of a link, say $lk(w)$, contains any bold letter $\boldsymbol{a}$ then it indicates that $\mathbf{a}$ is nonadjacent to w. For example, in $lk(u_i)$ vertices $\boldsymbol{a}$ and $\boldsymbol{c}$ are nonadjacent to u_i. In this article, we consider permutation of vertices in $lk(u_i)$ of a vertex u_i counterclockwise locally at u_i.

Definition. *Let* $P_2 := P(\cdots, v_{i-1}, v_i, v_{i+1}, \cdots)$ *be a path in the edge graph of* M *for which* v_i, v_{i+1} *are two consecutive inner vertices of* P_2 *or an extended path of* P_2. *We say* P_2 *is of type* A_2 *if* $lk(v_i) = C(\boldsymbol{a}, v_{i-1}, \boldsymbol{b}, c, v_{i+1}, d, e)$ *implies* $lk(v_{i+1}) = C(\boldsymbol{a}_0, v_{i+2}, \boldsymbol{b}_0, d, v_i, c, p)$ *and* $lk(v_i) = C(\boldsymbol{x}, v_{i+1}, \boldsymbol{z}, l, v_{i-1}, k, m)$ *implies* $lk(v_{i+1}) = C(\boldsymbol{l}, v_i, \boldsymbol{m}, x, v_{i+2}, g, z)$. *At least one of the former two conditions must occur for each vertex.*

Definition. *Let* $P_3 := P(\cdots, w_{i-1}, w_i, w_{i+1}, \cdots)$ *be a path in the edge graph of* M *for which* w_i, w_{i+1} *are two inner vertices of* P_3 *or an extended path of* P_3. *We say* P_3 *is of type* A_3 *if* $lk(w_i) = C(\boldsymbol{a}, w_{i-1}, \boldsymbol{b}, c, d, w_{i+1}, e)$ *implies* $lk(w_{i+1}) = C(\boldsymbol{a}_1, w_{i+2}, \boldsymbol{b}_1, p, e, w_i, d)$ *and* $lk(w_i) = C(\boldsymbol{a}_2, w_{i+1}, \boldsymbol{b}_2, p, e, w_{i-1}, d)$ *implies* $lk(w_{i+1}) = C(\boldsymbol{p}, w_i, \boldsymbol{d}, a_2, z_1, w_{i+2}, b_2)$.

Let Q be a maximal path (path of maximal length) of type A_t for a fixed $t \in \{1, 2, 3\}$. We show that there is an edge e in M such that $Q \cup e$ is a cycle of type A_t.

Lemma 4.1. *If* $P(u_1, \cdots, u_r)$ *is a maximal path of type* A_1*,* A_2*, or* A_3 *in* M *then there is an edge* $u_r u_1$ *in* M *such that* $C(u_1, u_2, \cdots, u_r)$ *is a cycle.*

Proof. Let $Q = P(u_1, \cdots, u_r)$ be of type A_1 and $lk(u_r) = C(\boldsymbol{x}, y, \boldsymbol{z}, w, v, u, u_{r-1})$. If $w = u_1$ then $Q = C(u_1, u_2, \cdots, u_r)$ is a cycle. If $w \neq u_1$. Then either $w = u_i$ for some $2 \leq i \leq r$ or $w \neq u_i$ for all $2 \leq i \leq r$. Suppose $w = u_i$ for some $2 \leq i \leq r$. Observe that $L = P(u_{i-1}, u_i, u_{i+1}) \subset Q$ and $L' = P(u_r, w, x)$ are two paths of type A_1 through u_i. By Definition 4.1, through each vertex in M we have only one path of this particular type A_1. So, $L = L'$. This implies that $u_r = u_{i-1}$ or $u_r = u_{i+1}$. This is a contraction since, by assumption, Q is a path and $u_i \neq u_j$ for all $i \neq j, 1 \leq i, j \leq r$. Therefore, $w \neq u_i$ for all $2 \leq i \leq r$. If $w \neq u_i$ for all $1 \leq i \leq r$ then by Definition 4.1, u_r is an inner vertex in the extended path of Q. Thus, we get a path, namely Q_1, which is extended from Q. Hence, length(Q) <

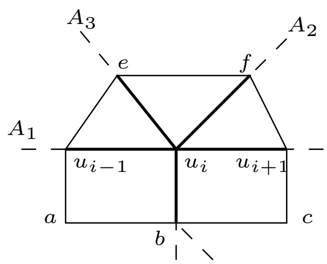

FIGURE 13. $lk(u_i)$

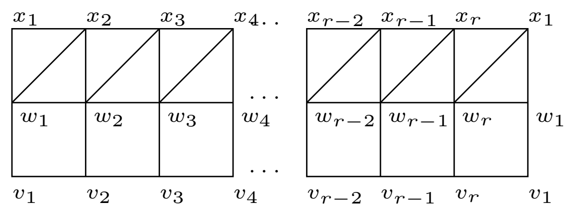

FIGURE 14. Cylinder

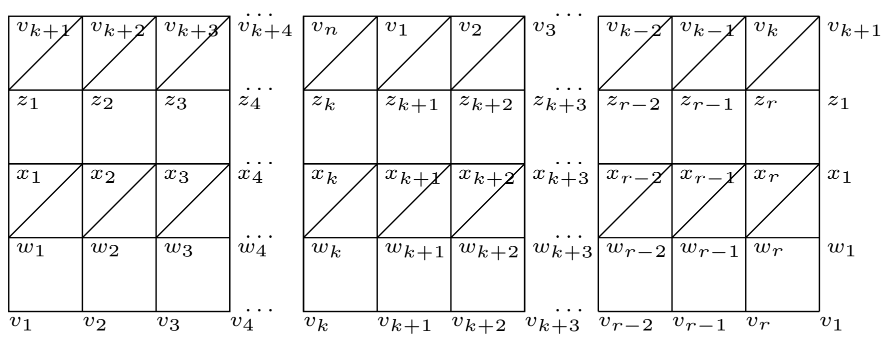

FIGURE 15. $T(r, 4, k)$

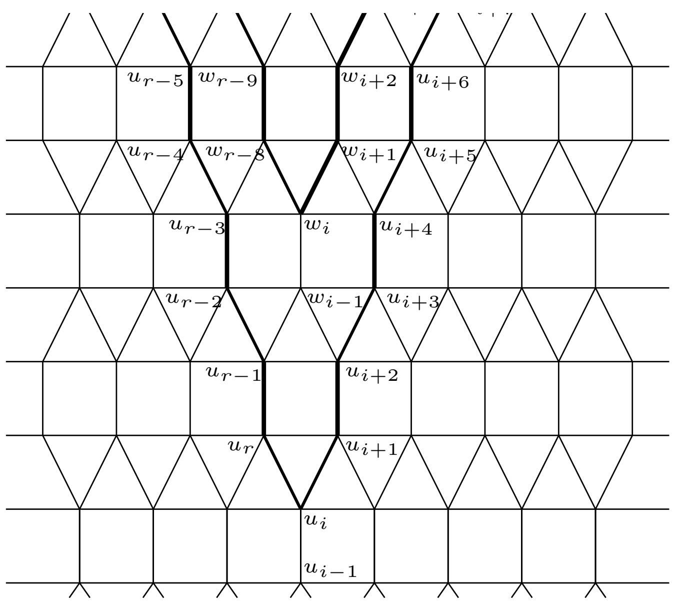

FIGURE 16.

length(Q_1). This is a contradiction as Q is maximal. Therefore, $w = u_1$ and $Q \cup u_r u_1 = C(u_1, u_2, \cdots, u_r)$ is a cycle.

Let $W = P(u_1, u_2, \cdots, u_r)$ be of type A_2. We follow a similar argument from [1, Theorem 1]. Let $lk(u_{r-1}) = (\boldsymbol{v}, u_r, \boldsymbol{z}, p, q, u_{r-2}, x)$ and $lk(u_r) =$

$(\boldsymbol{p}, u_{r-1}, \boldsymbol{x}, v, e, u_{r+1}, z)$. If $u_{r+1} = u_1$ then $C(u_1, u_2, \cdots, u_r)$ is a cycle. If $u_{r+1} \neq u_1$, then either $u_{r+1} = u_i$ for some $2 \leq i \leq r$ or $u_{r+1} \neq u_i$ for all $2 \leq i \leq r$. Suppose $u_{r+1} = u_i$ for some $2 \leq i \leq r$. Then $u_{r+1} = u_i$ defines a cycle $R = C(u_i, u_{i+1}, \cdots, u_r)$. Now by assumption, W is a path. That is, $u_i \neq u_j$ for all $1 \leq i, j \leq r$ and $i \neq j$. By Definition 4.2, through each vertex in M we have exactly two paths of type A_2. Hence, we have either $lk(u_i) = C(\boldsymbol{a}, u_{i-1}, \boldsymbol{b}, c, u_{i+1}, d, e)$ or $lk(u_i) = C(\boldsymbol{a}, u_{i-1}, \boldsymbol{b}, c, z, u_{i+1}, e)$. If $lk(u_i) = C(\boldsymbol{a}, u_{i-1}, \boldsymbol{b}, c, z, u_{i+1}, e)$ then $u_{i+1} = u_r$. However, $u_i \neq u_j$ for all $1 \leq i, j \leq r$ and $i \neq j$. Hence $lk(u_i) = C(\boldsymbol{a}, u_{i-1}, \boldsymbol{b}, c, u_{i+1}, d, e)$. Thus from the cycles $lk(u_r)$ and $lk(u_i)$, $z = u_{i+1}$, $p = u_{i+2}$, $d = u_r$, and $u_i u_{i+1} u_r$ is a triangle (see Figure 16). Consider cycle R and faces incident to it. These faces

$$u_{r-3}w_i w_{r-8}, u_{r-3}w_{r-8}u_{r-4}, [u_{r-4}, w_{r-8}, w_{r-9}, u_{r-5}],$$
$$u_{r-5}w_{r-9}w_{r-10}, u_{r-5}w_{r-10}u_{r-6}, \cdots,$$
$$w_{i+3}u_{i+7}u_{i+6}, w_{i+3}u_{i+6}w_{i+2}, [w_{i+2}, u_{i+6}, u_{i+5}, w_{i+1}],$$
$$w_{i+1}u_{i+5}u_{i+4}, w_{i+1}u_{i+4}w_i$$

define a new cycle $R' = C(w_i, w_{i+1}, \cdots, w_{r-9}, w_{r-8})$ (see Figure 16). Observe that, R' is the same type of cycle as R since the faces $[u_{r-2}, u_{r-3}, w_i, w_{i-1}]$ and $[u_{i+3}, u_{i+4}, w_i, w_{i-1}]$ have a common edge $w_{i-1}w_i$, and length(R') $<$ length(R). Similarly, we consider cycle R' and repeat the process as above. Thus, we get a sequence of cycles of the same type as R. However, in this sequence, the length of cycles is gradually decreasing. After a finite number of steps, the cycle of type R may no longer exist since the map is finite. Therefore, $u_{r+1} \neq u_i$ for all $2 \leq i \leq r$. By Definition 4.2, $lk(u_r) = (\boldsymbol{z}, u_{r-1}, \boldsymbol{x}, v, u_{r+1}, w)$ implies $lk(u_{r+1}) = C(\boldsymbol{y}, u_r, \boldsymbol{x}, a, w, b, c)$ for some vertices b, w. Hence, we define a new path $L := P(u_1, \cdots, u_r) \cup P(u_r, u_{r+1})$ which is of type A_2. So, we have a path Q with length(Q) $>$ length(P). This gives a contradiction as P is maximal. Therefore, $u_{r+1} = u_1$, that is, $C(u_1, u_2, \cdots, u_r)$ is cycle of type A_2.

We use similar argument for the maximal path of type A_3. Similarly, we get an edge which defines a cycle of type A_3. This completes the proof. □

Every maximal path of type A_1, A_2 or A_3 is a cycle. In this article, we use the terminology cycle in place of maximal path since maximal paths are also cycles.

Lemma 4.2. *Let C_1 and C_2 be two cycles of type A_t for a fixed $t \in \{1, 2, 3\}$.*
(a) *If $t = 1$ and $C_1 \cap C_2 \neq \emptyset$, then $C_1 = C_2$.*
(b) *If $t = 2$ or 3 and $E(C_1) \cap E(C_2) \neq \emptyset$ then $C_1 = C_2$.*

Proof. Let $C_1 := C(u_{1,1}, u_{1,2}, \cdots, u_{1,r})$ and $C_2 := C(u_{2,1}, u_{2,2}, \cdots, u_{2,s})$ be two cycles of type A_1. If $C_1 \cap C_2 \neq \emptyset$ then $V(C_1 \cap C_2) \neq \emptyset$. Let $w \in V(C_1 \cap C_2)$. The cycles C_1 and C_2 are both well defined at the common vertex w. Let $lk(w) = C(\boldsymbol{w}_1, w_2, \boldsymbol{w}_3, w_4, w_5, w_6, w_7)$. By Definition 4.1, $w_4, w_7 \in V(C_1 \cap C_2)$. So, $P(w_4, w, w_7)$ is part of C_t for $t \in \{1, 2\}$. Let

$w = u_{1,t_1} = u_{2,t_2}$. Then $w_4 = u_{1,t_1-1} = u_{2,t_2-1}$ and $w_7 = u_{1,t_1+1} = u_{2,t_2+1}$ for some $t_1 \in \{1, \cdots, r\}$ and $t_2 \in \{1, \cdots, s\}$. We can argue for w_4 and w_7 as we did for w to get two vertices, $u_{1,t_1-2} = u_{2,t_2-2}$ and $u_{1,t_1+2} = u_{2,t_2+2}$. This process stops after a finite number of steps as r and s both are finite. Let $r < s$. Then $u_{1,1} = u_{2,I+1}, u_{1,2} = u_{2,I+2}, \cdots, u_{1,r} = u_{2,I+r}$, and $u_{1,1} = u_{2,I+r+1}$ for some $I \in \{1, \cdots, s\}$. Hence $u_{1,1} = u_{2,I+1} = u_{2,I+r+1}$. This implies that $I + 1 = I + r + 1$ and the cycle C_2 contains a cycle of length r. This gives $r = s$. Hence $C_1 = C_2$.

Let C_1, C_2 be two cycles of type A_2 and $E(C_1) \cap E(C_2) \neq \emptyset$. Let $uv \in E(C_1) \cap E(C_2)$. We proceed with the vertex u of edge uv in a similar manner to the one used for the cycles of type A_1. (We have also used this argument in Lemma 5.2.) Thus, we get $C_1 = C_2$. Similarly we argue for the case of cycles of type A_3 to show that $C_1 = C_2$. This completes the proof. □

Now, we show that the cycle of type A_t for each $t \in \{1, 2, 3\}$ is noncontractible.

Lemma 4.3. *If a cycle C is of type A_t for some $t \in \{1, 2, 3\}$ in M then C is noncontractible.*

Proof. Let C be a cycle of type A_t for a fixed $t \in \{1, 2, 3\}$ in M. We claim that the cycle C is noncontractible. Let the cycle C be of type A_1. Suppose C is contractible. Let D_C be a 2-disk bounded by the cycle C. Let f_0, f_1, and f_2 denote the number of vertices, edges, and faces of D_C, respectively. Let there be n internal vertices and m boundary vertices. So, $f_0 = n + m$, $f_1 = (5n + 3m)/2$, and $f_2 = n + (n + m)/2$ if quadrangles are incident with C, and $f_0 = n + m$, $f_1 = (5n + 4m)/2$ and $f_2 = 3n/2 + m$ if triangles are incident with C in D_C. In both the cases, $f_0 - f_1 + f_2 = 0$. This is not possible since the Euler characteristic of the 2-disk D_C is 1. Therefore, C is noncontractible.

We use a similar argument for the cycles of types A_2 and A_3. Suppose a cycle W of type A_2 is contractible. Let D_W be a 2-disk which is bounded by the cycle W. As above, calculate f_0, f_1, and f_2 to get $f_0 - f_1 + f_2 = 0$, a contradiction.

Finally, we again use a similar argument for cycles of type A_3. This completes the proof. □

Let C be a cycle of type A_t for a fixed $t \in \{1, 2, 3\}$ in M. Let S be a set of faces which are incident at u for all $u \in V(C)$. The geometric carrier $|S|$ is a cylinder since C is noncontractible. Let $S_C := |S|$. Observe that a *cylinder (or an annulus)* in M is a subcomplex of M with two boundary cycles. If the boundary cycles of a cylinder are same, it is a torus and we say that the cylinder has identical boundary components. Clearly, the S_C is a cylinder and has two boundary cycles. Let $\partial S_C = \{C_1, C_2\}$. Then $\text{length}(C) = \text{length}(C_1) = \text{length}(C_2)$ by Lemma 4.4.

Lemma 4.4. *If C is a cycle of type A_i for a fixed $i \in \{1, 2, 3\}$ such that S_C is a cylinder and $\partial S_C = \{C_1, C_2\}$, then $\text{length}(C) = \text{length}(C_1) = \text{length}(C_2)$.*

Proof. Let C be a cycle of type A_1. Let $F_1,\ F_2,\cdots,F_r$ be a sequence of faces in order which are incident with C and lie on one side of C. These faces $F_1,\ F_2,\cdots,F_r$ are also incident with C_t and lie on one side of C_t for a fixed $t\in\{1,2\}$. Without loss of generality we assume that $C_t=C_1$. Let $\widehat{F_1},\ \widehat{F_2},\cdots,\widehat{F_r}$ denote the sequence of faces incident with C that lie on the other side of C. Let $\widehat{F_i}$ be a $\widehat{d_i}$-gon. Then, we get the sequence $T_1:=\{\widehat{d_1},\widehat{d_2},\cdots,\widehat{d_r}\}$ of face-types corresponding to the sequence $\widehat{F_1},\ \widehat{F_2},\cdots,\widehat{F_r}$. Again, let $W_1,\ W_2,\cdots,W_r$ denote the sequence of faces incident with C_1 that lie on the other side of C_1. Similarly, let $T_2:=\{d_1,d_2,\cdots,d_r\}$ for the d_i-gon W_i, $i=1,2,\cdots,r$. Since $F_1,\ F_2,\cdots,F_r$ is a sequence of faces that lie on one side of both C and C_1, there exists a j such that $\widehat{d_1}=d_j$, $\widehat{d_2}=d_{j+1},\cdots,\widehat{d}_{k-j+1}=d_k,\ \widehat{d}_{k-j+2}=d_1,\cdots,\widehat{d_k}=d_{j-1}$. So, the cycle C_1 is of type A_1. Similarly we argue as above for C_2 and we get that the cycle C_2 is of type A_1. For example in Figure 15, the cycle $C=C(x_1,\cdots,x_r)$, $\partial S_C=\{C_1(w_1,\cdots,w_r),C_2(z_1,\cdots,z_r)\}$, and C, C_1, and C_2 are cycles of type A_1.

