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INDEPENDENCE COMPLEXES AND INCIDENCE GRAPHS

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ABSTRACT. We show that the independence complex of the incidence graph of a hypergraph is homotopy equivalent to the suspension of the combinatorial Alexander dual of the independence complex of the hypergraph, generalizing a result of Csorba. As an application, we refine and generalize a result of Kawamura on a relation between the homotopy types of the independence complex and the edge covering complex of a graph.

1. INTRODUCTION

In [2], Csorba showed that the independence complex of the edge subdivision of a simple graph is homotopy equivalent to the suspension of the combinatorial Alexander dual of the independence complex of the graph. (An alternative proof was given by Barmak [1].) Studying this homotopy equivalence from a poset point of view, we describe the homotopy type of the former as a certain kind of poset of intervals, generalize the result to the case of hypergraphs and give an explicit map giving the homotopy equivalence. The independence complex of a hypergraph (see Definition 4.5) is essentially a simplicial complex of transversals and this generalization might be of interest.

In Section 2, we collect some fundamental definitions and results. In Section 3, we study the homotopy type of the poset of intervals and prove two key results, Propositions 3.5 and 3.8. In Section 4, we define a kind of generalized hypergraph (see Definition 4.1) and describe the relation between the homotopy type of the independence complex of a generalized hypergraph and that of the incidence graph of the hypergraph. See Theorem 4.10 and Corollary 4.11. As an application, we show that the combinatorial Alexander dual of the independence complex of a hypergraph and that of its dual hypergraph are homotopy equivalent and as a special case of this result, we obtain a refinement of a result of Kawamura [4]. See Corollaries 4.15 and 4.17. As another application, in Section 5, we consider the homotopy types of independence complexes of bipartite graphs. The results of this section

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are known [5], [3]; however, our point of view would give a certain natural interpretation of these results.

This work grew out of my attempts to understand the result of Csorba. The author would like to thank K. Iriye for suggesting the generalization to hypergraphs.

2. Preliminaries

In this section, we collect some fundamental definitions, results, and fix notation. Given a poset P, we denote its order complex by $\Delta(P)$, and given a simplicial complex Δ, we denote its face poset by $P(\Delta)$. We denote the geometric realization of a simplicial complex Δ by the same symbol Δ by an abuse of notation; however, we sometimes denote the geometric realization of the order complex $\Delta(P)$ by $|P|$. The order complex of the face poset of a simplicial complex Δ is called the barycentric subdivision of Δ and their geometric realizations are homeomorphic: $\Delta \cong \Delta(P(\Delta))$. When we consider homotopical properties of posets, we consider those of geometric realizations of their order complexes, that is, say, a map $f\colon P \to Q$ of posets is called a homotopy equivalence if the induced map $|f|\colon |P| \to |Q|$ is a homotopy equivalence. We consider subsets of posets as posets via the induced order unless otherwise stated. For a poset P, we denote the dual of P by P^o, which is the poset on the same underlying set with the opposite ordering. A map $f\colon P \to Q$ of posets could be seen as a map of dual posets, which we denote by $f^o\colon P^o \to Q^o$, that is $f^o = f$ as a map of sets. If P has a maximum (resp. minimum) element, then we denote it by $\hat{1}$ or $\hat{1}_P$ (resp. $\hat{0}$ or $\hat{0}_P$) and call it the top (resp. bottom) element. A poset P is said to be bounded if P has both top and bottom elements and is said to be nontrivial if P has more than one element.

Definition 2.1. *Let P and Q be posets.*

(1) *The direct product $P \times Q$ is the poset whose underlying set is the Cartesian product of P and Q with the order given by $(p, q) \leq (p', q') \Leftrightarrow p \leq p'$ and $q \leq q'$.*
(2) *The join $P * Q$ is the poset whose underlying set is the disjoint union $P \amalg Q$ and whose order agrees with the given one on P and Q and $p < q$ for all $p \in P$ and $q \in Q$.*

Definition 2.2. *Let P be a poset, $p \in P$ an element and $S \subset P$ a subset.*

(1) *S is called an upper (resp. lower) set if S has the property that, if $x \in S$ and $x \leq y$ (resp. $x \geq y$), then $y \in S$.*
(2) *We define subposets of P as follows:*

$$P_{\geq p} = \{x \in P \mid x \geq p\}, \qquad P_{\leq p} = \{x \in P \mid x \leq p\},$$
$$P_{>p} = \{x \in P \mid x > p\}, \qquad P_{<p} = \{x \in P \mid x < p\},$$
$$P_{\geq S} = \bigcup_{s \in S} P_{\geq s}, \qquad P_{\leq S} = \bigcup_{s \in S} P_{\leq s}.$$

Note that $P_{\geq S}$ and $P_{\leq S}$ are defined by the union, not the intersection. Therefore, we have $P_{\geq S\cup T} = P_{\geq S} \cup P_{\geq T}$ and $P_{\geq\emptyset} = P_{\leq\emptyset} = \emptyset$.

