Contributions to Discrete Mathematics

Volume 12, Number 2, Pages 1–10
ISSN 1715-0868

THE FORCING STRONG METRIC DIMENSION OF A GRAPH

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Abstract. For any two vertices u, v in a connected graph G, the interval $I(u, v)$ consists of all vertices which are lying in some $u-v$ shortest path in G. A vertex x in a graph G strongly resolves a pair of vertices u, v if either $u \in I(x, v)$ or $v \in I(x, u)$. A set of vertices W of $V(G)$ is called a strong resolving set if every pair of vertices of G is strongly resolved by some vertex of W. The minimum cardinality of a strong resolving set in G is called the strong metric dimension of G and it is denoted by $\mathrm{sdim}(G)$. For a strong resolving set W of G, a subset S of W is called the forcing subset of W if W is the unique strong resolving set containing S. The forcing number $f(W, \mathrm{sdim}(G))$ of W in G is the minimum cardinality of a forcing subset for W, while the forcing strong metric dimension, $f_{\mathrm{sdim}}(G)$, of G is the smallest forcing number among all strong resolving sets of G. The forcing strong metric dimensions of some well-known graphs are determined. It is shown that for any positive integers a and b, with $0 \leq a \leq b$, there is nontrivial connected graph G with $\mathrm{sdim}(G) = b$ and $f_{\mathrm{sdim}}(G) = a$ if and only if $\{a, b\} \neq \{0, 1\}$.

1. Introduction

The distance between two vertices u, v, denoted by $d(u, v)$, in a connected graph G is the length of the shortest $u-v$ path in G. The *diameter* of G, $\mathrm{diam}(G)$, is given by $\max\{d(u, v) | u, v \in V(G)\}$. A vertex v is said to be extreme vertex if its neighbors induce a complete graph. For other terminology in graph theory, refer to [15]. The interval $I(u, v)$ consists of all vertices which are lying in some shortest $u-v$ path in G. For a set of vertices S of $V(G)$, the union of all $I(u, v)$ for $u, v \in S$ is denoted by $I(S)$. A set S is *convex* if $I(S) = S$, i.e., for every two vertices $u, v \in S$, the set $I(u, v)$ is contained in S. Clearly $V(G)$ is always convex. The *convexity number*, $\mathrm{con}(G)$, of a graph is defined in [4, 5] as the maximum cardinality of a proper convex set of G. A vertex $x \in V(G)$ resolves a pair of vertices $u, v \in V(G)$ if $d(u, x) \neq d(v, x)$.

Received by the editors November 30, 2014, and in revised form April 12, 2015.

2000 *Mathematics Subject Classification.* 05C12.

Key words and phrases. metric dimension, strong metric dimension, forcing strong metric dimension.

A set of vertices S of $V(G)$ is called a *resolving set* if every pair of distinct vertices of G is resolved by some vertex of S. The minimum cardinality of a resolving set of G is called the *metric dimension* of G and it is denoted by $\dim(G)$. For more information about metric dimension in graphs, see [1, 2, 3, 7, 8, 9, 10, 11].

A vertex x *strongly resolves* a pair $u, v \in V(G)$ if $u \in I(x, v)$ or $v \in I(x, u)$. A set of vertices S of $V(G)$ is called a *strong resolving set* if every pair of vertices of G is strongly resolved by some vertex of S. The minimum cardinality of a strong resolving set of G is called the *strong metric dimension*, $\mathrm{sdim}(G)$, was introduced by Sebö and Tannier in [14]. For more information about strong metric dimension in graphs, see [12, 13, 14, 16, 17, 18].

A vertex $x \in V(G)$ is *maximally distant* from $y \in V(G)$ if $d(x, y) \geq d(z, y)$, for every $z \in N(x)$, where $N(x) = \{v \in V(G) | xv \in E(G)\}$. If x is maximally distant from y and y is maximally distant from x, then we say that x and y are *mutually maximally distant* and denote this by $x \,\mathrm{MMD}\, y$. It is pointed out in [13] that if $x \,\mathrm{MMD}\, y$ in G, then any strong resolving set of G must contain either x or y.