We repeat the above argument for the other two types A_j for $j\in\{2,3\}$ and get the similar results. So, the boundary cycles of S_C for a cycle C of type A_i, $i\in\{1,2,3\}$ are also of type A_i.

Suppose length$(C)\neq$ length$(C_1)\neq$ length(C_2). Let $C:=C(u_1,\ u_2,\cdots,u_r)$, $C_1:=C(v_1,\ v_2,\ \cdots,v_s)$, and $C_2:=C(w_1,w_2,\cdots w_l)$ denote three cycles of type A_1. The link $lk(u_1)$ contains the vertices v_1 and w_1. Let $P(v_1,u_1,w_1)$ be a path of type either A_2 or A_3 through u_1. Without loss of generality, we assume $r<s$ and $r\neq l$ since $r\neq s\neq l$. Now, the path $P(v_1,u_1,w_1)$ is a shortest path between v_1 and w_1 via u_1. It follows that the path $P(v_i,u_i,w_i)$ is also a shortest path of type either A_2 or A_3 between v_i and w_i via u_i. Since by assumption $r<s$, the link $lk(u_r)$ contains the vertices v_r, v_{r+j} for $j>0$ and w_r. This gives that the link of u_r is different from the link of u_{r-1}. This is a contradiction as M is a semiequivelar map. Therefore, $r=s=l$, that is, length$(C)=$ length$(C_1)=$ length(C_2).

We use a similar argument for the cycles of type A_j for $j\in\{2,3\}$. This completes the proof. □

Let C_1 and C_2 be two cycles of the same type in a semiequivelar map M on the torus. We denote a cylinder by S_{C_1,C_2} if the boundary components are C_1 and C_2. We say that the cycle C_1 is *homologous* to C_2 if S_{C_1,C_2} exists. In particular, if $C_1=C_2$ then consider $S_{C_1,C_2}=C_1=C_2$, hence C_1 is homologous to C_2.

By the above lemma, the cycle C and the boundary cycles of S_C are homologous. Let $C_1,\ C_2,\cdots,C_m$ be a list of cycles which are homologous to C in M, Then the cycles have same length. That is:

Lemma 4.5. *If $C_1,C_2,\cdots,C_m$ are homologous cycles of type A_t for a fixed $t\in\{1,2,3\}$ then* length$(C_i)=$ length(C_j) *for* $1\leq i,j\leq m$.

Proof. Let C_i be a cycle of type A_1. Then we have a cylinder S_{C_i}. Let $C_{t_1}, C_{t_2}, \cdots, C_{t_m}$ denote a sequence of cycles $\{C_1, C_2, \cdots, C_m\}$ such that $\partial S_{C_{t_j}} = \{C_{t_{j-1}}, C_{t_{j+1}}\}$ for $2 \leq j \leq (m-1)$. By Lemma 4.4, length(C_{t_1}) = length(C_{t_j}) for $j \in \{1, 2, \cdots, m\}$. Thus, length$(C_i)$ = length(C_j) for $1 \leq i, j \leq m$. We argue similarly for the cycles of type A_j for $j \in \{2, 3\}$. This completes the proof. □

A (r, s, k)-representation in M. Let $v \in V(M)$. By Lemma 4.1, there are three cycles of types A_1, A_2, A_3 through v. Let L_1, L_2, L_3 be three cycles through the vertex v where the cycle L_i is of type A_i, $i = 1, 2, 3$. Let $L_1 := C(a_1, a_2, \cdots, a_r)$. We cut the map M along the cycle L_1. We get a cylinder which is bounded by an identical cycle L_1. We denote this cylinder by N_1. We call such a cycle a *horizontal cycle* if the cycle is L_1 or homologous to L_1. Similarly, we say that a cycle is a *vertical cycle* if the cycle is L_i or homologous to L_i for $i \in \{2, 3\}$. Observe that the horizontal and vertical cycles are noncontractible by Lemma 4.3. Again, we say that a path is a *vertical path* if the path is part of a vertical cycle. We consider N_1 and make another cut in N_1 starting through the vertex v along a path $Q \subset L_3$ until reaching L_1 again for the first time where the starting adjacent face to the horizontal cycle L_1. (For example in Figure 15, $v = v_1$.) Assume that along $P := P(w_1(= a_1), w_2, \cdots, w_m) \subset L_3$ we took the second cut in N_1. Thus, we get a planar representation which is denoted by N_2.

Claim 4.6. *The representation N_2 is connected.*

Proof. Observe that the N_1 is connected as L_1 is a noncontractible cycle. Suppose N_2 is disconnected. This implies that there exists a 2-disk, namely $D_{P_1 \cup Q_1}$, which is bounded by a cycle $P_1 \cup Q_1 = P(u_j, \cdots, u_i) \cup P(a_t \cdots, a_s)$ where $P_1 \subset L_3$, $Q_1 \subset L_1$, $u_i = a_t$, and $u_j = a_s$. We consider faces which are incident with P_1 and Q_1 in $D_{P_1 \cup Q_1}$. (In this article, $\square$ denotes a quadrilateral face and $\triangle$ denotes a triangular face.) Observe that if the quadrilateral faces are incident with Q_1 and $\square_i$, $\triangle_{i,1}$, $\triangle_{i,2}$, $\square_{i+1}$, $\triangle_{i+1,1}$, $\triangle_{i+1,2}, \cdots, \triangle_{j-1,1}$, $\triangle_{j-1,2}$, $\square_j$ are incident with P_1 in $D_{P_1 \cup Q_1}$ then as in Lemma 4.3, we calculate the number of vertices f_0, edges f_1, and faces f_2 of $D_{P_1 \cup Q_1}$. So we get $f_0 - f_1 + f_2 = 0$. Similarly, if the triangular faces are incident with Q_1 then also we calculate f_0, f_1, and f_2 of $D_{P_1 \cup Q_1}$. Similarly, we get $f_0 - f_1 + f_2 = 0$ which is a contradiction in both cases as $D_{P_1 \cup Q_1}$ is 2-disk. Hence N_2 is connected. □

Observe that N_2 is planer and bounded by L_1 and Q. Let s denote the number of cycles which are homologous to L_1 along P in N_2. (For example in Figure 15, we took a second cut along the path $P(v_1, w_1, x_1, z_1, v_{k+1})$ which is part of L_3 and $s = 4$.) Now in N_2, length$(L_1) = r$ and number of horizontal cycles along P is s. We call N_2 a (r, s)-*representation.*

Observe that N_2 is bounded a by a cycle L_1, a path Q, a cycle L_1' and a path Q' where $L_1 = L_1'$ and $Q = Q'$. We say that the cycle L_1 is a *lower* *(base) horizontal cycle* and L_1' is an *upper horizontal cycle* in the (r,s)-representation. Without loss of generality, we may assume that the incident faces of L_1 are quadrangles. (For example in Figure 15, $C(v_1, \cdots, v_r)$ is a lower horizontal cycle and $C(v_{k+1}, \cdots, v_k)$ is an upper horizontal cycle.) So, we get an identification of the vertical sides of N_2 in the natural manner but the identification of the horizontal sides needs some shifting so that a vertex in the lower(base) side is identified with a vertex in the upper side. Let $L_1' = C(a_{k+1}(= w_m), \cdots, a_k)$, then a_{k+1} is the starting vertex in L_1'. In the (r,s)-representation, let $k =$: length$(P(a_1, \cdots, a_{k+1}))$ if $P(a_1, \cdots, a_{k+1})$ is part of L_1. Thus, we get a new *(r,s,k)-representation* of the (r,s)-representation. We call the boundaries of the (r,s,k)-representation the cycles and paths along which we took the cuts to construct the (r,s,k)-representation of M. (For example in Figure 15, the vertex v_{k+1} is the starting vertex of the upper horizontal cycle $C(v_{k+1}, v_{k+2}, \cdots, v_k)$ and $k =$ length$(P(v_1, v_2, \cdots, v_{k+1}))$.)

By the above construction, we see that every map of type $\{3^3, 4^2\}$ has a (r,s,k)-representation. We use $T(r,s,k)$ to represent a (r,s,k)-representation. Therefore, we have the following lemma.

Lemma 4.7. *The map of type $\{3^3, 4^2\}$ on the torus has a $T(r,s,k)$ representation.*

Let $T(r,s,k)$ be a representation of M. It has two identical upper and lower horizontal cycles of type A_1, namely, $C_{lh} := C(u_1, u_2, \cdots, u_r)$ and $C_{uh} := C(u_{k+1}, u_{k+2}, \cdots, u_k)$ in $T(r,s,k)$, respectively. (For example in Figure 11, $C_{lh} = C(v_1, v_2, \cdots, v_7)$ and $C_{uh} = C(v_4, v_5, \cdots, v_3)$.) We define a new cycle in $T(r,s,k)$ using C_{lh} and C_{uh}. The vertex $u_{k+1} \in V(C_{lh})$ is the starting vertex of C_{uh} in $T(r,s,k)$. In $T(r,s,k)$, we define two paths $Q_2 = P(u_{k+1}, \cdots, u_{k_1})$ of type A_2 and $Q_3 = P(u_{k+1}, \cdots, u_{k_2})$ of type A_3 through u_{k+1} in $T(r,s,k)$ where $u_{k_1}, u_{k_2} \in V(C_{uh})$. Clearly, the paths Q_2 and Q_3 are not homologous to horizontal cycles, that is, these are not part of cycles of type A_1. We define two edge disjoint paths $Q_2' = P(u_{k_1}, \cdots, u_{k+1})$ and $Q_3' = P(u_{k_2}, \cdots, u_{k+1})$ in C_{uh} where $Q_2' \cup Q_3' := P(u_{k_1}, \cdots, u_{k+1}, \cdots, u_{k_2}) \subset C_{uh}$ is a path in $T(r,s,k)$. Let $C_{4,1} := Q_2' \cup Q_2 := C(u_{k+1}, \cdots, u_{k_1}, \cdots, u_{k+1})$ and $C_{4,2} := Q_3' \cup Q_3 := C(u_{k+1}, \cdots, u_{k_2}, \cdots, u_{k+1})$. We define a new cycle C_4 using lengths of $C_{4,1}$ and $C_{4,2}$ as follows:

$$C_4 := \begin{cases} C_{4,1}, & \text{if length}(C_{4,1}) \le \text{length}(C_{4,2}), \\ C_{4,2}, & \text{if length}(C_{4,1}) > \text{length}(C_{4,2}). \end{cases} \tag{4.1}$$

It follows from the definition of C_4 that length$(C_4) := \min\{\text{length}(Q_2') + \text{length}(Q_2), \text{length}(Q_3') + \text{length}(Q_3)\} = \min\{k+s, (r-(s/2)-k) \pmod r) + s\}$. We say that the cycle C_4 is of type A_4 in $T(r,s,k)$. (In this section, we use $(r + \frac{s}{2} - k)$ in place of $(r + (s/2) - k) \pmod r$.) (For example in Figure 11, $C_{4,1} = Q_2' \cup Q_2 = P(v_4, v_3, v_2) \cup P(v_2, z_5, x_5, w_4, v_4)$ and $C_{4,2} = Q_3' \cup Q_3 = P(v_4, v_5, v_6, v_7) \cup P(v_7, z_4, x_4, w_4, v_4)$.) We have cycles of four types A_1, A_2, A_3, and A_4 in $T(r,s,k)$.

We show that the cycles of type A_1 have same length and the cycles of type A_2 have at most two different lengths in M.

Lemma 4.8. *In M, the cycles of type A_1 have unique length and the cycles of type A_2 have at most two different lengths.*

Proof. Let C_1 be a cycle of type A_1 in M. By the preceding argument of this section, the geometric carrier S_{C_1} of the faces which are incident with C_1 is a cylinder and $\partial S_{C_1} := \{C_2, C_0\}$ where the cycle C_2 is homologous to C_1 and length(C_1) = length(C_2). Similarly, the S_{C_2} is a cylinder and which is bounded by two homologous cycles C_1 and another, say C_3 of type A_1. Again, we consider the cycle C_3 and continue with above process. In this process, let C_i denote a cycle at the ith step such that $\partial S_{C_i} = \{C_{i-1}, C_{i+1}\}$ and length(C_{i-1}) = length(C_i) = length(C_{i+1}). Since M consists of finite number of vertices, it follows that this process stops after, say $t+1$ number of steps when the cycle C_0 appears in this process. Thus, we get C_1, C_2, $\cdots$, C_t cycles of type A_1 which are homologous to C_1 and cover all the vertices of M as the vertices of S_{C_i} are the vertices of $C_{i-1} \cup C_i \cup C_{i+1}$ for $1 \le i \le t$. It is clear from the definition that there is only one cycle of type A_1 through each vertex in M. Therefore, the cycles C_1, C_2, $\cdots$, C_t are the only cycles of type A_1 in M. Since these cycles are homologous to each other, it follows that length(C_1) = length(C_i) for $i \in \{1, \cdots, t\}$ by Lemma 4.5. This implies that the cycles of type A_1 have unique length in M.

Let L_1, L_2, L_3 be three cycles through a vertex of types A_1, A_2, A_3 respectively in M. We repeat the above process and consider L_2 in place of C_1. Similarly, we get a sequence of cycles, namely, $R_1(= L_2), R_2, \cdots, R_k$ of type A_2 which are homologous to each other. Since R_i and R_j are homologous to each other for $1 \le i, j \le k$, it follows that l_1 = length(L_2) = length (R_i) for $1 \le i \le k$ by Lemma 4.5. Again we consider the cycle L_3 and repeat above argument. Let l_2 = length(L_3). Since the cycles L_2 and L_3 are mirror images of each other, it follows that they define the same type of cycles. The map M contains the cycles of type A_2 of lengths l_1 and l_2. Therefore, the cycles of type A_2 have at most two different lengths in M. This completes the proof. □

We define admissible relations among r, s, k of $T(r,s,k)$ such that $T(r,s,k)$ represents a map after identifying their boundaries.

Lemma 4.9. *The maps of type $\{3^3, 4^2\}$ of the form $T(r,s,k)$ exist if and only if the following holds:*

(i) $s \ge 2$ *and* s *even,*
(ii) $rs \ge 10$,
(iii) $2 \le k \le r-3$ *if* $s = 2$ *and* $0 \le k \le r-1$ *if* $s \ge 4$.

Proof. Let $T(r,s,k)$ be a representation of M. In $T(r,s,k)$, the s denotes the number of horizontal cycles of type A_1. By the preceding argument of this section and Lemma 4.8, the cycles of type A_1 are homologous to each other and cover all the vertices of M with length r. So, the number of vertices n of M equals the length of the cycle of type A_1 multiplied by the number of cycles of type $A_1 = rs$.

Let C be a cycle of type A_1. By the definition of A_1, the triangles incident with C lie on one side and 4-gons lie on the other side of C. If $s = 1$ then $T(r, 1, k)$ contains one horizontal cycle, namely C. Since the incident faces of C are either triangles or 4-gons, it implies that the faces of M are either only 3-gons or 4-gons. This is a contradiction as M consists of both types of faces. So, $s \geq 2$ for all r. If s is not an even integer and C is the base horizontal cycle in $T(r, s, k)$ then the incident faces of a vertex in C are all 3-gons or 4-gons after identification of the boundaries of $T(r, s, k)$. This is a contradiction as both 3-gons and 4-gons are incident at each vertex of M. So, s is even.

If $s = 2$ and $r < 5$ then the representation $T(4, 2, k)$ has two horizontal cycles. If $C(u_1, u_2, u_3, u_4)$ and $C(u_5, u_6, u_7, u_8)$ are two horizontal cycles in $T(4, 2, k)$ then the link $lk(u_6)$ is not a cycle. So, $r \neq 4$. Similarly one can see that $r \neq 1, 2, 3$. Thus, $r \geq 5$. If $s \geq 4$ and $r < 3$ then one can see as in the above that some vertex has link which is not a cycle. By combining above all cases, $r \geq 3$ and $rs \geq 10$.

If $s = 2$ and $k \in \{1, \cdots, r-1\} \setminus \{2, \cdots, r-3\}$ then we proceed as before and get some vertex whose link is not a cycle. Thus, $s = 2$ implies $k \in \{2, \cdots, r-3\}$. Similarly we repeat the above argument for $s \geq 4$ and we get that $k \in \{1, \cdots, r-1\}$ if $s \geq 4$. This completes the proof. □

Let M_1 and M_2 be two maps of type $\{3^3, 4^2\}$ with same number of vertices on the torus and $T_i := T(r_i, s_i, k_i)$, $i \in \{1, 2\}$ denote M_i. If $a_{i,1}$ equals the length of the cycle of type A_1, $a_{i,2}$ equals the length of the cycle of type A_2, $a_{i,3}$ equals the length of the cycle of type A_3, and $a_{i,4}$ equals the length of the cycle of type A_4 in T_i then we say that $T(r_i, s_i, k_i)$ has cycle-type $(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4})$ if $a_{i,2} \leq a_{i,3}$ or $(a_{i,1}, a_{i,3}, a_{i,2}, a_{i,4})$ if $a_{i,3} < a_{i,2}$. Now, we show the following isomorphism lemma.