(3) *If P has the top or bottom element, we use the following notation:*

$$S_+ := P_{>\hat{0}} \cap S = S \setminus \{\hat{0}_P\},$$
$$S_- := P_{<\hat{1}} \cap S = S \setminus \{\hat{1}_P\}.$$

Note that, for posets P, Q and subsets $S \subset P$, $T \subset Q$, we have

$$(P \times Q)_{\geq S\times T} = P_{\geq S} \times Q_{\geq T},$$
$$(P \times Q)_{>S\times T} = (P_{\geq S} \times Q_{>T}) \cup (P_{>S} \times Q_{\geq T}).$$

We call a pair of maps

$$P \underset{f}{\overset{g}{\leftrightarrows}} Q$$

between posets an adjoint pair if the following holds: For all $p \in P$ and $q \in Q$, $f(p) \leq q$ if and only if $p \leq g(q)$. In this case, both f and g are maps of posets, namely, they preserve order. The map f is called a left adjoint or a lower adjoint of g and g is called a right adjoint or an upper adjoint of f. Note that, for any subsets $S \subset P$ and $T \subset Q$ satisfying $f(S) \subset T$ and $g(T) \subset S$, restrictions of f and g give an adjoint pair:

$$S \underset{f}{\overset{g}{\leftrightarrows}} T\ .$$

We give an order on the set of maps of posets from P to Q by

$$f \leq f' \Leftrightarrow f(p) \leq f'(p) \text{ for all } p \in P.$$

Our tools to construct and detect homotopy equivalences between posets are the classical ones due to Quillen:

Theorem 2.3 ([6])**.** *Let $f, f' \colon P \to Q$ be maps of posets.*

(1) *If $f \leq f'$, then f and f' are homotopic.*
(2) *If f has a right adjoint $g \colon Q \to P$, then f is a homotopy equivalence with g its inverse.*
(3) *If $f^{-1}(Q_{\geq q})$ is contractible for all $q \in Q$, then f is a homotopy equivalence.*

3. Posets of intervals

Definition 3.1.

(1) *Let P be a poset. We define a poset $I(P)$ by*

$$I(P) = \{(x, y) \in P \times P \mid x \leq y\}$$

with the order given by

$$(x', y') \leq (x, y) \Leftrightarrow x' \leq x \text{ and } y \leq y'.$$

Note that $I(P)$ is a lower subset of $P \times P^o$. If $S \subset P$ is a subposet, we consider $I(S)$ as a subposet of $I(P)$. One may call $I(P)$ the *poset of closed intervals or the poset of twisted arrows. Note that our ordering on $I(P)$ is the dual of that of Walker's in* [7].

(2) *We relativize the construction above. Let $f\colon X \to P$ and $g\colon Y \to P$ be maps of posets. We define a subposet of $X \times Y^o$ as follows:*
$$I(f,g) = \{(x,y) \in X \times Y^o \mid f(x) \leq g(y)\}\,.$$
Clearly, $I(f,g) = (f \times g^o)^{-1}(I(P))$ and is a lower subset of $X \times Y^o$.

The following two results are well known and can be proved using Theorem 2.3.

Proposition 3.2. *Let P be a poset. Then projections to the first and the second factors give homotopy equivalences:*
$$P \underset{\simeq}{\xleftarrow{p_1}} I(P) \underset{\simeq}{\xrightarrow{p_2}} P^o.$$