For a strong resolving set W of G, a subset S of W is called the *forcing subset* of W if W is the unique strong resolving set containing S. The *forcing number* $f(W, \mathrm{sdim}(G))$ of W in G is the minimum cardinality of a forcing subset for W, while the *forcing strong metric dimension*, $f_{\mathrm{sdim}}(G)$, of G is the smallest forcing number among all strong resolving sets of G. In [6], Chartrand and Zhang studied the forcing dimension number of a graph.

For any connected graph G, $0 \leq f_{\mathrm{sdim}}(G) \leq \mathrm{sdim}(G)$. For example we consider the graph G depicted in Figure 1.

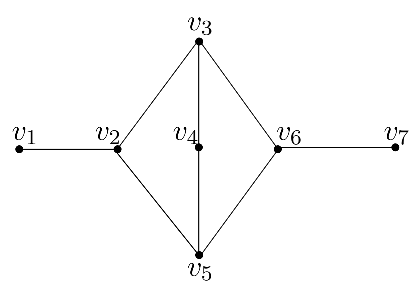

FIGURE 1. A graph G with $\mathrm{sdim}(G) = 3$ and $f_{\mathrm{sdim}}(G) = 3$.

Let S be any strong resolving set of G. Since $v_1 \,\mathrm{MMD}\, v_4, v_1 \,\mathrm{MMD}\, v_7$, and $v_4 \,\mathrm{MMD}\, v_7, |S \cap \{v_1, v_4, v_7\}| \geq 2$. Furthermore, since $v_3 \,\mathrm{MMD}\, v_5, S \cap \{v_3, v_5\} \neq \emptyset$. So, $\mathrm{sdim}(G) \geq 3$. On the other hand, $\{v_1, v_4, v_5\}$ forms a strong resolving set of G, and hence $\mathrm{sdim}(G) \leq 3$. Therefore, $\mathrm{sdim}(G) = 3$. Since any minimum strong resolving set must contain exactly two elements from $\{v_1, v_4, v_7\}$ and exactly one element from $\{v_3, v_5\}$, no two vertices that belong to a minimum strong resolving set S of G fixes the remaining element that belongs to S. Thus, $f_{\text{sdim}}(G) = 3$.

Lemma 1.1. *For any connected graph G, $f_{\text{sdim}}(G) = 0$ if and only if G has unique strong resolving set of G; $f_{\text{sdim}}(G) = 1$ if and only if G has at least two distinct strong resolving sets of G and some vertices belong to exactly one of them; $f_{\text{sdim}}(G) = \text{sdim}(G)$ if and only if no strong resolving set of G is the unique strong resolving set of G containing any of its proper subsets.*

2. Forcing strong metric dimension of certain graphs

In this section we determine the forcing strong dimensions of certain graphs. First we give the strong metric dimension of some well known graphs.

Theorem 2.1 ([14, 17])**.** *Let G be a connected graph of order $n \geq 2$. Then,*

a) $\text{sdim}(G) = 1$ *if and only if* $G = P_n$,
b) $\text{sdim}(G) = n - 1$ *if and only if* $G = K_n$,
c) *for* $G = C_n, n \geq 3$, $\text{sdim}(G) = \lceil n/2 \rceil$,
d) *if G is a tree T, then* $\text{sdim}(T) = k - 1$, *where k is the number of end vertices of T.*

Proposition 2.2. *Let G be a connected graph of order $n \geq 2$.*

a) *If $G = K_n$, then* $f_{\text{sdim}}(K_n) = \text{sdim}(K_n) = n - 1$.
b) *If $G = C_n$, then* $f_{\text{sdim}}(C_n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$

Proof. Let G be the complete graph K_n of order $n \geq 2$. Since every set W of $n-1$ vertices in K_n is a strong resolving set, W is not a unique strong resolving set containing any of its proper subset of G. By Lemma 1.1, $f_{\text{sdim}}(K_n) = \text{sdim}(K_n) = n - 1$.