Lemma 4.10. *The map $M_1 \cong M_2$ if and only if they have same cycle-type.*

Proof. We first assume that the maps M_1 and M_2 have the same cycle-type. This gives $a_{1,1} = a_{2,1}$, $\{a_{1,2}, a_{1,3}\} = \{a_{2,2}, a_{2,3}\}$, and $a_{1,4} = a_{2,4}$. The maps M_i for $i \in \{1, 2\}$ have a $T_i = T(r_i, s_i, k_i)$ representation.

Claim $T_1 \cong T_2$.

First, T_1 has s_1 horizontal cycles of type A_1, namely,

$$\begin{aligned} C(1,0) &:= C(u_{0,0}, u_{0,1}, \cdots, u_{0,r_1-1}), \\ C(1,1) &:= C(u_{1,0}, u_{1,1}, \cdots, u_{1,r_1-1}), \\ &\vdots \\ C(1,s_1-1) &:= C(u_{s_1-1,0}, u_{s_1-1,1}, \cdots, u_{s_1-1,r_1-1}) \end{aligned}$$

in order. Similarly, T_2 has s_2 horizontal cycles of type A_1, namely,

$$C(2,0) := C(v_{0,0}, v_{0,1}, \cdots, v_{0,r_2-1}),$$
$$C(2,1) := C(v_{1,0}, v_{1,1}, \cdots, v_{1,r_2-1}, v_{1,0}),$$
$$\vdots$$
$$C(2, s_2-1) := C(v_{s_2-1,0}, v_{s_2-1,1}, \cdots, v_{s_2-1,r_2-1})$$

in order. Now we have the following cases.

Case 1: $(r_1, s_1, k_1) = (r_2, s_2, k_2)$.

In this case, $r_1 = r_2, s_1 = s_2, k_1 = k_2$. Define a map $f_1 : V(T(r_1, s_1, k_1)) \to V(T(r_2, s_2, k_2))$ such that $f_1(u_{t,i}) = v_{t,i}$ for $0 \le t \le s-1$ and $0 \le i \le r-1$. Observe that

$$lk(u_{t,i}) = C(\boldsymbol{u}_{t-1,i-1}, u_{t-1,i}, \boldsymbol{u}_{t-1,i+1}, u_{t,i+1}, u_{t+1,i+1}, u_{t+1,i}, u_{t,i-1})$$

is the link of the vertex $u_{t,i}$ in $T(r_i, s_i, k_i)$. By f_1, $f_1(lk(u_{t,i})) = C(f_1(\boldsymbol{u}_{t-1,i-1}), f_1(u_{t-1,i}), f_1(\boldsymbol{u}_{t-1,i+1}), f_1(u_{t,i+1}), f_1(u_{t+1,i+1}), f_1(u_{t+1,i}), f_1(u_{t,i-1})) = C(\boldsymbol{v}_{t-1,i-1}, v_{t-1,i}, \boldsymbol{v}_{t-1,i+1}, v_{t,i+1}, v_{t+1,i+1}, v_{t+1,i}, v_{t,i-1})$. So, $f_1(lk(u_{t,i})) = lk(v_{t,i})$. This implies that the map f_1 sends vertices to vertices, edges to edges, faces to faces and also, preserves incidents. Therefore, the map f_1 defines an isomorphism map between $T(r_1, s_1, k_1)$ and $T(r_2, s_2, k_2)$. Thus, $T_1 \cong T_2$ by f_1.

Case 2: $r_1 \neq r_2$.

In this case, length$(C_{1,1}) \neq$ length$(C_{2,1})$. This implies that $a_{1,1} \neq a_{2,1}$, a contradiction since $a_{1,1} = a_{2,1}$. So, $r_1 = r_2$.

Case 3: $s_1 \neq s_2$.

In this case, $n_1 = r_1 s_1 \neq r_1 s_2 = n_2$ as $r_1 = r_2$ by Case 2. This is a contradiction since $n_1 = n_2$. So, $s_1 = s_2$.

Case 4: $k_1 \neq k_2$.

By assumption, $a_{1,4} = a_{2,4}$, length$(C_{1,4})$ = length$(C_{2,4})$. This implies that $\min\{k_1 + s_1, r_1 + s_1/2 - k_1\} = \min\{k_2 + s_2, r_2 + (s_2/2) - k_2\}$. It follows that $k_1 + s_1 \neq k_2 + s_2$ since $k_1 \neq k_2$ and $s_1 = s_2$. This gives us $k_1 + s_1 = r_2 + (s_2/2) - k_2 = r_1 + (s_1/2) - k_2$ as $r_1 = r_2$ and $s_1 = s_2$. That is, $k_2 = r_1 - k_1 + (s_1/2) - s_1 = r_1 - k_1 - (s_1/2)$. In this case, identify T_2 along vertical identical boundary $P(v_{0,0}, v_{1,0}, \cdots, v_{s_2-1,0}, v_{0,k_1})$ of T_2 and then cut along the path $Q = P(v_{0,0}, v_{1,0}, v_{2,1}, \cdots, v_{s_2-1,(s_2/2)-1}, v_{0,(s_2/2)+k_2})$. Observe that the path Q is of type A_2 and through the vertex $v_{0,0}$. So, we get a new (r,s,k)-representation of M_2 and we denote it by R. This process defines the map $f_2 : V(T(r_2, s_2, k_2)) \to V(R)$ such that $f_2(v_{t,i}) = v_{t,(r_2-i+[t/2]) \pmod{r_2}}$ for $0 \le t \le s_2-1$ and $0 \le i \le r_2-1$. In R, the base horizontal cycle is $C'(2,0) := C(v_{0,0}, v_{0,r_2-1}, \cdots, v_{0,1})$ and the upper horizontal cycle is $C(v_{0,k_2+(s_2/2)}, v_{0,k_2+(s_2/2)-1}, \cdots, v_{0,k_2+(s_2/2)+1})$ where the length of the path $P(v_{0,0}, v_{0,r_2-1}, \cdots, v_{0,k_2+s_2/2})$ in $C'(2,0)$ is $r_2 - (s_2/2) - k_2$. In this process, we are not changing the length of the horizontal cycles or number of horizontal cycles which are homologous to the cycle $C'(2,0)$. So, we get $R = T(r_2, s_2, r_2 - k_2 - (s_2/2))$. Now

$r_2-k_2-(s_2/2) = r_2-(r_1-k_1-(s_1/2))-(s_2/2) = k_1$ since $r_1 = r_2, s_1 = s_2$ and $k_2 = r_1 - k_1 - (s_1/2)$. Thus, by f_1, $T(r_2, s_2, r_2 - k_2 - (s_2/2)) \cong T(r_1, s_1, k_1)$. So, $T_1 \cong T_2$.

By Cases 1–4, the claim follows. Therefore, by f_1, $M_1 \cong M_2$. Conversely, let $M_1 \cong M_2$. Then there is an isomorphism $f : V(M_1) \to V(M_2)$. Let $C_{1,j}$ be a cycle of type A_j for $j = 1, 2, 3, 4$ in M_1. By f, consider $C_{2,j} := f(C_{1,j})$ for $j = 1, 2, 3, 4$. So, length$(C_{1,j})$ = length$(f(C_{1,j}))$ = length$(C_{2,j})$ for $j = 1, 2, 3, 4$ since f is an isomorphism. Hence, M_1 and M_2 have the same cycle-type. □

Thus, we state the following corollary.

Corollary 4.11. *The following holds:*

(i) $T(r_1, s_1, k_1) \ncong T(r_2, s_2, k_2)$ *for all* $r_1 \neq r_2$,
(ii) $T(r_1, s_1, k_1) \ncong T(r_2, s_2, k_2)$ *for all* $s_1 \neq s_2$,
(iii) $T(r_1, s_1, k_1) \ncong T(r_1, s_1, k_2)$ *if* $s_1 = 2$, *and* $k_2 \in \{2, 3, \cdots, r_1 - 3\} \setminus \{k_1, r_1 - k_1 - 1\}$,
(iv) $T(r_1, s_1, k_1) \ncong T(r_1, s_1, k_2)$ *if* $s_1 \geq 4$ *and* $k_2 \in \{0, 1, \cdots, r_1 - 1\} \setminus \{k_1, r_1 - k_1 - s_1/2\}$,
(v) $T(r_1, s_1, k_1) \cong T(r_1, s_1, r_1 - k_1 - 1)$ *if* $s_1 = 2$ *and* $r_1 \geq 5$,
(vi) $T(r_1, s_1, k_1) \cong T(r_1, s_1, r_1 - s_1/2 - k_1)$ *if* $s_1 \geq 4$ *and* $r_1 \geq 3$.

Proof.

(i) If $r_1 \neq r_2$ then it follows that $a_{1,1} \neq a_{2,1}$. This implies that $T(r_1, s_1, k_1) \ncong T(r_2, s_2, k_2)$ by Lemma 4.10. So, $T(r_1, s_1, k_1) \ncong T(r_2, s_2, k_2)$ for all $r_1 \neq r_2$.
(ii) Again, $s_1 \neq s_2$ implies $r_1 \neq r_2$ since $r_1 s_1 = r_2 s_2$. This implies that $T(r_1, s_1, k_1) \ncong T(r_2, s_2, k_2)$ for all $s_1 \neq s_2$.
(iii) If $k_1 \neq k_2$, $r_1 = r_2$ and $s_1 = s_2$ then by the argument in the proof of Lemma 4.10, $T(r_1, s_1, k_1) \cong T(r_1, s_1, k_2)$ if and only if $k_2 = r_1 - k_1 - s_1/2$. So, $T(r_1, s_1, k_1) \cong T(r_1, s_1, k_2)$ if $s_1 = 2$ and $k_2 \neq r_1 - k_1 - 1$. Thus, $T(r_1, s_1, k_1) \ncong T(r_1, s_1, k_2)$ if $s_1 = 2$ and $k_2 \in \{2, 3, \cdots, r_1 - 3\} \setminus \{k_1, r_1 - k_1 - 1\}$.
(iv) From the argument in (iii), $T(r_1, 2, k_1) \cong T(r_1, 2, r_1-k_1-1)$ if $r_1 \geq 5$ (by Lemma 4.9).
(v) Again, $T(r_1, s_1, k_1) \cong T(r_1, s_1, k_2)$ if $s_1 \geq 4$ and $k_2 \neq r_1 - s_1/2 - k_1$. So, $T(r_1, s_1, k_1) \ncong T(r_1, s_1, k_2)$ if $s_1 \geq 4$ and $k_2 \in \{0, 1, \cdots, r_1 - 1\} \setminus \{k_1, r_1 - k_1 - s_1/2\}$, and $T(r_1, s_1, k_1) \cong T(r_1, s_1, r_1 - s_1/2 - k_1)$ if $s_1 \geq 4$ and $r_1 \geq 3$ (by Lemma 4.9).

□

We calculate all possible $T(r, s, k)$ representations on n vertices by Lemma 4.9. Then, we calculate lengths of the cycles of type A_i for $i \in \{1, 2, 3, 4\}$. Next, we classify all $T(r, s, k)$ representation by Lemma 4.10 up to isomorphism. So, by the Lemmas 4.9, 4.10, maps of type $\{3^3, 4^2\}$ can be classified up to isomorphism. We repeat this same argument in the Sections 5, 6, 7, 8, 9, 10, 11. We have done the above calculations for vertices up to $n \leq 22$.

TABLE 1. Maps of type $\{3^3, 4^2\}$

n	Equivalence classes	Length of cycles	$i(n)$
10	T(5, 2, 2)	(5, $\{10, 10\}$, 4)	1(10)
12	T(6, 2, 2), T(6, 2, 3)	(6, $\{6, 4\}$, 4)	3(12)
	T(3, 4, 0), T(3, 4, 1)	(3, $\{4, 12\}$, 4)	
	T(3, 4, 2)	(3, $\{12, 12\}$, 6)	
14	T(7, 2, 2), T(7, 2, 4)	(7, $\{14, 14\}$, 4)	2(14)
	T(7, 2, 3)	(7, $\{14, 14\}$, 5)	
16	T(8, 2, 2), T(8, 2, 5)	(8, $\{8, 16\}$, 4)	5(16)
	T(8, 2, 3), T(8, 2, 4)	(8, $\{16, 4\}$, 5)	
	T(4, 4, 0), T(4, 4, 2)	(4, $\{4, 8\}$, 4)	
	T(4, 4, 1)	(4, $\{16, 16\}$, 5)	
	T(4, 4, 3)	(4, $\{16, 16\}$, 7)	
18	T(9, 2, 2), T(9, 2, 6)	(9, $\{18, 6\}$, 4)	5(18)
	T(9, 2, 3), T(9, 2, 5)	(9, $\{6, 18\}$, 5)	
	T(9, 2, 4)	(9, $\{18, 18\}$, 6)	
	T(3, 6, 0)	(3, $\{6, 6\}$, 6)	
	T(3, 6, 1), T(3, 6, 2)	(3, $\{18, 18\}$, 7)	
20	T(10, 2, 2), T(10, 2, 7)	(10, $\{10, 20\}$, 4)	6(20)
	T(10, 2, 3), T(10, 2, 6)	(10, $\{20, 10\}$, 5)	
	T(10, 2, 4), T(10, 2, 5)	(10,$\{10, 4\}$, 6)	
	T(5, 4, 0), T(5, 4, 3)	(5, $\{4, 20\}$, 4)	
	T(5, 4, 1), T(5, 4, 2)	(5, $\{20, 20\}$, 5)	
	T(5, 4, 4)	(5, $\{20, 20\}$, 8)	
22	T(11, 2, 2), T(11, 2, 8)	(11, $\{22, 22\}$, 4)	4(22)
	T(11, 2, 3), T(11, 2, 7)	(11, $\{22, 22\}$, 5)	
	T(11, 2, 4), T(11, 2, 6)	(11, $\{22, 22\}$, 6)	
	T(11, 2, 5)	(11, $\{22, 22\}$, 7)	

We list the resulting objects in the form of their (r, s, k)-representation in Table 1. In Table 1, we use n to denote the number of vertices of a map. We put $T(r_1, s_1, k_1)$ and $T(r_2, s_2, k_2)$ in a single equivalence class if $T(r_1, s_1, k_1)$ and $T(r_2, s_2, k_2)$ are isomorphic. We write $(a_1, \{a_2, a_3\}, a_4)$ to denote a permutation of lengths of cycles where $a_j = \text{length}(C_{1,j})$ for $j \in \{1, 2, 3, 4\}$ and $\{a_2, a_3\}$ denotes a set of lengths of the cycles $C_{1,2}$ and $C_{1,3}$ of type A_2. We also use $i(n)$ where i denotes the number of nonisomorphic objects of type $\{3^3, 4^2\}$ on n vertices up to isomorphism. This notation is also used in Tables 2, 3, 4, 5, 6, 7, 8.

5. MAPS OF TYPE $\{3^2, 4, 3, 4\}$

Let M be a map of type $\{3^2, 4, 3, 4\}$ on the torus. Through each vertex in M there is a path as follows.

Definition. *Let $P(\cdots, u_{i-1}, u_i, u_{i+1}, \cdots)$ be a path in the edge graph of M. We say the path P is of type B_1 if*

(i) $lk(u_i) = C(\boldsymbol{a}, u_{i+1}, b, c, \boldsymbol{d}, u_{i-1}, e)$ *implies* $lk(u_{i-1}) = C(\boldsymbol{f}, g, e, u_i, \boldsymbol{c}, d, u_{i-2})$ *and* $lk(u_{i+1}) = C(\boldsymbol{e}, a, k, u_{i+2}, \boldsymbol{l}, b, u_i)$*;*

(ii) $lk(u_i) = C(\boldsymbol{e}, h, k, u_{i+1}, \boldsymbol{l}, b, u_{i-1})$ *implies* $lk(u_{i-1}) = C(\boldsymbol{h}, u_i, b, c, \boldsymbol{d}, u_{i-2}, e)$ *and* $lk(u_{i+1}) = C(\boldsymbol{s}, u_{i+2}, t, l, \boldsymbol{b}, u_i, k)$*.*

In Figure 17, $\text{lk}(u_i) = C(\boldsymbol{a}, b, c, u_{i+1}, \boldsymbol{f}, e, u_{i-1})$ and the path $P(u_{i-1}, u_i, u_{i+1})$ is part of a path of type B_1. Let P be a maximal path of type B_1. Then by the following lemma, P defines a cycle.

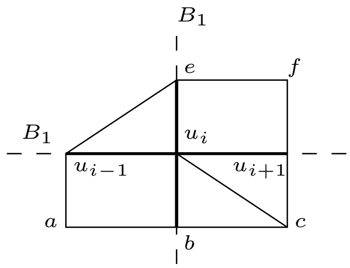

Figure 17. $lk(u_i)$

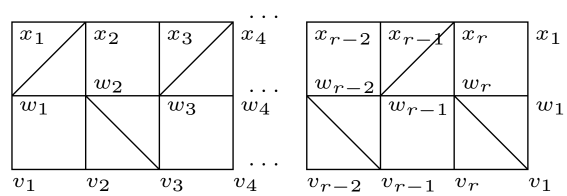

Figure 18. Cylinder

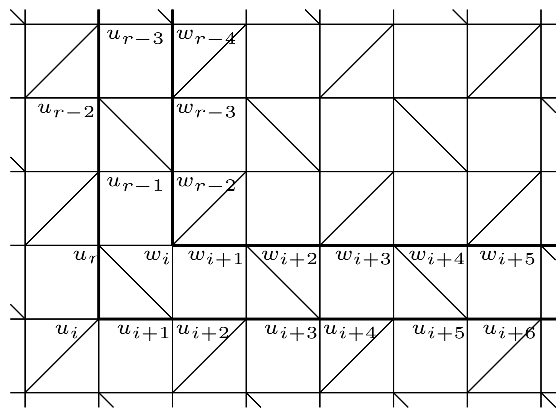

Figure 20.