Proof. We show that the first projection is a homotopy equivalence. For $p \in P$, we have
$$p_1^{-1}(P_{\geq p}) = \{(x,y) \in I(P) \mid p \leq x\}\,.$$
Define two maps
$$(P_{\geq p})^o \underset{j_p}{\overset{p_2}{\rightleftarrows}} p_1^{-1}(P_{\geq p})$$
by $p_2(x,y) = y$ and $j_p(y) = (p,y)$. Clearly, p_2 and j_p are well-defined maps of posets and give an adjoint pair: $j_p(y') \leq (x,y) \Leftrightarrow y' \leq_{P^o} p_2(x,y)$, whence homotopy equivalences. Since $(P_{\geq p})^o$ has the maximum element p, it is contractible and so is $p_1^{-1}(P_{\geq p})$. Therefore, by the Quillen fibre lemma (Theorem 2.3(3)), the projection p_1 is a homotopy equivalence. □

Lemma 3.3. *Let P be a poset and $T, U \subset P$ upper subsets. We define a map*
$$\phi\colon T \cup U \to \{0,1\} * (T \cap U)$$
of posets by
$$\phi(x) = \begin{cases} 0, & x \notin T \\ 1, & x \notin U \\ x, & x \in T \cap U \end{cases}$$
where we consider $T \cup U$ as a subposet of P and the set $\{0,1\}$ is equipped with the discrete order. If both T and U are contractible, then the map ϕ is a homotopy equivalence.

Proof. Since T and U are upper sets, one easily sees that the map ϕ preserves order.

We denote $\{0,1\} * (T \cap U)$ by S.

For $p \in T \cap U \subset S$, we have $\phi^{-1}(S_{\geq p}) = (T \cap U)_{\geq p}$ which is contractible since it has the minimum element p. Clearly, we have $\phi^{-1}(S_{\geq 0}) = U$ and $\phi^{-1}(S_{\geq 1}) = T$.

Therefore, if both T and U are contractible, the map ϕ is a homotopy equivalence by the Quillen fibre lemma (Theorem 2.3(3)). $\square$

Remark 3.4: Under the assumption of Lemma 3.3, we have $\Delta(T \cup U) = \Delta(T) \cup \Delta(U)$ and $\Delta(T) \cap \Delta(U) = \Delta(T \cap U)$.

Combining Proposition 3.2 and Lemma 3.3, we obtain the following simple observation, which is a variant of a result of Walker [7, Theorem 6.1 (c)] and is one of our key results.

Proposition 3.5. *Let P be a bounded poset, $U \subset P$ an upper subset and $L \subset P$ a lower subset. We define maps*

$$\varphi_1 \colon I(U) \cup I(L) \to \{0,1\} * (U \cap L)\,,$$
$$\varphi_2 \colon I(U) \cup I(L) \to \{0,1\} * (U \cap L)^o$$

of posets by

$$\varphi_i(x_1, x_2) = \begin{cases} 0, & x_1 \notin U \\ 1, & x_2 \notin L \\ x_i, & x_1 \in U \text{ and } x_2 \in L \end{cases}$$

where we consider $I(U) \cup I(L)$ as a subposet of $I(P)$ and the set $\{0,1\}$ is equipped with the discrete order. If U and L are nonempty, then the maps φ_1 and φ_2 are homotopy equivalences.

Remark 3.6: (1) Note that if $L = \emptyset$, then $I(U) \cup I(L) = I(U) \simeq U$ which is either empty or contractible. On the other hand, $\{0,1\} * (U \cap L) = \{0,1\} * \emptyset = \{0,1\}$.
(2) Walker [7, Theorem 6.1 (c)] showed that $|I(P_{>\hat{0}}) \cup I(P_{<\hat{1}})|$ is homeomorphic to $|\{0,1\} * \left(P_{>\hat{0}} \cap P_{<\hat{1}}\right)|$. In general, $|I(U) \cup I(L)|$ and $|\{0,1\} * (U \cap L)\,|$ are not homeomorphic.

Proof. It is straightforward to see that both $I(U)$ and $I(L)$ are upper subsets of $I(P)$, $I(U) \cap I(L) = I(U \cap L)$ and the map φ_1 is the composite:

$$I(U) \cup I(L) \xrightarrow{\phi} \{0,1\} * I\,(U \cap L) \xrightarrow{id * p_1} \{0,1\} * (U \cap L)\,.$$

We have $I(U) \simeq U$, $I(L) \simeq L$ and $I(U \cap L) \underset{\simeq}{\xrightarrow{p_1}} U \cap L$ by Proposition 3.2. It is easy to see that the map $id * p_1$ is also a homotopy equivalence. If U and L are nonempty, they are contractible since P is bounded. Therefore, by Lemma 3.3, the map ϕ is a homotopy equivalence, whence so is the map φ_1. $\square$