Assume that G is a cycle C_n with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Suppose n is even. Then $v_i \,\text{MMD}\, v_{(n/2)+i}$, $i = 1, 2, \ldots, n/2$ and every strong resolving set of G contains either v_i or $v_{(n/2)+i}$. Let W be a strong resolving set of C_n, and let S be a proper subset of W such that $|S| \leq |W| - 1$. Now consider the set $X = W - S$. Then $W' = (W - X) \cup X'$, where $X' = \{x | x \,\text{MMD}\, y, y \in X\}$. Furthermore, W' is a strong resolving set of C_n containing S. Since $W \neq W'$, W is not a unique strong resolving set of C_n containing S. Therefore $f_{\text{sdim}}(C_n) = n/2$.

Suppose n is odd. Then $v_i \,\text{MMD}\, v_{\lceil n/2 \rceil + (i-1)}$ and $v_i \,\text{MMD}\, v_{\lceil n/2 \rceil + i}$, $i = 1, 2, \ldots, \lceil n/2 \rceil$, where the indices taken modulo n. For every vertex v_i, the set $\{v_i, v_{i+1}, \ldots, v_{i+\lceil n/2 \rceil - 1}\}$ and $\{v_{i+\lceil n/2 \rceil}, v_{i+\lceil n/2 \rceil+1}, \ldots, v_i\}$ with indices taken modulo n are strong resolving sets of C_n containing v_i and $f_{\text{sdim}}(C_n) \geq 2$. Let $W = \{v_1, v_2, \ldots, v_{\lceil n/2 \rceil}\}$. Then W is a strong resolving set of C_n. Also, every strong resolving set of C_n induces a connected subgraph of C_n. Hence W is the unique strong resolving set containing $\{v_1, v_{\lceil n/2 \rceil}\}$ and $f_{\text{sdim}}(C_n) = 2$. □

Next we determine the forcing strong dimension of hypercubes Q_n, $n \geq 2$.

Proposition 2.3 ([12]). *For $n \geq 2$, $\mathrm{sdim}(Q_n) = 2^{n-1}$.*

Proposition 2.4 ([4]). *For $n \geq 2$, a set S is convex in Q_n if and only if S induces Q_{n-1} in Q_n.*

Proposition 2.5. *For $n \geq 2$, a set S is the strong resolving set in Q_n if and only if S induces a Q_{n-1} in Q_n.*

Proof. Assume that S is a set of vertices of Q_n such that S induces a hypercube Q_{n-1} in Q_n. For any two vertices $x, y \in V(Q_n) - S$, we have $d(x, y) \leq n - 1$. Then there exists a vertex $v \in N(x) \cap S$, such that the pair x, y is strongly resolved by v. Hence S is a strong resolving set of Q_n.

Conversely, assume that S is a strong resolving set of Q_n. Therefore we have that $d(x, y) \leq n - 1$ for all $x, y \in V(Q_n) - S$. Thus, for every pair u, v in S, the interval $I(u, v)$ is contained in S and hence S is a convex set in Q_n. According to Proposition 2.4 the strong resolving set S induces a hypercubes Q_{n-1} in Q_n. □

Every strong resolving set of Q_n is also a convex set of Q_n and it was proved in [5] that the forcing convexity number of Q_n is 2 and hence we have

Proposition 2.6. *For $n \geq 2$, $f_{\mathrm{sdim}}(Q_n) = 2$.*

Proposition 2.7. *Let G be a connected graph of order at least 2.*

a) *If $G = K_{m,n}, m, n \geq 1$, then $f_{\mathrm{sdim}}(G) = \mathrm{sdim}(G)$.*
b) *If $G = K_1 + (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r})$, then $f_{\mathrm{sdim}}(G) = \mathrm{sdim}(G) = n-2$.*
c) *If G is a tree with k end vertices, then $f_{\mathrm{sdim}}(G) = \mathrm{sdim}(G) = k - 1$.*