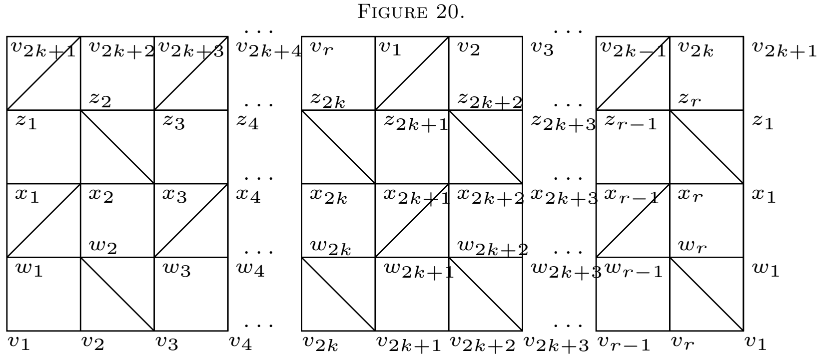

Figure 19. $T(r, 4, 2k)$

Lemma 5.1. *If P is a maximal path of type B_1 in M then there exists an edge e such that $P \cup e$ is a cycle.*

Proof. Let $P(u_1, u_2, \cdots, u_r)$ be a maximal path of type B_1 and $lk(u_r) = C(u_{r-1}, \boldsymbol{a}, b, c, d, \boldsymbol{f}, e)$. If $d = u_1$ then $C(u_1, u_2, \cdots, u_r)$ is a cycle. Suppose $d \neq u_1$ and $d = u_i$ for some $2 \leq i \leq r$. Then it defines a cycle $L = C(u_i, u_{i+1}, \cdots, u_r)$. By a similar argument to the one used in Lemma 4.1 and by Definition 5.1, either $f = u_{i+1}$, $d = u_i$, and $c = u_{i-1}$ or $c = u_{i+1}$, $d = u_i$, and $f = u_{i-1}$. In both the cases, by considering faces incident with the cycle, we get a new cycle $C(w_i, w_{i+1}, \cdots, w_{r-2})$ (see Figure 20) of the same type as L with lesser length. By induction, it is impossible similarly as in Lemma 4.1. Therefore, $d \neq u_i$ for $2 \leq i \leq r$. So, we get a path Q which is extended from P with length(P) < length(Q). This is a contradiction as P is maximal. Therefore, $d = u_1$ and the path P defines the cycle $C(u_1, u_2, \cdots, u_r)$. So, every maximal path of type B_1 is a cycle. □

In Figure 19, the path $P(v_1, v_2, \cdots, v_r)$ is of type B_1 and the cycle $C(u_1, u_2, \cdots, u_r)$ is of type B_1. Let C_1 and C_2 be two cycles of type B_1. We claim that

Lemma 5.2. *If C_1 and C_2 are two cycles of type B_1 and $E(C_1) \cap E(C_2) \neq \emptyset$ then $C_1 = C_2$.*

Proof. Let $C_1 := C(u_{1,1}, u_{1,2}, \cdots, u_{1,r})$, $C_2 := C(u_{2,1}, u_{2,2}, \cdots, u_{2,s})$, and $E(C_1) \cap E(C_2) \neq \emptyset$. Then there is an edge $e \in E(C_1 \cap C_2)$. Let $e = yx$. The cycles C_1, C_2 are both well defined at the vertices y and x. Let $lk(x) = C(\boldsymbol{a}, b, c, w, \boldsymbol{d}, e, y)$. By Definition 5.1, $w \in V(C_1 \cap C_2)$. So, the path $P(y, x, w)$ is a part of both C_1 and C_2. This implies that $y = u_{1,t_1-1} = u_{2,t_2-1}$, $x = u_{1,t_1} = u_{2,t_2}$, and $w = u_{1,t_1+1} = u_{2,t_2+1}$ for some $t_1 \in \{1, \cdots, r\}$ and $t_2 \in \{1, \cdots, s\}$. We argue similarly for the edge xw as we did for the edge e and we continue with the above process stopping after r steps. Let $t_2 > t_1$ and $t_2 - t_1 = m$ for some m. By this process we get $u_{1,1} = u_{2,m+1}$, $u_{1,2} = u_{2,m+2}, \cdots, u_{1,r} = u_{2,m+r}$, and $u_{1,1} = u_{2,m+r+1}$. This implies that $m + 1 = m + r + 1$ and $r = s$ since $u_{1,m+r+1} = u_{2,m+1}$ and C_2 is a cycle. Hence, $C_1 = C_2$. Again, let $lk(x) = C(\boldsymbol{a}, b, c, w, \boldsymbol{d}, y, e)$. By Definition 5.1, $b \in V(C_1 \cap C_2)$. We repeat above argument and get $C_1 = C_2$. Therefore, by combining above two cases, $E(C_1) \cap E(C_2) \neq \emptyset$ implies $C_1 = C_2$. This completes the proof. □

Let C be a cycle of type B_1. Similarly we argue as in Lemma 4.3 for the cycles of type B_1 and so, the cycle C is noncontractible. Let $S := \{F \in F(M) \mid V(C) \cap V(F) \neq \emptyset\}$. The cylinder $S_C = |S|$ has two boundary cycles which are either disjoint or identical by Lemma 5.3.

Lemma 5.3. *Let C be a cycle of type B_1, $\partial S_C = \{C_1, C_2\}$. If $C_1 \cap C_2 \neq \emptyset$ then $C_1 = C_2$.*

Proof. The cycle C is of type B_1 and $\partial S_C = \{C_1, C_2\}$. We argue similarly as in Lemma 4.1 for the cycle C and the cylinder S_C. So, the cycles C, C_1, and C_2 are of the same type B_1. Let $C_1 \cap C_2 \neq \emptyset$ and $u \in V(C_1 \cap C_2)$. Suppose $C_1 \cap C_2$ does not contain any edge which is incident at u. By Definition 5.1, the number of incident edges that lie on one side of the cycle C_i is two and on the other side is one at each vertex of C_i. Let d_i denote the number of incident edges which are incident at u and does not belong to $E(C_i)$. Then $d_1 + d_2 = 3$. The vertex $u \in V(C_1)$ and $u \in V(C_2)$. Since $C_1 \cap C_2$ does not contain any edge at u, the cycles C_1 and C_2 both contain two different edges which are incident at the vertex u. This implies that degree(u) $\geq (d_1 + d_2 + 4) = 7$. This is a contradiction as the degree of u is five. Therefore, $C_1 \cap C_2$ contains an edge at the vertex u. This implies that

$C_1 = C_2$ by Lemma 5.2. Again, if $C_1 \cap C_2$ contains an edge then by Lemma 5.2, $C_1 = C_2$. Therefore, boundary cycles of a cylinder are either identical or disjoint. □

We show that the cycles of type B_1 have at most two different lengths in the next Lemma 5.4.

Lemma 5.4. *In M, the cycles of type B_1 have at most two different lengths.*

Proof. We proceed as in the case of Lemma 5.3. There are two cycles of type B_1 through each vertex of M (by the definition of cycle of type B_1). Let $u \in V(M)$. Let C_1 and C_1' denote two cycles through a vertex u. Consider the cylinder S_{C_1} which is defined by the cycle C_1. Let $\partial S_{C_1} = \{C_2, C_0\}$. The cycles C_1, C_2, and C_0 are homologous to each other and length(C_1) = length(C_2) = length(C_0) by a similar argument of Lemma 4.5. Again, we proceed with the above argument for the cycle C_2 in place of C_1 and continue. In this process, let C_i denote a cycle at ith step where $\partial S_{C_i} = \{C_{i+1}, C_{i-1}\}$ and length(C_{i-1}) = length(C_i) = length(C_{i+1}). Let $k+1$ be the number of steps until process stops and the cycle C_1 appears. Thus, the cycles C_i, C_j are homologous for every $1 \leq i, j \leq k$ where $\cup_{i=1}^{k} V(C_i) = V(M)$ and l_1 =length(C_1) = length(C_i) for all $1 \leq i \leq k$. Again, we proceed with the above process for C_1' in place of C_1. Similarly, we get a sequence of homologous cycles, namely, C_1', C_2', $\cdots$, C_{k_1}' such that $\cup_{i=1}^{k_1} V(C_i') = V(M)$ and l_2 =length(C_i') for all $1 \leq i \leq k_1$. So, M contains cycles of type B_1 of at most two different lengths l_1 and l_2. This completes the proof of the lemma. □

As in Section 4, observe that every map of type $\{3^2, 4, 3, 4\}$ on the torus has a $T(r, s, k)$ representation for some r, s, k. We define admissible relations among r, s, k of $T(r, s, k)$ such that $T(r, s, k)$ represents a map after identifying their boundaries in next lemma. We omit the proof of next lemma as its argument is similar to the one used in Lemma 4.9.

Lemma 5.5. *Maps of type $\{3^2, 4, 3, 4\}$ of the form $T(r, s, k)$ exist if and only if the following holds:*

(i) $s \geq 2$, *s is even,*

(ii) $2 \mid r$,

(iii) $rs \geq 16$,

(iv)
$$\begin{cases} k \in \{2t+4 : 0 \leq t \leq (r-8)/2\} & \text{if } s = 2, \\ k \in \{2t : 0 \leq t < r/2\} & \text{if } s \geq 4. \end{cases}$$

Let $T_i = T(r_i, s_i, k_i)$, $i \in \{1, 2\}$ denote M_i of type $\{3^2, 4, 3, 4\}$ on the torus with n_i vertices and $n_1 = n_2$. Let $C_{i,1}$ and $C_{i,2}$ be two nonhomologous cycles of type B_1 in M_i for $i = 1, 2$ and $a_{i,j}$ = length($C_{i,j}$).

Lemma 5.6. *The map $M_1 \cong M_2$ if and only if $(a_{1,1}, a_{1,2}) = (a_{2,t_1}, a_{2,t_2})$ for $t_1 \neq t_2 \in \{1, 2\}$.*

Proof. We first assume that $(a_{1,1}, a_{1,2}) = (a_{2,t_1}, a_{2,t_2})$ where $t_1, t_2 \in \{1, 2\}$ and $t_1 \neq t_2$. This implies that $\{a_{1,1}, a_{1,2}\} = \{a_{2,1}, a_{2,2}\}$.

Claim. $T_1 \cong T_2$.

From the definition of a (r_i, s_i, k_i)-representation, T_1 has s_1 horizontal cycles of type B_1:

$$\begin{aligned} C(1,0) &:= C(u_{0,0}, u_{0,1}, \cdots, u_{0,r_1-1}),\\ C(1,1) &:= C(u_{1,0}, u_{1,1}, \cdots, u_{1,r_1-1}),\\ &\vdots\\ C(1,s_1-1) &:= C(u_{s_1-1,0}, u_{s_1-1,1}, \cdots, u_{s_1-1,r_1-1}), \end{aligned}$$

and T_2 has s_2 horizontal cycles of type B_1:

$$\begin{aligned} C(2,0) &:= C(v_{0,0}, v_{0,1}, \cdots, v_{0,r_2-1}),\\ C(2,1) &:= C(v_{1,0}, v_{1,1}, \cdots, v_{1,r_2-1}, v_{1,0}),\\ &\vdots\\ C(2,s_2-1) &:= C(v_{s_2-1,0}, v_{s_2-1,1}, \cdots, v_{s_2-1,r_2-1}). \end{aligned}$$

Case 1: $(r_1, s_1, k_1) = (r_2, s_2, k_2)$ (i.e., $r_1 = r_2, s_1 = s_2, k_1 = k_2$).

Similar to the proof of the Lemma 4.10, we define an isomorphism $f_1 : V(T(r_1, s_1, k_1)) \to V(T(r_2, s_2, k_2))$ such that $f(u_{t,i}) = v_{t,i}$ for $0 \leq t \leq s_1 - 1$ and $0 \leq i \leq r_1 - 1$. So, $T_1 \cong T_2$ by f_1.

Case 2: $r_1 = r_2, s_1 = s_2, k_1 \neq k_2$.

Since $r_1 = r_2$, it implies that the vertical cycles in T_1 and T_2 have same length.

We define a cycle in T_1 as in equation (1) of Section 4. Let C_{lh} denote the base horizontal cycle and C_{uh} denote the upper horizontal cycle in T_2. Let Q be a path through u_{k_1+1} of type B_1 and not homologous to C_{lh}. Let Q' and Q'' denote two edge disjoint paths in C_{uh} such that $C_{uh} = Q' \cup Q''$. Hence as in equation (1), we define a new cycle $C_3(1)$ using the above paths in T_1 of minimum length. Similarly, there is a cycle $C_3(2)$ as $C_3(1)$ in T_2. Since $\{a_{1,1}, a_{1,2}\} = \{a_{2,1}, a_{2,2}\}$ and $r_1 = r_2$, it follows that length$(C_3(1)) =$ length$(C_3(2))$.

Since length$(C_3(1)) =$ length$(C_3(2))$, this implies that $\min\{s_1+k_1, r_1+s_1-k_1\} = \min\{s_2+k_2, r_2+s_2-k_2\}$. It follows that $r_1+s_1-k_1 = s_2+k_2$ since $k_1 \neq k_2$. So, $k_2 = r_1 - k_1$ as $s_1 = s_2$. We proceed similarly to Lemma 4.10. In this process, identify $T(r_2, s_2, k_2)$ along the vertical boundary and cut along a path $Q := P(v_{0,i}, v_{1,i}, \cdots, v_{s_2-1,i}, v_{0,i+k_2})$ for some even $0 \leq i \leq r_1 - 1$. Thus, we get a new representation R of M_2 with a map $f_2 : V(T(r_2, s_2, k_2)) \to V(R)$ such that $f_2(v_{t,i'}) = v_{t,(i+r_2-i') \pmod{r_2}}$ for $0 \leq t \leq s_2 - 1$ and $0 \leq i' \leq r_2 - 1$. Clearly, f_2

maps the cycle $C(2,t) := C(v_{t,0}, v_{t,1}, \cdots, v_{t,r_2-1})$ to the cycle $C'(2,t) := C(v_{t,i}, v_{t,i-1}, \cdots, v_{t,r_2-1}, v_{t,0}, v_{t,1}, \cdots, v_{t,i+1})$. Since the path

$$Q_1 := P(v_{0,i}, v_{0,i-1}, \cdots, v_{0,i+k_2}) \subset C'(2,0) := C(v_{0,i}, v_{0,i-1}, \cdots, v_{0,i+1})$$

and length$(Q_1) = i + r_2 - k_2 - i = r_2 - k_2$, it follows that R has s_2 number of horizontal cycles of length r_2 and the cycle of type B_2 has length $r_2 - k_2$ as length$(Q_1) = r_2 - k_2$. Observe that the R is not of type $T(r,s,k)$ for some r, s, k because the sequence of the incident faces in R of the base horizontal cycle $C' = C(v_{0,i}, v_{0,i-1}, \cdots, v_{0,r_2-1}, v_{0,0}, v_{0,1}, \cdots, v_{0,i+1})$ starts with triangular faces. (For example, R in Figure 12 does not follow the definition of a (r,s,k)-representation since the sequence

$$v_1v_2w_1, w_1w_2v_2, [v_2, v_3, w_3, w_2], v_3v_4w_3, v_4w_3w_4,$$
$$[v_4, v_5, w_5, w_4], v_5v_6w_4, v_6w_5w_6, [v_1, v_6, w_6, w_1]$$

starts with the triangular faces $v_1v_2w_1, w_1w_2v_2$.) In this case, if $C'(2,0)$, $C'(2,1), \cdots, C'(2, s_2-1)$ denotes a sequence of horizontal cycles in R, then we identify R along $C(v_{0,i}, v_{0,i-1}, \cdots, v_{0,i+1})$ and cut along $C(v_{1,i}, v_{1,i-1}, v_{1,i-2}, \cdots, v_{1,i+2}, v_{1,i+1})$. Thus, we get a new representation of M_2, say R' where $C'(2,1) := C(v_{1,i}, v_{1,i-1}, \cdots, v_{1,i+1})$ denotes the base horizontal cycle in R'. In this process,

$$C'(2,1) \to C'(2,0), C'(2,2) \to C'(2,1), C'(2,3) \to C'(2,2), \cdots,$$
$$C'(2,s_2-1) \to C'(2,s_2-2), C'(2,0) \to C'(2,s_2-1).$$

This process defines a map $f_3 : R \to R'$ such that $f_3(C'(2,t)) = C'(2, t-1 \pmod{s_2}))$ for $0 \le t \le s_2 - 1$. Now observe that $C'(2,1), C'(2,2), C'(2,3), \cdots, C'(2,s_2-1), C'(2,0)$ denotes the sequence of horizontal cycles in R'. (Figure 5 is a $R' = T(6,4,2)$ representation which is defined from R in Figure 12. In Figure 12, we cut R along the cycle $C(v_1, v_2, \cdots, v_6)$ and identify along $C(w_1, w_2, \cdots, w_6)$. Hence, we get a representation R' in Figure 5.) In the above process, we are redefining R to a desired representation R'. The length of the horizontal cycles of type B_1 remain unchanged as we are only changing the order of the horizontal cycles. So, R' has a well defined $T(r_2, s_2, r_2-k_2)$ representation. Thus, $T(r_2, s_2, r_2-k_2) = T(r_1, s_1, k_1)$ since $r_1 = r_2, s_1 = s_2, k_2 = r_1 - k_1$. So, M_2 has a $T(r_1, s_1, k_1)$ representation. Therefore, by f_1, $T_1 \cong T_2$.