We consider the relative case. Observe that:

Lemma 3.7. *If $f \colon P \to Q$ is a map of posets and $T \subset Q$ is an upper set (resp. a lower set), then so is $f^{-1}(T)$.*

Moreover, if f has a right adjoint $g \colon Q \to P$, then the following hold:

(1) *If P has the bottom element, then so does Q and $f(\hat{0}_P) = \hat{0}_Q$. Dually, if Q has the top element, then so does P and $g(\hat{1}_Q) = \hat{1}_P$.*
(2) *For any subset $S \subset P$, we have $g^{-1}(P_{\geq S}) = Q_{\geq f(S)}$. Dually, for any subset $T \subset Q$, we have $f^{-1}(Q_{\leq T}) = P_{\leq g(T)}$. In particular, if $U \subset P$ is an upper set, then $g^{-1}(U) = Q_{\geq f(U)}$ and if $L \subset Q$ is an lower set, then $f^{-1}(L) = P_{\leq g(L)}$.*
 Therefore, restrictions of f and g give adjoint pairs of maps of posets:

$$P_{\geq S} \cap P_{\leq g(T)} \underset{f}{\overset{g}{\rightleftarrows}} Q_{\geq f(S)} \cap Q_{\leq T} \ ,$$

$$U \cap f^{-1}(L) \underset{f}{\overset{g}{\rightleftarrows}} Q_{\geq f(U)} \cap L \ .$$

(3) *The map $f^o \colon P^o \to Q^o$ is a right adjoint of $g^o \colon Q^o \to P^o$: $g^o(q) \leq_{P^o} p \Leftrightarrow q \leq_{Q^o} f^o(p)$.*

Proof. The proof is straightforward. □

Proposition 3.8. *Let X, Y, P be bounded posets and $f \colon X \to P$, $g \colon Y \to P$ maps of posets. Assume that f has a right adjoint h and g has a left adjoint k:*

$$X \underset{f}{\overset{h}{\rightleftarrows}} P \underset{k}{\overset{g}{\rightleftarrows}} Y.$$

Then we have an adjoint pair $X \times Y^o \underset{f \times g^o}{\overset{h \times k^o}{\rightleftarrows}} P \times P^o$ *and the following hold:*

(1) *The restrictions give an adjoint pair of maps of posets:*

$$I(f,g)_{++} \underset{f \times g^o}{\overset{h \times k^o}{\rightleftarrows}} I\left(P_{\geq f(X_+)} \cap P_{\leq g(Y_-)}\right)$$

 where $I(f,g)_{++} = I(f,g) \cap (X_+ \times (Y_-)^o)$.
 In particular, we have homotopy equivalences

$$P_{\geq f(X_+)} \cap P_{\leq g(Y_-)} \xleftarrow[\simeq]{fp_1} I(f,g)_{++} \xrightarrow[\simeq]{g^o p_2} \left(P_{\geq f(X_+)} \cap P_{\leq g(Y_-)}\right)^o$$

 where p_i is the projection to the i-th factor.
(2) *The restrictions give an adjoint pair of maps of posets:*

$$I(f,g)_{+} \underset{f \times g^o}{\overset{h \times k^o}{\rightleftarrows}} I\left(P_{\geq f(X_+)}\right) \cup I\left(P_{\leq g(Y_-)}\right) \ .$$

In particular, if X and Y are nontrivial, then we have homotopy equivalences

$$\begin{tikzcd}
& \{0,1\} * \left(P_{\geq f(X_+)} \cap P_{\leq g(Y_-)}\right) \\
I(f,g)_+ \arrow[r, "f\times g^o", "\simeq"'] & I\left(P_{\geq f(X_+)}\right) \cup I\left(P_{\leq g(Y_-)}\right) \arrow[u, "\varphi_1", "\simeq"'] \arrow[d, "\varphi_2"', "\simeq"] \\
& \{0,1\} * \left(P_{\geq f(X_+)} \cap P_{\leq g(Y_-)}\right)^o .
\end{tikzcd}$$

Proof. Clearly, the map $h \times k^o \colon P \times P^o \to X \times Y^o$ is a right adjoint of $f \times g^o \colon X \times Y^o \to P \times P^o$.

One can show (1) in the same way as (2).