Proof. Assume that $G = K_{m,n}$ with partite sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then $\mathrm{sdim}(G) = n - 2$. Let W be a strong resolving set of G. Then $W = W_1 \cup W_2$, $W_i \subseteq V_i$ $(i = 1, 2)$ with $|W_1| = m - 1$ and $|W_2| = n - 1$. Assume $W = V(G) - \{u_m, v_n\}$. Let S be a proper subset of W. Then $S = S_1 \cup S_2$, $S_i \subseteq W_i$ $(i = 1, 2)$ and $|S_1| \leq m - 2$ or $|S_2| \leq n - 2$, say, $|S_1| \leq m - 2$. Thus there exists a vertex $u_i \in W, 1 \leq i \leq m - 1$, such that $u_i \notin S$. Then $W' = (W - \{u_i\}) \cup \{u_m\}$ is a strong resolving set of G containing S. Since $W' \neq W$, W is not a unique strong resolving set of G containing S. Therefore $f_{\mathrm{sdim}}(G) = \mathrm{sdim}(G)$.

Now, let $G = K_1 + (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r})$. Assume $V(K_1) = \{x\}$. Since $\mathrm{sdim}(G) = n-2$, the vertex x does not belong to any of the strong resolving set of G. Let $W = V(G) - \{x, y\}$ where $y \in V(K_{n_i})$. Let S be a proper subset of W with $|S| \leq |W| - 1$. Then there exists a vertex $z \in W - S$. Let $W' = (W - \{z\}) \cup \{y\}$. Then W' is a strong resolving set of G containing S and $W' \neq W$. Thus W is not a unique strong resolving set of G containing S. Therefore $f_{\mathrm{sdim}}(G) = \mathrm{sdim}(G)$.

Next, let G be a tree with k end vertices. Let $\mathcal{E}(G)$ be the set of end vertices of G. Let $W \subseteq \mathcal{E}(G)$ with $|W| = k - 1$ and assume $y \in \mathcal{E}(G) - W$.

Then W is a strong resolving set of G. Let S be a proper subset of W with $|S| \leq |W| - 1$. Then there exists a vertex $x \in W - S$. Let $W' = (W - \{x\}) \cup \{y\}$. Then W' is a strong resolving set of G containing S and $W' \neq W$. Thus W is not a unique strong resolving set of G containing S. Therefore $f_{\text{sdim}}(G) = \text{sdim}(G)$. □

The Nordhaus–Gaddum-type results for the strong metric dimension of unicyclic graphs are studied in [17], also [16] investigated the strong metric dimension of unicyclic graphs.

Proposition 2.8. *Let G be a unicyclic graph with p end vertices. If the cycle C of G has length k, l is the greatest order of a path P on C, and every vertex on P has degree 2 in G, then*

$$\text{sdim}(G) = \begin{cases} p-1, & \text{if } \ l \leq \frac{k-2}{2}; \\ p+l-\lfloor \frac{k}{2} \rfloor, & \text{if } \ \frac{k-1}{2} \leq l \leq k-2; \\ p+\lceil \frac{k}{2} \rceil - 1, & \text{if } \ l = k-1. \end{cases}$$

Proof. Let C be the cycle $v_1v_2\ldots v_kv_1$ and X be the set of all end vertices of G. Assume without loss of generality that P is the path $v_1v_2\ldots v_l$, where $\deg v_i = 2$ for $i = 1, 2, \ldots, l$. So $\deg v_k \geq 3$ and $\deg v_{l+1} \geq 3$.

Case 1: $l \leq (k-2)/2$.

Let $W \subseteq X$ and $|W| = p-1$. Let $x \in X - W$ with the property that either $v_k \in I(x, v_1)$ or $v_{l+1} \in I(x, v_1)$. Then W is a strong resolving set of G and $\text{sdim}(G) = p-1$.

Case 2: $(k-1)/2 \leq l \leq k-2$.

Then $W = X \cup \{v_1, v_2, \ldots, v_{l-\lfloor k/2 \rfloor}\}$ is strong resolving set of G and $\text{sdim}(G) = p + l - \lfloor k/2 \rfloor$.

Case 3: $l = k-1$.

Then $W = X \cup \{v_1, v_2, \ldots, v_{\lceil k/2 \rceil - 1}\}$ is a strong resolving set of G and $\text{sdim}(G) = p + \lceil k/2 \rceil - 1$.