Case 3: $r_1 \neq r_2$.

This case implies that $a_{1,1} \neq a_{2,1}$. By assumption $\{a_{1,1}, a_{1,2}\} = \{a_{2,1}, a_{2,2}\}$, we get that $a_{1,1} = a_{2,2}$. We identify boundaries of $T(r_2, s_2, k_2)$ and cut along the hole cycle $C(2,2)$ in place of $C(2,1)$. Then take another cut along $C(2,1)$ until we reaching $C(2,2)$ again for the first time. Hence, we get $r_1 = \text{length}(C(1,1)) = \text{length}(C(2,2)) = r_2$. Thus, $r_1s_1 = r_2s_2$ implies that $s_1 = s_2$. Since $r_1 = r_2$ and $s_1 = s_2$, this implies that we are in Case 2. Similarly to Case 2, we define maps f_1, f_2, and f_3. Thus, by f_1, f_2, and f_3, $T_1 \cong T_2$.

Case 4: $s_1 \neq s_2$.

TABLE 2. Maps of type $\{3^2, 4, 3, 4\}$

n	Equivalence classes	Length of cycles	$i(n)$
16	T(8, 2, 4), T(4, 4, 2)	(8, 4)	2(16)
	T(4, 4, 0)	(4, 4)	
20	T(10, 2, 4), T(10, 2, 6)	(10, 10)	1(20)
24	T(12, 2, 4), T(12, 2, 8), T(6, 4, 2), T(6, 4, 4)	(12, 6)	3(24)
	T(12, 2, 6), T(4, 6, 2)	(4, 12)	
	T(6, 4 ,0), T(4, 6, 0)	(4, 6)	
28	T(14, 2, 4), T(14, 2, 10) T(14, 2, 6), T(14, 2, 8)	(14, 14)	1(28)
32	T(16, 2, 4), T(16, 2, 12), T(8, 4, 2), T(8, 4, 6)	(16, 8)	5(32)
	T(16, 2, 6), T(16, 2, 10)	(16, 16)	
	T(16, 2, 8), T(4, 8, 2)	(4, 16)	
	T(8, 4, 4)	(8, 8)	
	T(8, 4, 0), T(4, 8, 0)	(4, 8)	

This case implies that $n_1 = r_1 s_1 \neq r_2 s_2 = n_2$ if $r_1 = r_2$. This is a contradiction as $n_1 = n_2$. If $r_1 \neq r_2$ then we are in Case 3. By combining Case 2 and 3, we get an isomorphism f_1 if $s_1 \neq s_2$. Again, let $k_1 \neq k_2$. Here, we have the following cases. If $r_1 \neq r_2$ then we are in Case 3. Similarly, if $s_1 \neq s_2$ then we are in Case 4. If $r_1 = r_2$, $s_1 = s_2$ and $k_1 \neq k_2$ then we are in Case 2. Thus, $(a_{1,1}, a_{1,2}) = (a_{2,t_1}, a_{2,t_2})$ where $t_1, t_2 \in \{1, 2\}$ defines $T_1 \cong T_2$.

By Cases 1, 2, 3, and 4, the claim follows. Thus, by f_1, $M_1 \cong M_2$.

Conversely, let $M_1 \cong M_2$. Similarly to Lemma 4.10, let $f : V(M_1) \to V(M_2)$ such that $C_{2,j} := f(C_{1,j})$ for $j \in \{1, 2\}$. So, $\{a_{1,1}, a_{1,2}\} = \{a_{2,1}, a_{2,2}\}$. Thus, this implies that $(a_{1,1}, a_{1,2}) = (a_{2,t_1}, a_{2,t_2})$ where $t_1 \neq t_2 \in \{1, 2\}$. □

As in Section 4, by Lemmas 5.5 and 5.6, the maps of type $\{3^2, 4, 3, 4\}$ can be classified up to isomorphism. We have done calculation for the vertices up to 32 and listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 2.

6. MAPS OF TYPE $\{3, 6, 3, 6\}$

Let M be a semiequivelar map of type $\{3, 6, 3, 6\}$ on the torus. We define a path in M as follows. Through each vertex in M there are two paths of type X_1 as shown in Figure 20.

Definition. *Let $Q_1 := P(\cdots, u_{i-1}, u_i, u_{i+1}, \cdots)$ be a path in the edge graph of M. Let $A(v)$ denote a set of incident edges through v in M. We say the path Q_1 of type X_1 if $A(u_i) \setminus E(Q_1)$ is a set of two edges where one edge lies on one side and remaining one lies on the other side of $P(u_{i-1}, u_i, u_{i+1})$.*

For example in Figure 20, $P(u_{i-1}, u_i, u_{i+1})$ and $P(b, u_i, g)$ are two paths through u_i and both are part of paths of type X_1. Let $P(u_1, \cdots, u_r)$ be a maximal path of type X_1 in M. Now consider a vertex u_r, $lk(u_r)$ and argue similarly as in Lemma 4.1. We get an edge $u_1 u_r$ such that $P \cup \{u_1 u_r\}$ is a cycle of type X_1. Thus, there are two cycles of type X_1 through a vertex. Let $L_1(v)$ and $L_2(v)$ be two cycles of type X_1 through a vertex v.

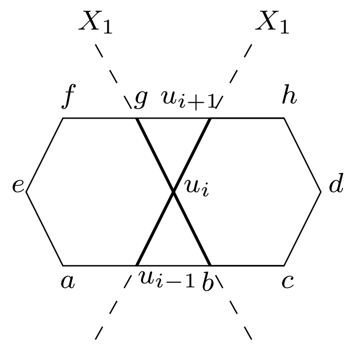

FIGURE 20. $lk(u_i)$

We proceed with a similar argument from Section 4 and get a connected $T(r,s,k)$ representation of M. In this process, we cut M along the cycles $L_1(v)$ and $L_2(v)$ where we take second cut along the cycle L_2 and the starting adjacent face to the base horizontal cycle L_1 is a 3-gon. Thus, every map M has a $T(r,s,k)$ representation. Figure 10 is an example of $T(8,2,6)$ representation of a map with 24 vertices on the torus.

Now, we show that map of type $\{3,6,3,6\}$ contains three cycles of type X_1 up to homologous.

Lemma 6.1. *The map M contains at most three cycles of type X_1 of different lengths.*

Proof. Let $\triangle(u,v,w)$ be a 3-gon in M. Then $\triangle(u,v,w)$ has three edges $e_1 = uv$, $e_2 = vw$, and $e_3 = uw$. By the definition of a cycle of type X_1, M contains at least three cycles, say C_1, C_2, and C_3 where C_i contains edge e_i for $i \in \{1,2,3\}$ and C_i does not contain e_j for $j \neq i$. Since C_i does not contain e_j for $j \neq i$, cycles are not identical. Again, since the cycles are not identical and $V(C_i) \cap C(C_j)$ is a vertex of $\triangle$ for $i \neq j$, the cycles are not homologous. (In Figure 10, $v_1v_2v_{1,2,16,9}$ denotes a face, and the cycles which are of type X_1 and contains the edges v_1v_2, $v_1v_{1,2,16,9}$ and $v_{1,2,16,9}v_2$ are namely, $L_1 = C(v_1, v_2, \cdots, v_8)$, $L_2 = C(v_1, v_{1,2,16,9}, v_9, v_{9,10,6,7}, \cdots, v_{11,12,8,1})$, and $L_3 = C(v_2, v_{1,2,16,9}, v_{16}, v_{15,16,4,5}, \cdots, v_{13,14,2,3})$, respectively. The cycles L_1, L_2, and L_3 in Figure 10 are not homologous to each other.) Let $\triangle_1$ be a 3-gon in $T(r,s,k)$ and $\triangle \neq \triangle_1$. Observe that there is a cycle of type X_1 through an edge $\triangle_1$ and homologous to C_i for some i. That is, there is a cylinder which is bounded by two cycles of type X_1 and containing edges of $\triangle$ and $\triangle_1$; this is true for any 3-gon in $T(r,s,k)$. We proceed as in the case of Lemma 4.5 and thus, the homologous cycles of type X_1 have the same length. Thus, there are three different cycles C_1, C_2, C_3 of type X_1 up to homologous in M. So, the map M contains at most three cycles of type X_1 of different lengths. (In Figure 10, consider the face $v_{11}v_{12}v_{11,12,8,1}$ and the cycles of type X_1 which contain the edges $v_{11}v_{12}$, $v_{11}v_{11,12,8,1}$, and $v_{12}v_{11,12,8,1}$ are $L_1' = C(v_9, v_{10}, \cdots, v_{16})$, $L_2' = C(v_{11}, v_{11,12,8,1}, v_1, v_{1,2,16,9}, \cdots, v_{3,4,10,11})$, and $L'_3 = C(v_{12}, v_{11,12,8,1}, v_8, v_{7,8,14,15}, \cdots, v_{5,6,12,13})$, respectively. The cycles which contain the edges v_1v_2 and $v_{11}v_{12}$ are L_1 and L'_1, respectively, and the cycles L_1 and L'_1 are homologous. The cycles which contain $v_1v_{1,2,16,9}$ and $v_{11}v_{11,12,8,1}$ are L_2 and L'_2 respectively and $C_2 = C'_2$. Also, the cycles which contain $v_2v_{1,2,16,9}$ and $v_{12}v_{11,12,8,1}$ are L_3 and L'_3, respectively, and $L_3 = L'_3$.) □

We define admissible relations among r, s, k of $T(r, s, k)$ in the next lemma. In this lemma we omit some cases since similar cases are discussed in the previous sections.

Lemma 6.2. *The maps of type $\{3, 6, 3, 6\}$ of the form $T(r, s, k)$ exist if and only if the following holds:*

(i) $s \geq 1$,
(ii) $2 \mid r$,
(iii) *there are* $3rs/2 \geq 21$ *vertices of* $T(r, s, k)$,
(iv)

$$r \geq \begin{cases} 14 & \text{if } s = 1, \\ 8 & \text{if } s = 2, \\ 6 & \text{if } s \geq 3, \end{cases}$$

(v)

$$\begin{cases} k \in \{2t + 6 \colon 0 \leq t \leq \frac{r-10}{2}\} \setminus \{2(\frac{r-10}{4}) + 6\} & \text{if } s = 1, \\ k \in \{2t + 6 \colon 0 \leq t \leq \frac{r-8}{2}\} & \text{if } s = 2 \\ k \in \{2t \colon 0 \leq t < \frac{r}{2}\} & \text{if } s \geq 3. \end{cases}$$

Proof. Let

$$\begin{gathered} C_0(u_{0,0}, u_{0,1}, \cdots, u_{0,r-1}), \\ C_1(u_{1,0}, u_{1,1}, \cdots, u_{1,r-1}), \\ \vdots \\ C_{s-1}(u_{s-1,0}, u_{s-1,1}, \cdots, u_{s-1,r-1}) \end{gathered}$$

be horizontal cycles of type X_1 in $T(r, s, k)$. By the definition of $T(r, s, k)$, $T(r, s, k)$ contains s horizontal cycles of type X_1. Observe that the number of adjacent vertices which lie on one side of a horizontal cycle and do not belong to any horizontal cycles is $r/2$. So, the total number of vertices in $T(r, s, k)$ is $(r + r/2)s$. This implies that $n = (r + r/2)s = 3rs/2$.

By Euler's formula, the number of 6-gons in $T(r, s, k)$ is $2n/6$ and it is an integer. This implies that $6 \mid 2n$. Hence $3 \mid 3rs/2$ as $n = 3rs/2$. Thus, $2 \mid r$ if $s = 1$. Again, if $s \geq 2$ and $2 \nmid r$, the link $lk(u_1)$ is not of type $\{3, 6, 3, 6\}$, which is a contradiction. Therefore, $2 \mid r$ for all $s \geq 1$.

Let $s = 1$. If $r < 14$ then $r \in \{2, 4, 6, 8, 10, 12\}$ and there is a vertex in $T(r, s, k)$ whose link is not a cycle. So, $r \geq 14$ if $s = 1$. Similarly, we get that $r \geq 8$ if $s = 2$ and $r \geq 6$ if $s \geq 3$. So, $3rs/2 \geq 21$.

If $s = 1$ and

$$k \in \{t\colon 0 \le t \le r-1\} \setminus \left(\left\{ 2t + 6\colon 0 \le t \le \frac{r-10}{2} \right\} \setminus \left\{ 2\frac{r-10}{4} + 6 \right\} \right)$$

then similar to the above we get some vertex whose link is not a cycle. We repeat the same argument as above for other two cases when $s = 2$ and $s \ge 3$. □

Lemma 6.3. *Let M_i, $i = 1, 2$, be maps of type $\{3, 6, 3, 6\}$ on n_i vertices and $n_1 = n_2$. Let $C_{i,j}$, $j = 1, 2, 3$, denote cycles which are of type X_1 and nonhomologous in $T_i = T(r_i, s_i, k_i)$. Let $a_{i,j}$ = length$(C_{i,j})$ for $i = 1, 2$ and $j = 1, 2, 3$. Then $M_1 \cong M_2$ if and only if $(a_{1,1}, a_{1,2}, a_{1,3}) = (a_{2,t_1}, a_{2,t_2}, a_{2,t_3})$ for $t_1 \ne t_2 \ne t_3 \in \{1, 2, 3\}$.*

Proof. We first assume that $(a_{1,1}, a_{1,2}, a_{1,3}) = (a_{2,t_1}, a_{2,t_2}, a_{2,t_3})$ where $t_i \in \{1, 2, 3\}$ and $t_i \ne t_j$. This implies that $\{a_{1,1}, a_{1,2}, a_{1,3}\} = \{a_{2,1}, a_{2,2}, a_{2,3}\}$.

Claim. $T_1 \cong T_2$.

Case 1: $(r_1, s_1, k_1) = (r_2, s_2, k_2)$.

In this case, $T(r_1, s_1, k_1) = T(r_2, s_2, k_2) = T(r, s, k)$. Let

$$C(1,0) := C(u_{0,0}, u_{0,1}, \cdots, u_{0,r-1}),$$
$$C(1,1) := C(u_{1,0}, u_{1,1}, \cdots u_{1,r-1}),$$
$$\vdots$$
$$C(1,s) := C(u_{s,0}, u_{s,1}, \cdots, u_{s,r-1}\}$$

denote a sequence of horizontal cycles of type X_1 in T_1. Let $G_1(t, t+1) := \{w_{t,0}, w_{t,1}, \cdots, w_{t,r-2/2})$ be a set of vertices such that $w_{t,i}$ is adjacent to both $u_{t,2i}$ and $u_{t+1,2i}$ in T_1 and does not belong to both $C(1,t)$ and $C(1,t+1)$ for $0 \le t \le s$. (For example in Figure 10, $G_1(0,1) = \{v_{1,2,16,9}, v_{3,4,10,11}, v_{5,6,12,13}, v_{7,8,14,15}\}$ where

$$x_{0,0} = v_{1,2,16,9}, x_{0,1} = v_{3,4,10,11}, x_{0,2} = v_{5,6,12,13}, x_{0,3} = v_{7,8,14,15}).$$

Similarly, let

$$C(2,0) := C(v_{0,0}, v_{0,1}, \cdots, v_{0,r-1}),$$
$$C(2,1) := C(v_{1,0}, v_{1,1}, \cdots v_{1,r-1}),$$
$$\vdots$$
$$C(2,s) := C(v_{s,0}, v_{s,1}, \cdots v_{s,r-1})$$

denote a sequence of horizontal cycles of type X_1 in T_2 and let $G_2(t, t+1) := \{x_{t,0}, x_{t,1}, \cdots, x_{t,r-2/2}\}$ be a set of vertices such that the vertex $x_{t,i}$ is adjacent to both $v_{t,2i}$ and $v_{t+1,2i}$ in T_2 for $0 \le t \le s$. Now define an isomorphism $f : V(T(r_1, s_1, k_1)) \to V(T(r_2, s_2, k_2))$ such that $f(u_{t,i}) = v_{t,i}$ for all $0 \le i \le r-1$, $0 \le t \le s-1$ and $f(w_{t,i}) = x_{t,i}$ for the vertices of $G_1(t, t+1)$ and $G_2(t, t+1)$ for all $0 \le t \le s-1$. By f,

TABLE 3. Maps of type $\{3, 6, 3, 6\}$

n	Equivalence classes	Length of cycles	$i(n)$
21	T(14, 1, 6), T(14, 1, 10)	(14, 14, 14)	1(21)
24	T(16, 1, 6), T(16, 1, 12)	(16, 16, 8)	2(24)
	T(8, 2, 6)		
	T(16, 1, 8), T(16 , 1, 10)	(16, 16, 4)	
27	T(18, 1, 6), T(18, 1, 8)	(18, 18, 6)	2(27)
	T(18, 1, 12), T(18, 1, 14)		
	T(6, 3, 2), T(6, 3, 4)		
	T(6, 3, 0)	(6, 6, 6)	
30	T(20, 1, 6), T(20, 1, 8)	(20, 20, 10)	2(30)
	T(20, 1, 14), T(20, 1, 16)		
	T(10, 2, 2), T(10, 2, 6)		
	T(10, 2, 8)		
	T(20, 1, 10), T(20, 1, 12)	(20, 4, 10)	
	T(10, 2, 0), T(10, 2, 4)		

the link $lk(u_{t,i})$ maps to the link $lk(v_{t,i})$ and $lk(w_{t,i})$ maps to $lk(x_{t,i})$, therefore $T_1 \cong T_2$.

Case 2: $(r_1, s_1, k_1) \neq (r_2, s_2, k_2)$.