We show (2). Recall that $I(f,g) = (f \times g^o)^{-1}(I(P))$ and $I(P)$ is a lower subset of $P \times P^o$. We also note that $(X \times Y^o)_+$ is an upper subset of $X \times Y^o$ and $I(f,g)_+ = (X \times Y^o)_+ \cap I(f,g)$. Thus, we may apply Lemma 3.7(2).

We have

$$\begin{aligned}
(f \times g^o)((X \times Y^o)_+) &= (f \times g^o)\left((X_+ \times Y^o) \cup (X \times (Y^o)_+)\right) \\
&= \left(f(X_+) \times g^o(Y)\right) \cup \left(f(X) \times g^o(Y_-)\right), \\
(P \times P^o)_{\geq (f\times g^o)((X\times Y^o)_+)} &= (P \times P^o)_{\geq f(X_+) \times g^o(Y)} \cup (P \times P^o)_{\geq f(X) \times g^o(Y_-)} \\
&= \left(P_{\geq f(X_+)} \times P^o{}_{\geq g^o(Y)}\right) \cup \left(P_{\geq f(X)} \times P^o{}_{\geq g^o(Y_-)}\right) \\
&= \left(P_{\geq f(X_+)} \times P^o\right) \cup \left(P \times P^o{}_{\geq g^o(Y_-)}\right)
\end{aligned}$$

where the last equality holds since $\hat{0}_P = f(\hat{0}_X) \in f(X)$ and $\hat{1}_P = g(\hat{1}_Y) \in g(Y)$ by Lemma 3.7(1). Note that $P^o{}_{\geq g^o(Y_-)} = P_{\leq g(Y_-)}$ as subsets of P, whence

$$\begin{aligned}
\left(P_{\geq f(X_+)} \times P^o\right) \cap I(P) &= I(P_{\geq f(X_+)}), \\
\left(P \times P^o{}_{\geq g^o(Y_-)}\right) \cap I(P) &= I(P_{\leq g(Y_-)}),
\end{aligned}$$

and thus

$$(P \times P^o)_{\geq (f\times g^o)((X\times Y^o)_+)} \cap I(P) = I(P_{\geq f(X_+)}) \cup I(P_{\leq g(Y_-)}).$$

Therefore, by Lemma 3.7(2), we have an adjoint pair of maps of posets:

$$I(f,g)_+ \underset{f\times g^o}{\overset{h\times k^o}{\rightleftarrows}} I(P_{\geq f(X_+)}) \cup I(P_{\leq g(Y_-)}) \ .$$

If X and Y are nontrivial, X_+ and Y_- are nonempty and so are $P_{\geq f(X_+)}$ and $P_{\leq g(Y_-)}$. Therefore, in this case, the map φ_1 is a homotopy equivalence by Proposition 3.5. □

Given a Boolean lattice $P = (P, \wedge, \vee, \neg, \hat{0}, \hat{1})$ and a subset $S \subset P$, we define a subset $\neg S \subset P$ by $\neg S = \{\neg x | x \in S\} = \{x | \neg x \in S\}$ and given maps

$f\colon X \to P$, $g\colon P \to Y$ of sets, we define maps $\neg f\colon X \to P$, $g\neg\colon P \to Y$ by $(\neg f)(x) = \neg(f(x))$, and $(g\neg)(p) = g(\neg p)$. Note that we have $\neg(P - S) = P - \neg S$, $\neg(P_{\geq S}) = P_{\leq \neg S}$ and $\neg(S_+) = (\neg S)_-$.

Corollary 3.9. *Let P be a Boolean lattice. Given a subset $S \subset P$, we define a subposet of $P \times P$ by*

$$D(S) = \left\{(x, y) \in S \times \neg S \;\middle|\; x \wedge y = \hat{0}\right\}.$$

Then the following hold:

(1) *Projections to the first and the second factors give homotopy equivalences:*

$$S \xleftarrow[\simeq]{p_1} D(S) \xrightarrow[\simeq]{p_2} \neg S.$$

(2) *Given an upper subset $U \subset P$ and a lower subset $L \subset P$, the maps*

$$\psi_1\colon D(U) \cup D(L) \to \{0,1\} * (U \cap L),$$
$$\psi_2\colon D(U) \cup D(L) \to \{0,1\} * \neg(U \cap L)$$

of posets defined by

$$\psi_i(x_1, x_2) = \begin{cases} 0, & x_1 \notin U \\ 1, & x_2 \notin \neg L \\ x_i, & x_1 \in U \text{ and } x_2 \in \neg L \end{cases}$$

are homotopy equivalences if U and L are nonempty.