□

Proposition 2.9. Let G be a unicyclic graph with p end vertices. If the cycle C of G has length k, l is the greatest order of a path P on C, and every vertex on P has degree 2 in G, then

$$f_{\text{sdim}}(G) = \begin{cases} 1, & \text{if } \ l = k-1 \text{ and if } \ \frac{k-1}{2} \leq l \leq k-2; \\ p-1, & \text{if } \ l = 0; \\ m+n-1, & \text{if } \ 1 \leq l \leq \frac{k-2}{2}. \end{cases}$$

Proof. Let C be the cycle $v_1v_2\ldots v_kv_1$ and X be the set of all end vertices of G. Assume without loss of generality that P is the path $v_1v_2\ldots v_l$, where $\deg \ v_i = 2$ for $i = 1, 2, \ldots, l$. So $\deg \ v_k \geq 3$ and $\deg \ v_{l+1} \geq 3$.

Case 1: $l = k-1$.

Since $\text{diam}(C_k) = \lfloor k/2 \rfloor$, the only strong resolving sets are $X \cup \{v_1, v_2, \ldots, v_{\lceil k/2 \rceil - 1}\}$ and $X \cup \{v_{k-1}, v_{k-2}, \ldots, v_{\lfloor k/2 \rfloor + 1}\}$. According to Lemma 1.1, we have that $f_{\text{sdim}}(G) = 1$. Assume that $(k-1)/2 \leq l \leq k-2$. Then

$X \cup \{v_1, v_2, \ldots, v_{\lceil k/2 \rceil - 1}\}$ and $X \cup \{v_{\lfloor k/2 \rfloor + 1}, \ldots, v_l\}$ are the only strong resolving set of G, and from Lemma 1.1, it follows that $f_{\text{sdim}}(G) = 1$.

Case 2: $l = 0$.

It is easy to see that $f_{\text{sdim}}(G) = p - 1$.

Case 3: $1 \leq l \leq (k-2)/2$.

Let $Y = \{y|\ y$ is an end vertex with the property that $v_k \in I(y, v_1)\}$ and $Z = \{z|\ z$ is an end vertex with the property that $v_l \in I(z, v_1)\}$ and assume that $|Y| = m$ and $|Z| = n$. Let $W \subseteq X$ with $|W| = p - 1$ and $W = X - \{u\}$, where the end vertex u has the property that either $u \in Y$ or $u \in Z$. Then every strong resolving set contains $X - (Y \cup Z)$. Let $W = X - \{y\}$ where $y \in Y$. Let S be a subset of minimum strong resolving set W of G and assume $S = Y \cup Z - \{y\}$. Then W is a unique strong resolving set of G containing S and $f_{\text{sdim}}(G) = m + n - 1$.

□

3. Realization results

In this section, we determine which pair a, b of integers with $0 \leq a \leq b$ and $b \geq 1$ are realizable as the forcing strong metric dimension and strong metric dimension of some nontrivial connected graph.

Theorem 3.1. *For any positive integer a and b with $0 \leq a \leq b$, there is a nontrivial connected graph G with $\text{sdim}(G) = b$ and $f_{\text{sdim}}(G) = a$ if and only if $(a, b) \neq (0, 1)$.*

Proof. Since the path P_n is the only nontrivial connected graph with strong metric dimension 1 and with only two strong resolving sets in P_n, we have $f_{\text{sdim}}(P_n) = 1$. Hence there is no connected graph G with $\text{sdim}(G) = 1$ and $f_{\text{sdim}}(G) = 0$. Thus $(a, b) \neq (0, 1)$.

Assume $a = 0$ and $b = 2$. The required graph G is illustrated in Figure 2.

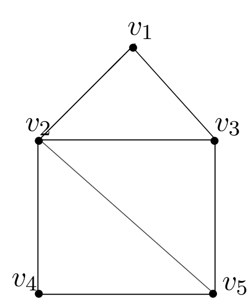

Figure 2. A graph G with $\text{sdim}(G) = 2$ and $f_{\text{sdim}}(G) = 0$.