If $r_1 \neq r_2$, we identify boundaries of $T(r_2, s_2, k_2)$ and cut M_2 along cycle of length r_1 then make another cut along a cycle of type X_1 to get a (r, s, k)-representation. Thus we get a new $T(r'_2, s'_2, k'_2)$ representation of M_2. This implies that $r_1 = r'_2$ and $s_1 = s'_2$ as $n_1 = 3r_1s_1/2 = 3r'_2s'_2/2 = n_2$. By this process, we get a new representation $T(r_1, s_1, k'_3)$ of M_2. If $k_1 = k'_3$ then $M_1 \cong M_2$ by f in Case 1. If $k_1 \neq k'_3$, similar to Lemma 5.6, we make a cut along a path which is homologous to the vertical boundary path and identify along the boundary path. Thus, we get an another representation $T(r_1, s_1, k''_3)$ of M_2 and $k_1 = k''_3$. So, the M_2 has a $T(r_1, s_1, k_1)$ representation since $k_1 = k''_3$. Therefore, there exists f and $T_1 \cong T_2$ by f.

This completes the Claim and by f, $M_1 \cong M_2$.

Conversely, let $M_1 \cong M_2$. We proceed as in the converse part of Lemma 5.6 and we get $\{a_{1,1}, a_{1,2}, a_{1,3}\} = \{a_{2,1}, a_{2,2}, a_{2,3}\}$. That is, $(a_{1,1}, a_{1,2}, a_{1,3}) = (a_{2,t_1}, a_{2,t_2}, a_{2,t_3})$ for $t_1 \neq t_2 \neq t_3 \in \{1, 2, 3\}$. ☐

As in Section 4, by Lemmas 6.2 and 6.3, the maps of type $\{3, 6, 3, 6\}$ can be classified up to isomorphism on different number of vertices. We have done the calculation for vertices up to 30. We have listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 3.

7. MAPS OF TYPE $\{3, 12^2\}$

Let M be a semiequivelar map of type $\{3, 12^2\}$ on the torus. We define a fixed type of path G_1 in the edge graph of M as shown in Figure 21. Let

$Q(i) := P(u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4})$ be a path in M where

$$\begin{aligned}
lk(u_i) &= C(u_{i-1}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{g'}, \boldsymbol{u}_{i+2},\\
&\quad u_{i+1}, u, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \mathbf{a'}, \mathbf{u}_{i-3}, \mathbf{u}_{i-2}),\\
lk(u_{i+1}) &= C(u_i, \boldsymbol{u}_{i-1}, \boldsymbol{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{g'}, u_{i+2},\\
&\quad \boldsymbol{u}_{i+3}, \boldsymbol{u}_{i+4}, \mathbf{o'}, \boldsymbol{o}, \mathbf{p}, \mathbf{q}, \boldsymbol{r}, \mathbf{s}, \mathbf{t}, u),\\
lk(u_{i+2}) &= C(u_{i+1}, \boldsymbol{u}_i, \boldsymbol{u}_{i-1}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, g',\\
&\quad u_{i+3}, \boldsymbol{u}_{i+4}, \mathbf{o'}, \boldsymbol{o}, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}),\\
lk(u_{i+3}) &= C(u_{i+2}, g', \mathbf{g}, \boldsymbol{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n},\\
&\quad \boldsymbol{u}_{i+6}, \boldsymbol{u}_{i+5}, u_{i+4}, \mathbf{o'}, \boldsymbol{o}, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}\boldsymbol{u}_{i+1}),\\
lk(u_{i+4}) &= C(u_{i+3}, \mathbf{g'}, \boldsymbol{g}, \mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n},\\
&\quad \boldsymbol{u}_{i+6}, u_{i+5}, o', \mathbf{o}, \boldsymbol{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \boldsymbol{u}_{i+1}, \boldsymbol{u}_{i+2}).
\end{aligned}$$

Definition. *Let $R_1 := P(\cdots, v_{i-1}, v_i, v_{i+1}, \cdots)$ be a path in the edge graph of M. We say R_1 is of type G_1 if $L_1 := P(v_t, v_{t+1}, v_{t+2}, v_{t+3}, v_{t+4})$ is a subpath of R_1 or L_1 is part of an extended path of R_1, then either $L_1 \mapsto Q(i)$ by $v_j \mapsto u_j$, $L_1 \mapsto Q(i+1)$ by $v_j \mapsto u_{j+1}$, $L_1 \mapsto Q(i+2)$ by $v_j \mapsto u_{j+2}$, or $L_1 \mapsto Q(i+3)$ by $v_j \mapsto u_{j+3}$ for $j \in \{t, t+1, t+2, t+3, t+4\}$.*

Definition. *Let $R_2 := P(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots)$ be a path in the edge graph of M. We say R_2 is of type G_1' if $L_2 := P(x_t, x_{t+1}, x_{t+2}, x_{t+3}, x_{t+4})$ is a subpath of R_2 or L_2 is part of the extended path of R_2, then either $L_2 \mapsto Q(i)$ by $x_j \mapsto u_{2t+4-j}$, $L_2 \mapsto Q(i+1)$ by $x_j \mapsto u_{2t+4-j}$, $L_2 \mapsto Q(i+2)$ by $x_j \mapsto u_{2t+4-j}$, or $L_1 \mapsto Q(i+3)$ by $v_j \mapsto u_{2t+4-j}$ for $j \in \{t, t+1, t+2, t+3, t+4\}$.*

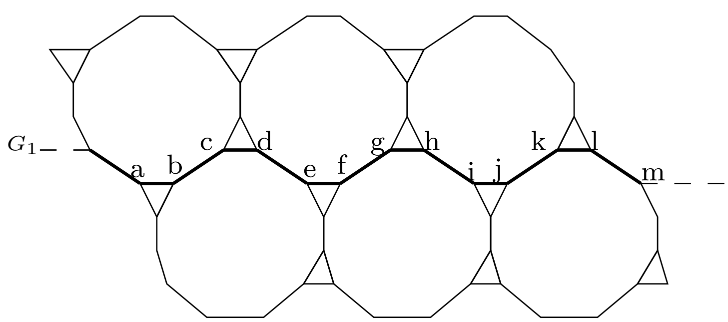

FIGURE 21. Cycle of type G_1

We use a similar argument from Lemma 4.1 for the path of types G_1 and G_1' hence every maximal path of types G_1 and G_1' is a cycle and non-contractible (by a similar argument from Lemma 4.3). Observe that the cycles of type G_1 and G_1' are mirror images of each other. Hence these types define the same type of cycle. (A similar argument is provided in detail in Section 8 for the type $\{3^4, 6\}$.) Clearly, there are two cycles of type G_1 through each vertex of M. Let uvw be a 3-gon in M. Let $L_1(u, uw)$,

$L_2(w, wv)$, and $L_3(v, vu)$ denote three cycles through u, w, and v, respectively, where $L_1(u, uw)$ contains the edge uw, $L_2(w, wv)$ contains the edge wv, and $L_3(v, vu)$ contains the edge vu. We repeat a similar argument from Section 4 and define a $T(r, s, k)$ representation of the map M for some r, s, k. In the process, we take the first cut along $L_1(u, uw)$, and then the second cut along $L_2(w, wv)$ where the starting adjacent face to the horizontal base cycle $L_1(u, uw)$ is a 12-gon. Let $\text{length}(L_1(u, uw)) = r$, s denote the number of homologous cycles of $L_1(u, uw)$ of type G_1, and k denote the distance of the starting vertex of upper horizontal cycle from the starting vertex w in $L_1(u, uw)$. By this process, we get a $T(r, s, k)$ representation of M. Now, we proceed with the process from Section 6 for the map of type $\{3, 12^2\}$. We show that a map of type $\{3, 12^2\}$ contains at most three nonhomologous cycles of type G_1 of different lengths.

Lemma 7.1. *The map M contains at most three cycles of type G_1 of different lengths.*

Proof. We proceed as in the case of Lemma 6.1 for the map of type $\{3, 12^2\}$. Consider the map of type $\{3, 12^2\}$ in place of $\{3, 6, 3, 6\}$, a cycle of type G_1 in place of X_1, and the 3-gon in the proof from Lemma 6.1. Thus, we get three nonhomologous cycles of type G_1 of different lengths. □

We define admissible relations among r, s, k of $T(r, s, k)$ in M.

Lemma 7.2. *The maps of type $\{3, 12^2\}$ of the form $T(r, s, k)$ exist if and only if the following holds:*

(i) $s \geq 1$,
(ii) $4 \mid r$,
(iii) *there are* $3rs/2 \geq 36$ *vertices of* $T(r, s, k)$,
(iv)

$$r \geq \begin{cases} 24 & \text{if } s = 1, \\ 16 & \text{if } s = 2, \\ 12 & \text{if } s \geq 3, \end{cases}$$

(v)

$$\begin{cases} k \in \{4t + 9 \colon 0 \leq t \leq \frac{r-20}{4}\} \setminus \{4(\frac{r}{8} - 3) + 9\} & \text{if } s = 1, \\ k \in \{4t + 5 \colon 0 \leq t \leq \frac{r-16}{4}\} & \text{if } s = 2 \\ k \in \{4t + 1 \colon 0 \leq t \leq \frac{r-4}{4}\} & \text{if } s \geq 3. \end{cases}$$

Proof. We proceed as in the proof of Lemma 6.2. We consider a map of type $\{3, 12^2\}$ in place of $\{3, 6, 3, 6\}$ and different values of r, s, k. Thus, we get all the cases. □

Lemma 7.3. *Let $T_i = T(r_i, s_i, k_i)$ be representations of M_i, $i = 1, 2$, on the same number of vertices. Similar to Section 6, let $b_{i,j} = \text{length}(L_{i,j})$, $j = 1, 2, 3$, then $M_1 \cong M_2$ if and only if $(b_{1,1}, b_{1,2}, b_{1,3}) = (b_{2,t_1}, b_{2,t_2}, b_{2,t_3})$ for $t_1 \neq t_2 \neq t_3 \in \{1, 2, 3\}$.*

Table 4. Maps of type $\{3, 12^2\}$

n	Equivalence classes	Length of cycles	$i(n)$
36	T(24, 1, 13)	(24, 12, 8)	1(36)
42	T(28, 1, 9), T(28, 1, 13) T(28, 1, 17)	(28, 28, 28)	1(42)
48	T(32, 1, 9), T(32, 1, 21) T(16, 2, 5)	(32, 32, 16)	2(48)
	T(32, 1, 17)	(32, 32, 8)	

As in Section 4, by Lemmas 7.2 and 7.3, maps of type $\{3, 12^2\}$ can be classified up to isomorphism on different number of vertices. We have done the calculation for vertices up to 48. We have listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 4.

8. Maps of type $\{3^4, 6\}$

Let M be a semiequivelar map of type $\{3^4, 6\}$ on the torus. Let $Q(i) := P(w_i, w_{i+1}, w_{i+2}, w_{i+3})$ be a path in M, where

$$\begin{aligned} lk(w_i) &= C(w_{i-1}, x_2, w_{i+1}, \boldsymbol{w}_{i+2}, \boldsymbol{x}_3, \boldsymbol{x}_4, x_5, x_6), \\ lk(w_{i+1}) &= C(w_i, x_2, x_7, x_8, w_{i+2}, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5), \\ lk(w_{i+2}) &= C(w_{i+1}, x_8, x_9, w_{i+3}, x_3, \boldsymbol{x}_4, \boldsymbol{x}_5, \boldsymbol{w}_i,), \\ lk(w_{i+3}) &= C(w_{i+2}, x_9, w_{i+4}, \boldsymbol{w}_{i+5}, \boldsymbol{x}_{10}, \boldsymbol{x}_{11}, x_{12}, x_3). \end{aligned}$$

We define two fixed types of paths Y_1 and Y_1' in the edge graph of M.

Definition. *Let $R_1 := P(\cdots, v_{i-1}, v_i, v_{i+1}, \cdots)$ be a path in the edge graph of M. We say R_1 is of type Y_1 if $L_1 := P(u_t, u_{t+1}, u_{t+2}, u_{t+3})$ is a subpath of R_1 or L_1 is in the extended path of R_1, then $L_1 \mapsto Q(i)$ by $u_j \mapsto w_j$, $L_1 \mapsto Q(i+1)$ by $u_j \mapsto w_{j+1}$, or $L_1 \mapsto Q(i+2)$ by $u_j \mapsto w_{j+2}$ for $j \in \{t, t+1, t+2, t+3\}$.*

Definition. *Let $R_2 := P(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots)$ be a path in the edge graph of M. We say R_2 of type Y_1' if $L_2 := P(x_t, x_{t+1}, x_{t+2}, x_{t+3})$ is a subpath of R_2 or L_2 is in the extended path of R_2, then $L_2 \mapsto Q(i)$ by $x_j \mapsto w_{2t+3-j}$, $L_2 \mapsto Q(i+1)$ by $x_j \mapsto w_{2t+3-j}$, or $L_2 \mapsto Q(i+2)$ by $x_j \mapsto w_{2t+3-j}$ for $j \in \{t, t+1, t+2, t+3\}$.*

Let P be a maximal path of type Y_1 or Y_1'. By a similar argument from Lemma 4.1, the path P defines a cycle of type Y_1 or Y_1', that is, there is an edge e in M such that $P \cup \{e\}$ is a cycle of type Y_1 or Y_1'. We show that the cycles of type Y_1 and Y_1' define same type of cycle. Let $C_1 := C(u_1, u_2, \cdots, u_r)$ of type Y_1 and $C_2(v_1, v_2, \cdots, v_r)$ of type Y_1' be two cycles of length r. Let $P_1 := P(u_{i-1}, u_i, u_{i+1})$ be a subpath of C_1 where the adjacent 6-gon lies on one side and all 3-gons lie on the other side of P_1 at the vertex u_i. Similarly, let $P_2 := P(v_{j-1}, v_j, v_{j+1})$ be a subpath of C_2 where the adjacent 6-gon lies on one side and all 3-gons lie on the other side of P_2 at the vertex v_j. Define a map $f : V(C_1) \to V(C_2)$ by $f(u_i) = v_j$, $f(u_{i+1}) = v_{j-1}$, $f(u_{i+2}) = v_{j-2}, \cdots, f(u_{i-1}) = v_{j+1}$. Let $P(u_t, \cdots, u_k)$ be a subpath of C_1. Then $P(u_t, \cdots, u_k)$ and $P(f(u_t), \cdots, f(u_k))$ divides the link of the vertices u_i and $f(u_i)$ for $t \leq i \leq k$ into the same ratio. This is true for every subpath of C_1. Therefore, cycles C_1 and C_2 are of type Y_1, and hence, $Y_1 = Y_1'$.

Let M be a map and C be a cycle of type Y_1 in M. By a similar argument from Lemma 4.3, C is noncontractible. As in Section 4, we get that the cycles of type Y_1 which are homologous to C have the same length by a similar argument from Lemma 4.5. There are three cycles of type Y_1 through each vertex of M. Let $v \in V(M)$ and $L_1(v)$, $L_2(v)$, $L_3(v)$ be three cycles of type Y_1 through the vertex v. We repeat a similar construction of the (r, s, k)-representation of a map as in Section 4 for M. In this process, we take the first cut along $L_1(v)$ and the second cut along $L_2(v)$ where the starting adjacent face to the horizontal base cycle L_1 is a 3-gon. This gives a $T(r, s, k)$ representation of the map M. Thus, $T(r, s, k)$ exists for every M.

Now, we show that map M of type $\{3^4, 6\}$ contains at most three nonhomologous cycles of type Y_1 of different lengths in Lemma 8.1.

Lemma 8.1. *The map M contains at most three nonhomologous cycles of type Y_1 of different lengths.*

Proof. Let $v \in V(M)$ and $T(r, s, k)$ denote a (r, s, k)-representation of M. We have three cycles, namely, C_1, C_2 and C_3 through v in M of type Y_1. The cycles C_1, C_2, and C_3 are not identical as C_i divides the link $lk(v)$ into a different ratio. Also, the cycles are not disjoint as $v \in V(C_i) \cap V(C_j)$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Hence C_1, C_2, and C_3 are not homologous to each other. Let $w \in V(M)$ and $v \neq w$. Consider cycles of type Y_1 at w in $T(r, s, k)$ and denoted by C_1', C_2', and C_3'. Now, by the definition of cycle of type Y_1 and considering the cylinder, C_i and C_j' are homologous for some $i, j \in \{1, 2, 3\}$ as we have seen from Lemma 6.1. This holds for any vertex of M. Therefore, M contains at most three nonhomologous cycles of type Y_1. We proceed as in the proof of Lemma 4.5 to show that the homologous cycles of type Y_1 have the same length. Thus, the map M contains at most three nonhomologous cycles of type Y_1 of different lengths. □

We define admissible relations among r, s, k of $T(r, s, k)$ such that representation $T(r, s, k)$ gives a map of type $\{3^4, 6\}$ after identifying their boundaries.

Lemma 8.2. *The maps of type $\{3^4, 6\}$ of the form $T(r, s, k)$ exist if and only if the following holds:*

(i) *$s \geq 2$ even,*
(ii) $3 \mid r$,
(iii) *there are $rs \geq 18$ vertices of $T(r, s, k)$,*
(iv)

$$r \geq \begin{cases} 9 & \text{if } s = 2, \\ 6 & \text{if } s \geq 4, \end{cases}$$

(v)

$$\begin{cases} k \in \{3t+5 : 0 \le t \le \frac{r-9}{3}\} & \textit{if } s = 2, \\ k \in \{2+3t \ : \ 0 \le t \le \frac{r-3}{3}\} & \textit{if } s \ge 4. \end{cases}$$

Proof. Let $T(r,s,k)$ be a representation of M. It has s disjoint horizontal cycles of type Y_1 of length r by the definition of a (r,s,k)-representation. These cycles cover all the vertices of M. So, $n = rs$. By Euler's formula, $n - 5n/2 + 4n/3 + n/6 = 0$. Hence the number of 6-gons in M is $n/6$ and is an integer. This implies that $6 \mid n$. That is, $6 \mid rs$ as $n = rs$. So, $3 \mid r$ for $s = 2$. Let $s \ge 3$. If s is an odd integer then $T(r,s,k)$ contains an odd number of horizontal cycles of type Y_1. Consider a vertex v of base horizontal cycle which is belongs to only triangles which is a contradiction. Therefore, $2 \mid s$. Similarly, for $3 \mid r$, we get a vertex whose link does not follow the type $\{3^4, 6\}$. So, $n = rs$ where $6 \mid n$, $2 \mid s$ and $3 \mid r$.