Proof. Clearly, the map $id \times \neg\colon P \times P \to P \times P^o$ sending $(x, y) \in P \times P$ to $(x, \neg y) \in P \times P^o$ gives an isomorphism of posets $D(S) \cong I(S)$ and we have $\psi_1 = \varphi_1(id \times \neg)$, $\psi_2 = (id{*}\neg)\varphi_2(id \times \neg)$. The results follow from Propositions 3.2 and 3.5. $\square$

Corollary 3.10. *Let P be a Boolean lattice, X_1, X_2 bounded posets and $f_i\colon X_i \to P$ maps of posets which have right adjoints $g_i\colon P \to X_i$. We define subposets of $X_1 \times X_2$ as follows:*

$$D(f_1, f_2) = \left\{(x_1, x_2) \in X_1 \times X_2 \;\middle|\; f_1(x_1) \wedge f_2(x_2) = \hat{0}\right\},$$
$$D(f_1, f_2)_{++} = D(f_1, f_2) \cap (X_{1+} \times X_{2+}).$$

Then the following hold:

(1) *We have an adjoint pair of maps of posets:*

$$D(f_1, f_2)_{++} \underset{f_1 \times f_2}{\overset{g_1 \times g_2}{\rightleftarrows}} D\left(P_{\geq f_1(X_{1+})} \cap P_{\leq \neg f_2(X_{2+})}\right).$$

In particular, we have homotopy equivalences:

$$P_{\geq f_1(X_{1+})} \cap P_{\leq \neg f_2(X_{2+})} \xleftarrow[\simeq]{f_1 p_1} D(f_1, f_2)_{++} \xrightarrow[\simeq]{f_2 p_2} P_{\leq \neg f_1(X_{1+})} \cap P_{\geq f_2(X_{2+})}.$$

(2) *We have an adjoint pair of maps of posets:*

$$D(f_1, f_2)_+ \underset{f_1\times f_2}{\overset{g_1\times g_2}{\rightleftarrows}} D\left(P_{\geq f_1(X_{1+})}\right) \cup D\left(P_{\leq \neg f_2(X_{2+})}\right) .$$

Furthermore, if X_1 and X_2 are nontrivial, then we have homotopy equivalences:

$$\begin{tikzcd} & \{0,1\} * \left(P_{\geq f_1(X_{1+})} \cap P_{\leq \neg f_2(X_{2+})}\right) \\ D(f_1, f_2)_+ \arrow[r, "f_1\times f_2", "\simeq"'] & D\left(P_{\geq f_1(X_{1+})}\right) \cup D\left(P_{\leq \neg f_2(X_{2+})}\right) \arrow[u, "\psi_1", "\simeq"'] \arrow[d, "\psi_2"', "\simeq"] \\ & \{0,1\} * \left(P_{\leq \neg f_1(X_{1+})} \cap P_{\geq f_2(X_{2+})}\right) . \end{tikzcd}$$

Proof. We have maps of posets

$$X_1 \underset{f_1}{\overset{g_1}{\rightleftarrows}} P \underset{g_2^o\neg}{\overset{\neg f_2^o}{\rightleftarrows}} X_2^o$$

which satisfy the assumption of Proposition 3.8. Note that $(X_2^o)_- = X_{2+}$ as sets, $\neg\left(P_{\geq f_1(X_{1+})} \cap P_{\leq \neg f_2(X_{2+})}\right) = P_{\leq \neg f_1(X_{1+})} \cap P_{\geq f_2(X_{2+})}$ as subsets of P and $D(f_1, f_2) = I(f_1, \neg f_2^o) \subset X_1 \times X_2 = X_1 \times (X_2^o)^o$. The results follow from Proposition 3.8. $\square$

4. Independence complexes and incidence graphs

Let $P = (P, \wedge, \vee, \neg, \hat{0}, \hat{1})$ be a Boolean lattice which is complete, namely, every subset $S \subset P$ has a supremum which we denote by $\bigvee S$ or $\bigvee_{p\in S} p$. What we have in mind as P is the power set of a (finite) set V: $P = \mathcal{P}(V)$ (Example 4.4 and Corollaries 4.11 and 4.13). While the present formulation in a general set up makes it easy to distinguish the complements in V from the complements in $\mathcal{P}(V)$.