Let $W = \{v_1, v_4\}$. Then W is a strong resolving set of G and $\text{sdim}(G) \leq 2$. Since G is not a path, then $\text{sdim}(G) = 2$. Also W is the unique strong resolving set of G and $f_{\text{sdim}}(G) = 0$.

Suppose $a = 0$ and $b \geq 3$. Let G be a graph obtained from the path P_6 with vertices $v_1v_2v_3v_4v_5v_6$ and adding new edges, $v_2v_4, v_3v_5, v_3v_6, v_4v_6$ and u_iv_4, u_iv_6, $1 \leq i \leq b-1$, where $u_1, \ldots, u_{b-1}$ are new vertices. The graph G is shown in Figure 3.

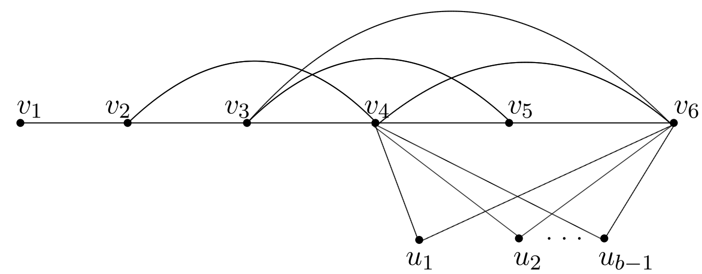

FIGURE 3

Let $W = \{u_1, u_2, \ldots, u_{b-1}, v_1\}$. Then W is a strong resolving set of G and $\mathrm{sdim}(G) \leq b$. Since any minimum strong resolving set contains at least $b-2$ vertices from the set $\{u_1, u_2, \ldots, u_{b-1}\}$, $\mathrm{sdim}(G) \geq b-2$. Assume $\mathrm{sdim}(G) = b-1$ and W' is a minimum strong resolving set of G. Then there exists a vertex u_k such that $u_k \notin W'$. Suppose $v_1 \in W'$. Then the pair (v_3, u_k) is not strongly resolved by any vertex of W'. Suppose $v_i \in W'$, for some $i \neq 1$. Then certainly W' is not a strong resolving set of G. Therefore $\mathrm{sdim}(G) \geq b$ and hence $\mathrm{sdim}(G) = b$.

Next we show that W is a unique minimum strong resolving set of G. Assume there exists a strong resolving set W' such that $W \neq W'$. Then there exists a vertex $u_k \in W$ and $u_k \notin W'$. Suppose $v_1 \notin W'$. Then the pair (v_1, u_k) is not strongly resolved by any vertex of W'. Therefore $v_1 \in W'$. Furthermore, W' contains exactly one vertex v_i ($i \neq 1$). Suppose $v_2 \in W'$. Then the pair (v_5, u_k) is not strongly resolved by any vertex of W'. Suppose $v_3 \in W'$. Then the pair (v_5, u_k) is not strongly resolved by any vertex of W'. Suppose $v_4 \in W'$. Then the pair (v_5, u_k) is not strongly resolved by any vertex of W'. Suppose $v_5 \in W'$. Then the pair (v_3, u_k) is not strongly resolved by any vertex of W'. Suppose $v_6 \in W'$. Then the pair (v_3, u_k) is not strongly resolved by any vertex of W'. Hence W' is not a strong resolving set of G. Therefore W is a unique strong resolving set of G and $f_{\mathrm{sdim}}(G) = 0$.

Now, assume $a > 0$.

Case 1: $a = b$.

When $a = b = 1$, the path P_n has the desired property. When $a = b = 2$, the star $K_{1,3}$ has the required property. When $a = b \geq 3$, the complete graph K_{a+1} has the desired property.

Case 2: $a < b$.

We investigate two subcases:

Subcase 1. $b = a + 1$.

Let G be the graph obtained from the 4-cycle $u_1u_2u_3u_4u_1$ by adding a new edge u_2u_4 and joining b new vertices $v_1, v_2, \ldots, v_b$ to u_2 and u_3. The graph G is shown in Figure 4.