For $r \ge 9$, we proceed with a similar argument to the proof of Lemma 4.9. We also proceed as in the proof of Lemma 4.9 to show $r \ge 6$ and $3 \mid r$ and the other remaining cases. This completes the proof. □

TABLE 5. Maps of type $\{3^4, 6\}$

n	Equivalence classes	Length of cycles	$i(n)$
18	T(9, 2, 5)	(9, 9, 9)	1(18)
24	T(12, 2, 5), T(12, 2, 8)	(12, 6, 12)	2(24)
	T(6, 4, 2)		
	T(6, 4, 5)	(6, 6, 6)	
30	T(15, 2, 5), T(15, 2, 8)	(15, 15, 15)	1(30)
	T(15, 2, 11)		
36	T(18, 2, 5), T(18, 2, 14)	(18, 9, 18)	2(36)
	T(9, 4, 2)		
	T(18, 2, 8), T(18, 2, 11)	(18, 6, 9)	
	T(9, 4, 5), T(9, 4, 8)		
	T(6, 6, 2), T(6, 6, 5)		
42	T(21, 2, 5), T(21, 2, 8)	(21, 21, 21)	1(42)
	T(21, 2, 11), T(21, 2, 14)		
	T(21, 2, 17)		

Lemma 8.3. *Let M_1 and M_2 be two maps of type $\{3^4, 6\}$ on the same number of vertices. Let $a_{i,j}$ = length($C_{i,j}$) where $C_{i,k}$ for $i = 1,2,3$ denote three nonhomologous cycles of type Y_1 in $T(r_i, s_i, k_i)$ of M_i. Then $M_1 \cong M_2$ if and only if $(a_{1,1}, a_{1,2}, a_{1,3}) = (a_{2,t_1}, a_{2,t_2}, a_{2,t_3})$ for $t_1 \ne t_2 \ne t_3 \in \{1,2,3\}$.*

Proof. Let $T(r_i, s_i, k_i)$ be a representation of M_i. If $r = r_1, s = s_1$, and $k = k_1$, then consider the horizontal cycles in $T(r_i, s_i, k_i)$ of type Y_1. Proceed a similar argument from Lemma 4.10. We get a map which defines an isomorphism between $T(r_1, s_1, k_1)$ and $T(r_2, s_2, k_2)$. Hence $M_1 \cong M_2$. Again, if $(r,s,k) \ne (r_1, s_1, k_1)$ then proceed with a similar argument to the proof of the Lemma 5.6. The converse of the lemma follows from a similar argument from the converse of Lemma 4.10. This completes the proof. □

As in Section 4, by Lemmas 8.2 and 8.3, the maps of type $\{3^4, 6\}$ can be classified up to isomorphism on different number of vertices. We have done the calculation for vertices up to 42. We have listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 5.

9. Maps of type $\{4, 6, 12\}$

Let M be a semiequivelar map of type $\{4, 6, 12\}$ on the torus. We define a fixed type of path H_1 in the edge graph of M. Let $Q(i) := P(u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}, u_{i+5}, u_{i+6})$ be a path in M where

$$lk(u_i) = C(u_{i-1}, \mathbf{b}, \mathbf{c}, \boldsymbol{u}_{i+2}, u_{i+1}\mathbf{p}, q, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}, \boldsymbol{u}_{i-5}, \boldsymbol{u}_{i-4}, \boldsymbol{u}_{i-3}, \boldsymbol{u}_{i-2}),$$
$$lk(u_{i+1}) = C(u_i, \boldsymbol{u}_{i-1}\mathbf{b}, \mathbf{c}, u_{i+2}, \boldsymbol{u}_{i+3}, \boldsymbol{u}_{i+4}, \boldsymbol{u}_{i+5}, \boldsymbol{u}_{i+6}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, p, \mathbf{q}),$$
$$lk(u_{i+2}) = C(u_{i+1}, \boldsymbol{u}_i, \boldsymbol{u}_{i-1}\mathbf{b}, c, \mathbf{d}, u_{i+3}, \boldsymbol{u}_{i+4}, \boldsymbol{u}_{i+5}, \boldsymbol{u}_{i+6}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}),$$
$$lk(u_{i+3}) = C(u_{i+2}, \boldsymbol{c}, d, \mathbf{e}, \mathbf{f}, \mathbf{g}, u_{i+4}\boldsymbol{u}_{i+5}\boldsymbol{u}_{i+6}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \boldsymbol{u}_{i+1}),$$
$$lk(u_{i+4}) = C(u_{i+3}, \boldsymbol{d}, \mathbf{e}, \mathbf{f}, g, \mathbf{h}, u_{i+5}\boldsymbol{u}_{i+6}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \boldsymbol{u}_{i+1}, \boldsymbol{u}_{i+2}),$$
$$lk(u_{i+5}) = C(u_{i+4}, \boldsymbol{g}, h, \mathbf{i}, \boldsymbol{u}_{i+8}, \boldsymbol{u}_{i+7}, u_{i+6}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \boldsymbol{u}_{i+1}, \boldsymbol{u}_{i+2}, \boldsymbol{u}_{i+3}),$$
$$lk(u_{i+6}) = C(u_{i+5}, \boldsymbol{h}, i, \boldsymbol{u}_{i+8}, u_{i+7}, \mathbf{j}, k, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \boldsymbol{u}_{i+1}, \boldsymbol{u}_{i+2}, \boldsymbol{u}_{i+3}, \boldsymbol{u}_{i+4}).$$

Definition. *Let $P_1 := P(\dots, v_{i-1}, v_i, v_{i+1}, \dots)$ be a path in the edge graph of M. We say P_1 is of type H_1 if $L_1 := P(v_t, v_{t+1}, v_{t+2}, v_{t+3}, v_{t+4}, v_{t+5}, v_{t+6})$ is a subpath of P_1 or L_1 lies in the extended path of P_1, then either $L_1 \mapsto Q(i)$ by $v_j \mapsto u_j$, $L_1 \mapsto Q(i+1)$ by $v_j \mapsto u_{j+1}$, $L_1 \mapsto Q(i+2)$ by $v_j \mapsto u_{j+2}$, $L_1 \mapsto Q(i+3)$ by $v_j \mapsto u_{j+3}$, $L_1 \mapsto Q(i+4)$ by $v_j \mapsto u_{j+4}$, or $L_1 \mapsto Q(i+5)$ by $v_j \mapsto u_{j+5}$ for $j \in \{t, t+1, t+2, t+3, t+4, t+5, t+6\}$.*

Definition. *Let $P_2 := P(\dots, x_{i-1}, x_i, x_{i+1}, \dots)$ be a path in the edge graph of M. We say P_2 is of type H_1' if $L_2 := P(x_t, x_{t+1}, x_{t+2}, x_{t+3}, x_{t+4}, x_{t+5}, x_{t+6})$ is a subpath of P_2 or K_2 lies in the extended path of P_2, then either $L_2 \mapsto Q(i)$ by $x_j \mapsto u_{2t+6-j}$, $L_2 \mapsto Q(i+1)$ by $x_j \mapsto u_{2t+6-j}$, $L_2 \mapsto Q(i+2)$ by $x_j \mapsto u_{2t+6-j}$, $L_1 \mapsto Q(i+3)$ by $v_j \mapsto u_{2t+6-j}$, $L_1 \mapsto Q(i+4)$ by $v_j \mapsto u_{2t+6-j}$, or $L_1 \mapsto Q(i+5)$ by $v_j \mapsto u_{2t+6-j}$ for $j \in \{t, t+1, t+2, t+3, t+4, t+5, t+6\}$.*

Let P be a maximal path of type H_1 or H_1' in M. By a similar argument from Lemma 4.1, the map M contains an edge e which defines a cycle $P \cup \{e\}$ of type H_1 or H_1'. The cycle $C := P \cup \{e\}$ is a noncontractible cycle (by a similar argument from Lemma 4.3). Observe that the cycles of types H_1 and H_1' are mirror images of each other. It follows that they define the same type of cycles as in Section 8. Hence we consider only cycles of type H_1. Let $C_1, C_2, \cdots, C_m$ be a sequence of homologous cycles of type H_1 in M. We use a similar argument from Lemma 4.5 and get that length(C_i) = length(C_j) for $1 \le i, j \le m$. By Definition 9.1, there are three cycles of type H_1 through each vertex of M. Let $v \in V(M)$ and $L_1(v)$, $L_2(v)$, $L_3(v)$ denote three cycles through v. Define a $T(r, s, k)$ representation of M by a similar construction given in Section 4. In this process, we first cut M

along L_1 and then take a second cut along the cycle L_3 where the starting adjacent face to the base horizontal cycle L_1 is a 6-gon. So, every map M has a $T(r,s,k)$ representation. In Lemma 9.1, we show that the map of type $\{4,6,12\}$ contains at most three nonhomologous cycles of type H_1 of different lengths.

Lemma 9.1. *The map M contains at most three nonhomologous cycles of type H_1 of different lengths.*

Proof. We proceed as in the proof of Lemma 8.1. Consider the map $\{4,6,12\}$ in place of $\{3^4,6\}$ and a cycle of type H_1 in place of Y_1. Let $u \in V(M)$ and $T(r,s,k)$ denote a (r,s,k)-representation of M. Let L_1, L_2, and L_3 denote three cycles of type H_1 through u in M. They are not identical as L_i divides the link $lk(u)$ into different ratios. Also, cycles are not disjoint as $u \in V(L_i) \cap V(L_j)$ for $i \neq j$. Therefore, cycles are not homologous to each other. Again, let $v \in V(M)$, $u \neq v$ and consider cycles of type H_1 at v in $T(r,s,k)$. Let L_1', L_2', and L_3' denote three cycles through v of type H_1. Then, by the definition of a cycle of type H_1 and considering cylinder, L_i and L_j' are homologous for some $i,j \in \{1,\ 2,\ 3\}$. This holds for any vertex v of M. Thus, M contains at most three nonhomologous cycles of type H_1. We proceed as in the proof of Lemma 4.5 to show that the homologous cycles of type H_1 have the same length. This completes the proof. □

We define admissible relations among r, s, k of $T(r,s,k)$ such that the representation $T(r,s,k)$ gives a map of type $\{4,6,12\}$ after identifying their boundaries.

Lemma 9.2. *Maps of type $\{4,6,12\}$ of the form $T(r,s,k)$ exist if and only if the following holds:*

(i) *$s \geq 2$ even,*
(ii) *$6 \mid r$,*
(iii) *there are $rs \geq 36$ vertices of $T(r,s,k)$,*
(iv)

$$r \geq \begin{cases} 18 & \text{if } s = 2, \\ 12 & \text{if } s \geq 4, \end{cases}$$

(v)

$$\begin{cases} k \in \{6t+9\colon 0 \leq t \leq \frac{r-18}{6}\} & \text{if } s = 2, \\ k \in \{6t+3\colon 0 \leq t \leq \frac{r-6}{6}\} & \text{if } s \geq 4. \end{cases}$$

Proof. We proceed as in the case of proof of the Lemma 8.2. We prove this lemma by considering link of some vertices in $T(r,s,k)$. We consider a map of type $\{4,6,12\}$ in place of type $\{3^4,6\}$ and different values of r, s, and k in the proof of Lemma 8.2. Thus, we get all possible ranges of r, s and k of $T(r,s,k)$. This completes the proof. □

Table 6. Maps of type $\{4, 6, 12\}$

n	Equivalence classes	Length of cycles	$i(n)$
36	T(18, 2, 9)	(18, 18, 18)	1(36)
48	T(24, 2, 9), T(12, 4, 9)	(24, 12, 24)	2(48)
	T(24, 2, 15)		
	T(12, 4, 3)	(12, 12, 12)	
60	T(30, 2, 9), T(30, 2, 15)	(30, 30, 30)	1(60)
	T(30, 2, 21)		

Lemma 9.3. *Let M_1 and M_2 be two maps of type $\{4, 6, 12\}$ on the same number of vertices. Let $T(r_i, s_i, k_i)$ denote a (r_i, s_i, k_i)-representation of M_i. Similar to Sections 4 and 5, by Lemma 9.1, let $b_{i,j} = \text{length}(L_{i,j})$ where $L_{i,j}$, $j = 1, 2, 3$ denotes nonhomologous cycles of type H_1 in $T(r_i, s_i, k_i)$. Then $M_1 \cong M_2$ if and only if $(b_{1,1}, b_{1,2}, b_{1,3}) = (b_{2,t_1}, b_{2,t_2}, b_{2,t_3})$ for $t_1 \neq t_2 \neq t_3 \in \{1, 2, 3\}$.*

Proof. We proceed as in the proof of Lemma 8.3. Let $r = r_1, s = s_1$, and $k = k_1$. Consider horizontal cycles in $T(r_i, s_i, k_i)$ of type H_1. We proceed with a similar argument from Lemma 4.10. We get $M_1 \cong M_2$. If $(r, s, k) \neq (r_1, s_1, k_1)$ then we proceed as in the proof of Lemmas 5.6 and 6.3. The converse follows a similar argument from the converse of Lemma 4.10. This completes the proof. □

As in Section 4, by Lemmas 9.2 and 9.3, the maps of type $\{4, 6, 12\}$ can be classified on different number of vertices. We have done the calculation for vertices up to 60. We have listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 6.

10. Maps of type $\{3, 4, 6, 4\}$

Let M be a semiequivelar map of type $\{3, 4, 6, 4\}$ on the torus. We define a path W_1 in the edge graph of M on the torus. Let $Q(i) := P(u_i, u_{i+1}, u_{i+2}, u_{i+3})$ be a path in M, where

$$
\begin{aligned}
lk(u_i) &= C(u_{i-1}, \mathbf{a}, b, u_{i+1}, \mathbf{i}, j, \mathbf{k}, \mathbf{l}, \boldsymbol{u}_{i-2}),\\
lk(u_{i+1}) &= C(u_i, b, \mathbf{c}, u_{i+2}, \boldsymbol{u}_{i+3}, \mathbf{g}, \mathbf{h}, i, \mathbf{j}),\\
lk(u_{i+2}) &= C(u_{i+1}, \mathbf{b}, c, d, \boldsymbol{e}, u_{i+3}, \mathbf{g}, \mathbf{h}, \mathbf{i}),\\
lk(u_{i+3}) &= C(u_{i+2}, \mathbf{d}, e, u_{i+4}, \boldsymbol{f}, g, \mathbf{h}, \mathbf{i}, \boldsymbol{u}_{i+1}).
\end{aligned}
$$

Definition. *Let $P_1 := P(\dots, v_{i-1}, v_i, v_{i+1}, \dots)$ be a path in the edge graph of M. We say P_1 is of type W_1 if $L_1 := P(v_t, v_{t+1}, v_{t+2}, v_{t+3})$ is a subpath of P_1 or L_1 is in a path containing P_1. In this case, either $L_1 \mapsto Q(i)$ by $v_j \mapsto u_j$, $L_1 \mapsto Q(i+1)$ by $v_j \mapsto u_{j+1}$, or $L_1 \mapsto Q(i+2)$ by $v_j \mapsto u_{j+2}$ for $j \in \{t, t+1, t+2, t+3\}$.*

Definition. *Let $P_2 := P(\dots, x_{i-1}, x_i, x_{i+1}, \dots)$ be a path in the edge graph of M. We say P_2 of type W_1' if $L_2 := P(x_t, x_{t+1}, x_{t+2}, x_{t+3})$ is a subpath of P_2 or L_2 is in a path containing P_2. In this case, either $L_2 \mapsto Q(i)$*

by $x_j \mapsto u_{2t+3-j}$, $L_2 \mapsto Q(i+1)$ *by* $x_j \mapsto u_{2t+3-j}$, *or* $L_2 \mapsto Q(i+2)$ *by* $x_j \mapsto u_{2t+3-j}$ *for* $j \in \{t, t+1, t+2, t+3\}$.

We consider only cycles of type W_1 as W_1 and W_1' define same type of cycle (by a similar argument from Section 8). Repeat a similar argument from Section 9 and define a $T(r,s,k)$ representation. In this process, we consider a path of type W_1 in place of H_1. By Definition 10.1, there are three cycles through each vertex of M. Let $v \in V(M)$ and L_1, L_2, and L_3 be three cycles through v. We first cut along L_1 and then take a second cut along L_3 where the starting adjacent face to base horizontal cycle L_1 is a 4-gon. So, every map has a $T(r,s,k)$ representation. In Lemma 10.1, we show that a map of type $\{3,4,6,4\}$ contains at most three nonhomologous cycles of type W_1 of different lengths.

Lemma 10.1. *The map M contains at most three nonhomologous cycles of type W_1 of different lengths.*

Proof. As above, proceed with a similar argument from Lemma 8.1. Consider a map of type $\{3,4,6,4\}$ in place of $\{3^4,6\}$ and a cycle of type W_1 in place of Y_1. Let $w_1 \neq w_2$ be two vertices of M. Let J_1, J_2, J_3 denote three cycles through w_1 and J_1', J_2', J_3' denote three cycles through w_2. Then by the definition of a cycle of type W_1, we get a cylinder which is bounded by J_i and J_j'. That is, J_i and J_j' are homologous for some $i, j \in \{1,2,3\}$. This holds for an arbitrary vertex of M. We proceed as in the case of proof of Lemma 4.5 to show that the homologous cycles of type W_1 have the same length. Thus, the map M contains at most three nonhomologous cycles of type W_1 of different lengths. This completes the proof. ☐

We define admissible relations among r, s, k of $T(r,s,k)$ such that representation $T(r,s,k)$ gives a map of type $\{3,4,6,4\}$ after identifying their boundaries.