Consider a triple $H = (P, f_1, f_2)$ where $f_i \colon V_i \to P$ is a map from a set V_i for each $i = 1, 2$. One may think of such a triple as a generalized hypergraph whose edges are indexed by $f_1 \colon V_1 \to P$ and whose "vertex set" is $f_2 \colon V_2 \to P$. We are mainly interested in the case when $P = \mathcal{P}(V)$ and f_2 is the inclusion $V \hookrightarrow \mathcal{P}(V)$. Then H is identified with a hypergraph possibly with multiple edges that are indexed by $f_1 \colon V_1 \to \mathcal{P}(V)$. The term "generalized hypergraph" may have various meanings in the literature, yet we will use the term only for the meaning given below.

Definition 4.1. (1) *A generalized hypergraph is a triple $H = (P, f_1, f_2)$ where $P = (P, \wedge, \vee, \neg, \hat{0}, \hat{1})$ is a complete Boolean lattice and $f_i \colon V_i \to P$ is a map from a set V_i for each $i = 1, 2$.*

(2) *Given a generalized hypergraph $H = (P, f_1, f_2)$, we define a bipartite graph $B(H)$ as follows:*

$$V(B(H)) = V_1 \amalg V_2,$$
$$E(B(H)) = \left\{ \{x_1, x_2\} \subset V_1 \amalg V_2 \;\middle|\; x_i \in V_i \text{ and } f_1(x_1) \wedge f_2(x_2) \neq \hat{0} \right\}.$$

We call the graph $B(H)$ the incidence graph of H.

Example 4.2. *Let $G = (V, E)$ be a simple graph, namely, a graph without loops and parallel edges. We may think of G as a generalized hypergraph:*

$$G = (\mathcal{P}(V), E \subset \mathcal{P}(V), V \subset \mathcal{P}(V)).$$

Then, clearly, the incidence graph $B(G)$ is the edge subdivision G_2 of G.

Example 4.3. *Let $H = (V, f_E \colon E \to \mathcal{P}(V))$ be a hypergraph whose vertex set is V and whose edges are indexed by the map f_E. We may think of H as a generalized hypergraph*

$$H = (\mathcal{P}(V), f_E \colon E \to \mathcal{P}(V), V \subset \mathcal{P}(V)).$$

The dual hypergraph $H^\vee$ is given by

$$H^\vee = (\mathcal{P}(E), f_V \colon V \to \mathcal{P}(E), E \subset \mathcal{P}(E))$$

where f_E and f_V determine each other by

$$e \in f_V(v) \Leftrightarrow v \in f_E(e).$$

Then the natural bijection $E \amalg V \cong V \amalg E$ gives an identification $B(H) = B(H^\vee)$.

Example 4.4. *Given a generalized hypergraph $H = (P, f_1, f_2)$, we define hypergraphs as follows:*

$$H_{V_2} = (\mathcal{P}(V_2), f_{V_1} \colon V_1 \to \mathcal{P}(V_2), V_2 \subset \mathcal{P}(V_2)),$$
$$H_{V_1} = (\mathcal{P}(V_1), f_{V_2} \colon V_2 \to \mathcal{P}(V_1), V_1 \subset \mathcal{P}(V_1))$$

where

$$f_{V_1}(x_1) = \left\{ x_2 \in V_2 \;\middle|\; f_1(x_1) \wedge f_2(x_2) \neq \hat{0} \right\},$$
$$f_{V_2}(x_2) = \left\{ x_1 \in V_1 \;\middle|\; f_1(x_1) \wedge f_2(x_2) \neq \hat{0} \right\}.$$

Clearly, we have $B(H) = B(H_{V_2}) = B(H_{V_1})$ and ${H_{V_2}}^\vee = H_{V_1}$. Moreover, if H is a hypergraph $(\mathcal{P}(V), f_E \colon E \to \mathcal{P}(V), V \subset \mathcal{P}(V))$, then $H_V = H$ and $H_E = H^\vee$.

Recall that, given a finite simple graph $G = (V, E)$, a subset $\sigma \subset V$ is called an independent set if σ contains no edge: $\sigma \notin \mathcal{P}(V)_{\geq E}$. All the independent sets in G form a simplicial complex $\mathrm{Ind}(G)$ called the independence complex of G. Note that we may define independence complexes of hypergraphs in the same way.