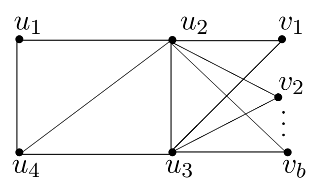

FIGURE 4. A graph G with $\mathrm{sdim}(G) = b$ and $f_{\mathrm{sdim}}(G) = a = b-1$.

Since G contains b extreme vertices, every strong resolving set of G contains at least $b-1$ vertices from the set $\{v_1, v_2, \ldots, v_b\}$. Let $W = \{v_1, v_2, \ldots, v_{b-1}, u_1\}$. Then W is a minimum strong resolving set of G and $\mathrm{sdim}(G) = b$. Also, every strong resolving set of G contains u_1, therefore $f_{\mathrm{sdim}}(G) \le b-1$. Let $S \subseteq W$ with $|S| \le b-2$. Assume $v_{b-1} \notin S$. Then $W' = (W - \{v_{b-1}\}) \cup \{v_b\}$ is a strong resolving set of G containing S and $W' \ne W$. Thus W is not a unique strong resolving set of G containing S. Hence $f_{\mathrm{sdim}}(G) = b-1 = a$.

Subcase 2. $b \ge a+2$.

Let $F = K_{2,b-a}$ be a complete bipartite graph with the vertex set $V(F) = \{u, v\} \cup \{u_1, u_2, \ldots, u_{b-a}\}$. The graph G is obtained from F by adding new vertices $x, v_1, v_2, \ldots, v_{a+1}$ and adding new edges $xu, xu_{b-a}, v_ix, v_iu_{b-a}, 1 \le i \le a+1$. The graph G is shown in Figure 5 for $a = 2$ and $b = 4$.

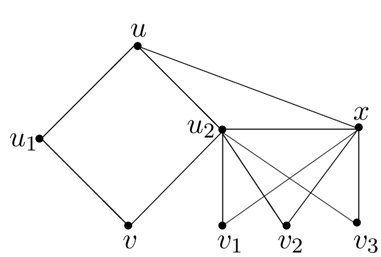

FIGURE 5. A graph G with $\mathrm{sdim}(G) = 4$ and $f_{\mathrm{sdim}}(G) = 2$.

First we note that every strong resolving set contains at least a vertices from $\{v_1, v_2, \ldots, v_{a+1}\}$. Let $W = \{v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_{b-a-1}, v\}$. Then W is a strong resolving set of G and $\mathrm{sdim}(G) \le b$.

Let W' be the minimum strong resolving set of G. Suppose there exist two vertices $u_i, u_j \in V(F) - W'$. Since $d(u_i, u_j) = 2$ and $\text{diam}(G) = 3$, the pair (u_i, u_j) is not strongly resolved by W'. Hence W' contains at least $b - a - 1$ vertices from $V(F)$. Also W' contains at least a vertices from $v_1, v_2, \ldots, v_{a+1}$ and $\text{sdim}(G) \geq b - 1$. Suppose $u \in W'$. Then the pair (v, x) is not strongly resolved by W'. Suppose $x \in W'$. Then the pair (u, v) is not strongly resolved by W'. Hence the only possibility is that v must belong to W'. Therefore $\text{sdim}(G) = b$.

Since every strong resolving set contains the vertices $v, u_1, u_2, \ldots, u_{b-a-1}$ then $f_{\text{sdim}}(G) \leq a$. Let $W = \{v, u_1, u_2, \ldots, u_{b-a-1}, v_1, v_2, \ldots, v_a\}$. Assume S is a proper subset of $W - \{v, u_1, u_2, \ldots, u_{b-a-1}\}$ with $|S| \leq a - 1$. Assume that $v_1 \notin S$. Then $W' = (W - \{v_1\}) \cup \{v_{a+1}\}$ is a minimum strong resolving set of G containing S and $W \neq W'$. Thus W is not a unique strong resolving set of G containing S. Therefore $f_{\text{sdim}}(G) = a$.

□

Acknowledgement

The authors wish to thank the referee of the original version of this manuscript for the helpful suggestions, modified proofs and comments that have improved the readability and quality of the present manuscript.

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