Lemma 10.2. *The maps of type $\{3,4,6,4\}$ of the form $T(r,s,k)$ exist if and only if the following holds:*

(i) $s \geq 2$ *even,*

(ii) $3 \mid r$,

(iii) *there are* $rs \geq 18$ *vertices of* $T(r,s,k)$,

(iv)
$$r \geq \begin{cases} 9 & \text{if } s = 2, \\ 6 & \text{if } s \geq 4, \end{cases}$$

(vi)
$$\begin{cases} k \in \{3t+4\colon 0 \leq t \leq \frac{r-9}{3}\} & \text{if } s = 2, \\ k \in \{3t+1\colon 0 \leq t \leq \frac{r-3}{3}\} & \text{if } s \geq 4. \end{cases}$$

Proof. We follow a similar argument from the proof of Lemma 8.2. We prove this lemma by considering a link of some vertices in $T(r,s,k)$ and by showing that the link of those vertices are not a cycle if we consider the

TABLE 7. Maps of type $\{3, 4, 6, 4\}$

n	Equivalence classes	Length of cycles	$i(n)$
18	T(9, 2, 4)	(9, 9, 9)	1(18)
24	T(12, 2, 4), T(12, 2, 7), T(6, 4, 4)	(12, 6, 12)	2(24)
	T(6, 4, 1)	(6, 6, 6)	
30	T(15, 2, 4), T(15, 2, 7), T(15, 2, 10)	(15, 15, 15)	1(30)
36	T(18, 2, 4), T(18, 2, 13), T(9, 4, 7)	(18, 9, 18)	2(36)
	T(18, 2, 7), T(18, 2, 10), T(9 , 4, 1)	(18, 9, 6)	
	T(9, 4, 4), T(6, 6, 4), T(6, 6, 1)		
42	T(21, 2, 4), T(21, 2, 7), T(21, 2, 10)	(21, 21, 21)	1(42)
	T(21, 2, 13), T(21, 2, 16)		
48	T(24, 2, 4), T(24, 2, 7), T(24, 2, 16)	(24, 24, 12)	3(48)
	T(24, 2, 19), T(12, 4, 4), T(12, 4, 10)		
	T(24, 2, 10), T(24, 2, 13), T(6, 8, 4)	(24, 24, 6)	
	T(12, 4, 1), T(12, 4, 7), T(6, 8, 1)	(12, 12, 6)	
54	T(27, 2, 4), T(27, 2, 13), T(27, 2, 22)	(27, 27, 27)	3(54)
	T(27, 2, 7), T(27, 2, 10), T(27, 2, 16)	(27, 27, 9)	
	T(27, 2, 19), T(9, 6, 4), T(9, 6, 7)		
	T(9, 6, 1)	(9, 9, 9)	

values of r, s, and k outside the given range in Lemma 10.2. Consider a map of type $\{3, 4, 6, 4\}$ in place of type $\{3^4, 6\}$ and different ranges of r, s, and k in the proof of Lemma 8.2. Thus, we get all the cases of this lemma. This completes the proof. □

Lemma 10.3. *Let M_i, for $i = 1, 2$, be maps of type $\{3, 4, 6, 4\}$ on the same number of vertices and $T_i = T(r_i, s_i, k_i)$ be (r_i, s_i, k_i)-representations of M_i. (By Lemma 10.1, there are at most three nonhomologous cycles of different lengths in T_i.) Let $c_{i,j} = \text{length}(N_{i,j})$ where $N_{i,j}$, $j = 1, 2, 3$ denote nonhomologous cycles of type W_1 in T_i. Then the map $M_1 \cong M_2$ if and only if $(c_{1,1}, c_{1,2}, c_{1,3}) = (c_{2,t_1}, c_{2,t_2}, c_{2,t_3})$ for $t_1 \neq t_2 \neq t_3 \in \{1, 2, 3\}$.*

Proof. We proceed as in the proof of Lemma 8.3. Let $r = r_1, s = s_1$, and $k = k_1$. We consider horizontal cycles of $T(r_i, s_i, k_i)$ of type W_1. Proceed with a similar argument from Lemma 4.10. Thus, we get $M_1 \cong M_2$. Again, if $(r, s, k) \neq (r_1, s_1, k_1)$ then we proceed as in the proof of Lemma 5.6 and Lemma 6.3. The converse follows from a similar argument from the converse of Lemma 4.10. This completes the proof. □

As in Section 4, by Lemmas 10.2 and 10.3, the maps of type $\{3, 4, 6, 4\}$ can be classified on different number of vertices. We have done calculation for up to 54 vertices. We have listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 7.

11. MAPS OF TYPE $\{4, 8^2\}$

Let M be a semiequivelar map of type $\{4, 8^2\}$ on the torus. We define a fixed type of path Z_1 in M. Let $Q(i) := P(u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4})$ be a path in M, where

$$\begin{aligned} lk(u_i) &= C(u_{i-1}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{i}, \boldsymbol{u}_{i+2}, u_{i+1}, \boldsymbol{s}, a, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{u}_{i-3}, \boldsymbol{u}_{i-2}), \\ lk(u_{i+1}) &= C(u_i, \boldsymbol{u}_{i-1}, \boldsymbol{f}, \mathbf{g}, \mathbf{h}, \mathbf{i}, u_{i+2}, \boldsymbol{u}_{i+3}, \boldsymbol{u}_{i+4}, \mathbf{p}, \boldsymbol{q}, \mathbf{r}, s, \boldsymbol{a}), \\ lk(u_{i+2}) &= C(u_{i+1}, \boldsymbol{u}_i, \boldsymbol{u}_{i-1}, \mathbf{f}, \mathbf{g}, \mathbf{h}, i, \mathbf{j}, u_{i+3}, \boldsymbol{u}_{i+4}, \mathbf{p}, \boldsymbol{q}, \mathbf{r}, \mathbf{s}), \\ lk(u_{i+3}) &= C(u_{i+2}, \mathbf{i}, j, \boldsymbol{k}, \mathbf{l}, \mathbf{m}, \boldsymbol{u}_{i+6}, \boldsymbol{u}_{i+5}, u_{i+4}, \mathbf{p}, \boldsymbol{q}, \mathbf{r}, \mathbf{s}, \boldsymbol{u}_{i+1}), \\ lk(u_{i+4}) &= C(u_{i+3}, \mathbf{j}, \boldsymbol{k}, \mathbf{l}, \mathbf{m}, \boldsymbol{u}_{i+6}, u_{i+5}, \mathbf{o}, p, \boldsymbol{q}, \mathbf{r}, \mathbf{s}, \boldsymbol{u}_{i+1}, \boldsymbol{u}_{i+2}). \end{aligned}$$

Definition. *Let $R_1 := P(\dots, v_{i-1}, v_i, v_{i+1}, \dots)$ be a path in the edge graph of M. We say R_1 of type Z_1 if $L_1 := P(v_t, v_{t+1}, v_{t+2}, v_{t+3}, v_{t+4})$ is a subpath of R_1 or L_1 is in the extended path of R_1. Then either $L_1 \mapsto Q(i)$ by $v_j \mapsto u_j$, $L_1 \mapsto Q(i+1)$ by $v_j \mapsto u_{j+1}$, $L_1 \mapsto Q(i+2)$ by $v_j \mapsto u_{j+2}$, or $L_1 \mapsto Q(i+3)$ by $v_j \mapsto u_{j+3}$ for $j \in \{t, t+1, t+2, t+3, t+4\}$.*

Definition. *Let $R_2 := P(\dots, x_{i-1}, x_i, x_{i+1}, \dots)$ be a path in the edge graph of M. We say R_2 of type Z_1' if $L_2 := P(x_t, x_{t+1}, x_{t+2}, x_{t+3}, x_{t+4})$ is a subpath of R_2 or L_2 is in the extended path of R_2. Then either $L_2 \mapsto Q(i)$ by $x_j \mapsto u_{2t+4-j}$, $L_2 \mapsto Q(i+1)$ by $x_j \mapsto u_{2t+4-j}$, $L_2 \mapsto Q(i+2)$ by $x_j \mapsto u_{2t+4-j}$, or $L_1 \mapsto Q(i+3)$ by $v_j \mapsto u_{2t+4-j}$ for $j \in \{t, t+1, t+2, t+3, t+4\}$.*

As in Section 8, we consider a path of type Z_1 as Z_1 and Z_1' define the same type of path (by a similar argument from Section 8). Let P be a maximal path of type Z_1. We use a similar argument from Lemma 4.1 for P. We get an edge e in M such that $P \cup e$ is a cycle of type Z_1. Therefore, every maximal path of type Z_1 is a cycle. Let $C = P \cup e$. The cycle C is of type Z_1 and noncontractible (by a similar argument from Lemma 4.3). Let C_1, $C_2, \cdots, C_t$ be a sequence of homologous cycles of type Z_1. Then we proceed with a similar argument from Lemma 4.5. Thus, length$(C_i) =$ length(C_j) for $1 \leq i, j \leq t$. By Definition 11.1, there are two cycles of type Z_1 through each vertex of M. Let $v \in V(M)$ and $L_1(v)$, $L_2(v)$ be two cycles through v. We repeat a similar construction of the (r, s, k)-representation from Section 4 for M. So, we get a $T(r, s, k)$ representation of M. In this process, we take the first cut along L_1 and then the second cut along L_2 where the starting adjacent face to the base horizontal cycle L_1 is a 4-gon. By this construction, every map of type $\{4, 8^2\}$ on the torus has a $T(r, s, k)$ representation.

We have two cycles of type Z_1 through each vertex of M. Therefore, by Lemma 11.1, map M contains at most two nonhomologous cycles of type Z_1 of different lengths.

Lemma 11.1. *The map M contains at most two nonhomologous cycles of type Z_1 of different lengths.*

Proof. Let v be a vertex in M. By the definition of a cycle of type Z_1, we have two cycles, namely, C_1 and C_2 through v. We proceed as in the proof of Lemma 8.1. We get that the map M contains at most two nonhomologous cycles of type Z_1 of different lengths. □

TABLE 8. Maps of type $\{4, 8^2\}$

n	Equivalence classes	Length of cycles	$i(n)$
20	T(20, 1, 6), T(20, 1, 14)	(20, 20)	1(20)
24	T(24, 1, 6), T(24, 1, 18)	(24, 8)	2(24)
	T(8, 3, 2), T(8, 3, 6)		
	T(24, 1, 14), T(24 , 1, 10)	(24, 24)	

We claim in the Lemma 11.2 that map of type $\{4, 8^2\}$ does not contain a cycle of type Z_1 which has length four. We use this result to classify the maps of type $\{4, 8^2\}$ on the torus.

Lemma 11.2. *The representation $T(r, s, k)$ does not contain a cycle of type Z_1 of length four.*

Proof. Suppose $T(r, s, k)$ has a cycle C of length four of type Z_1. Let C_{F_8} denote a boundary cycle of an 8-gon F_8. By the definition of a cycle of type Z_1, if $C \cap C_{F_8} \neq \emptyset$ then $C \cap C_{F_8}$ is a path of length three. Let $C \cap C_{F_8} = P(u_1, u_2, u_3, u_4)$. If C is a cycle of length four then $u_1 = u_4$ which is a contradiction as u_1 and u_4 are in C_{F_8}, and C_{F_8} is a cycle without a chord. Again, by the definition of cycle of type Z_1, C must intersect an 8-gon. Thus, length$(C) > 4$. So, a map M of type $\{4, 8^2\}$ does not contain a cycle C of type Z_1 of length four. This completes the proof. □

We define admissible relations among r, s, k of $T(r, s, k)$.

Lemma 11.3. *The maps of type $\{4, 8^2\}$ of the form $T(r, s, k)$ exist if and only if the following holds:*

(i) $4 \mid r$,
(ii) $s \geq 1$,
(iii) *there are* $rs \geq 20$ *vertices of* $T(r, s, k)$,
(iv)
$$r \geq \begin{cases} 20 & \text{if } s = 1, \\ 16 & \text{if } s = 2, \\ 8 & \text{if } s \geq 3, \end{cases}$$
(v)
$$\begin{cases} k \in \{4t + 6 \colon 0 \leq t \leq \frac{r-12}{4}\} & \text{if } s = 1, \\ k \in \{4t + 7 \colon 0 \leq t \leq \frac{r-16}{4}\} & \text{if } s = 2, \\ k \in \{4t - 1 \pmod{r} : 0 \leq t \leq \frac{r-4}{4}\} & \text{if } s \geq 3. \end{cases}$$

Proof. We proceed as in the proof of Lemma 8.2 and use Lemma 11.2 to prove this lemma. Consider the link of some vertices in $T(r, s, k)$, a map of type $\{4, 8^2\}$ in place of type $\{3^4, 6\}$, and different values of r, s, and k in the proof of Lemma 8.2. So, we get all possible ranges of r, s, and k of $T(r, s, k)$. This completes the proof. □

Lemma 11.4. *Let M_i, for $i = 1, 2$, be maps of type $\{4, 8^2\}$ on the same number of vertices and $T_i = T(r_i, s_i, k_i)$ be a representation of M_i. (By Lemma 11.1, there are at most two nonhomologous cycles of different lengths in T_i.) Let $a_{i,j}$ = length($C_{i,j}$) where $C_{i,j}$, $j = 1, 2$ denotes nonhomologous cycles of type Z_1 in T_i. Then $M_1 \cong M_2$ if and only if $(a_{1,1}, a_{1,2}) = (a_{2,t_1}, a_{2,t_2})$ for $t_1 \neq t_2 \in \{1, 2\}$.*

Proof. We proceed as in the proof of Lemma 8.3. Let $r = r_1, s = s_1$, and $k = k_1$. Consider horizontal cycles in $T(r_i, s_i, k_i)$ of type Z_1. Proceed with a similar argument from Lemma 4.10. So, we get $M_1 \cong M_2$. Again, if $(r, s, k) \neq (r_1, s_1, k_1)$ then we proceed as in the proof of Lemmas 5.6 and 6.3. The converse follows from a similar argument from the converse of Lemma 4.10. □

As in Section 4, by Lemmas 11.3, 11.4, the maps of type $\{4, 8^2\}$ can be classified up to isomorphism on different number of vertices. We have done the calculation for up to 24 vertices. We have listed the obtained objects in the form of their $T(r, s, k)$ representation in Table 8.

12. Semiequivelar maps

Proof of Theorem 1.1. The proof of the Theorem 1.1 follows from the Sections 4, 5, 6, 7, 8, 9, 10, and 11. Let M be a map on n vertices of type $\{3^3, 4^2\}$ on the torus. We consider all admissible $T(r, s, k)$ representations of M by Lemma 4.9. We calculate the length of the cycles of types A_1, A_2, A_3, and A_4. We classify them by Lemma 4.10. In Table 1, we have classified up them to 22 vertices. Similarly, we consider maps of types $\{3^2, 4, 3, 4\}$, $\{3, 6, 3, 6\}$, $\{3, 12^2\}$, $\{3^4, 6\}$, $\{4, 6, 12\}$, $\{3, 4, 6, 4\}$, and $\{4, 8^2\}$ on the torus. That is, we consider cycles of type B_1 in maps of type $\{3^2, 4, 3, 4\}$ and classify them by Lemma 5.6. Table 2 contains the classified maps up to 32 vertices. Consider cycles of type X_1 in maps of type $\{3, 6, 3, 6\}$ and classify them by Lemma 6.3. Table 3 contains the maps up to 30 vertices. Consider cycles of type G_1 in maps of type $\{3, 12^2\}$ and classify them by Lemma 7.3. Table 4 contains the maps up to 48 vertices. Consider cycles of type Y_1 in maps of type $\{3^4, 6\}$ and classify by Lemma 8.3. Table 5 contains the maps up to 42 vertices. Consider cycles of type H_1 in maps of type $\{4, 6, 12\}$ and classify by Lemma 9.3. Table 6 contains the maps up to 60 vertices. Consider cycles of type W_1 in maps of type $\{3, 4, 6, 4\}$ and classify them by Lemma 10.3. Table 7 contains the classified maps up to 54 vertices. Consider cycles of type Z_1 in maps of type $\{4, 8^2\}$ and classify them by Lemma 11.4. Table 8 contains the maps up to 24 vertices. This completes the proof. □

Acknowledgement

The authors are grateful to the anonymous referee whose comments led to a substantial improvement in the paper. The work of the second author is partially supported by SERB, DST grant No. SR/S4/MS:717/10.

References

1. A. Altshuler, *Construction and enumeration of regular maps on the torus*, Discrete Math. **4** (1973), 201–217.
2. J. A. Bondy and U. S. R. Murthy, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, 2008.
3. U. Brehm and W. Kühnel, *Equivelar maps on the torus*, European J. Combin. **29** (2008), 1843–1861.
4. U. Brehm and E. Schulte, *Handbook of discrete and computational geometry*, ch. Polyhedral maps, pp. 345–358, CRC Press, NY, 1997.
5. W. Kurth, *Enumeration of platonic maps on the torus*, Discrete Math. **61** (1986), 71–83.
6. S. Negami, *Uniqueness and faithfulness of embedding of toroidal graphs*, Discrete Math. **44** (1983), 161–180.
7. A. K. Tiwari and A. K. Upadhyay, *An enumeration of semi-equivelar maps on torus and klein bottle*, to appear.
8. A. K. Upadhyay, A. K. Tiwari, and D. Maity, *Semi-equivelar maps*, Beitr. Algebra Geom. **55** (2014), 229–242.

Department of Mathematics, Indian Institute of Technology Patna, Bihta, 801 103, India
E-mail address: `dipendumaity@gmail.com`

Department of Mathematics, Indian Institute of Technology Patna, Bihta, 801 103, India
E-mail address: `upadhyay@iitp.ac.in`