Definition 4.5. *Let $H = (V, f\colon E \to \mathcal{P}(V))$ be a hypergraph with V finite. We define a simplicial complex* $\mathrm{Ind}(H)$ *as follows:*

$$\mathrm{Ind}(H) = \left\{\sigma \subset V \;\middle|\; \sigma \notin \mathcal{P}(V)_{\geq f(E)}\right\}.$$

Recall that a subset $\sigma \subset V$ is called a transversal if σ has nonempty intersection with $f(e)$ for every $e \in E$. Clearly σ is independent if and only if $V - \sigma$ is transversal and we have

$$\mathrm{Ind}(H) = \{\sigma \subset V \mid V - \sigma \text{ is transversal}\}.$$

We also note that the face poset of $\mathrm{Ind}(H)$ is $\left(\mathcal{P}(V) - \mathcal{P}(V)_{\geq f(E)}\right)_+$.

Definition 4.6. *Let $H = (P, f_1, f_2)$ be a generalized hypergraph.*

(1) *Given a subset $S \subset P$, an element $p \in P$ is said to be independent with respect to S if $p \notin P_{\geq S}$ and be transversal if $p \notin P_{\leq \neg S}$. Note that p is independent if and only if $\neg p$ is transversal.*

(2) *We define subposets of P as follows:*

$$\begin{aligned}
\mathrm{ind}(H) &= \left(P - P_{\geq f_1(V_1)}\right) \cup \neg\left(P - P_{\geq f_2(V_2)}\right)\\
&= \left(P - P_{\geq f_1(V_1)}\right) \cup \left(P - P_{\leq \neg f_2(V_2)}\right),\\
\mathrm{Pind}(H) &= \left(P - P_{\geq f_1(V_1)}\right)_+ \cup \neg\left(\left(P - P_{\geq f_2(V_2)}\right)_+\right)\\
&= \left(P_+ - P_{\geq f_1(V_1)}\right) \cup \left(P_- - P_{\leq \neg f_2(V_2)}\right),
\end{aligned}$$

that is, $p \in \mathrm{ind}(H)$ if and only if it is independent with respect to $f_1(V_1)$ or transversal with respect to $f_2(V_2)$. We call $\mathrm{ind}(H)$ *the independence poset of H and* $\mathrm{Pind}(H)$ *its face poset. We denote the order complex of the face poset by $\mathfrak{Ind}(H)$:*

$$\mathfrak{Ind}(H) = \Delta\left(\mathrm{Pind}(H)\right).$$

Remark 4.7: Note that the poset structure of $\mathrm{ind}(H)$ does not determine that of $\mathrm{Pind}(H)$ which also depends on the description of $\mathrm{ind}(H)$ as the union of $P - P_{\geq f_1(V_1)}$ and $P - P_{\leq \neg f_2(V_2)}$.

Example 4.8. *In the case when H is a hypergraph*

$$H = (\mathcal{P}(V), f\colon E \to \mathcal{P}(V), V \subset \mathcal{P}(V)),$$

since $\mathcal{P}(V)_{\geq V} = \mathcal{P}(V)_+$, we have:

$$\begin{aligned}
\mathrm{ind}(H) &= \left(\mathcal{P}(V) - \mathcal{P}(V)_{\geq f(E)}\right) \cup \{V\},\\
\mathrm{Pind}(H) &= \left(\mathcal{P}(V) - \mathcal{P}(V)_{\geq f(E)}\right)_+.
\end{aligned}$$

Therefore, if V is finite, $\mathrm{ind}(H) = \mathrm{Ind}(H) \cup \{V\}$ *as subposets of $\mathcal{P}(V)$ and* $\mathrm{Pind}(H)$ *is the face poset of the independence complex:* $\mathrm{Pind}(H) = P(\mathrm{Ind}(H))$. *Hence, $\mathfrak{Ind}(H)$ is the barycentric subdivision of* $\mathrm{Ind}(H)$. *In particular, $\mathfrak{Ind}(H)$ and* $\mathrm{Ind}(H)$ *are homeomorphic.*

Note that if V is an infinite set, then $\mathrm{Pind}(H)$ *may contain "infinite dimensional" simplexes.*

Remark 4.9: The definition of $\mathrm{Pind}(H)$ is not symmetric with respect to f_1 and f_2. However, note that we have $\mathrm{Pind}(P, f_2, f_1) = \neg \mathrm{Pind}(P, f_1, f_2)$ as subsets of P, whence $\mathrm{Pind}(P, f_2, f_1) \cong \mathrm{Pind}(P, f_1, f_2)^o$ as posets.

We consider the relation between the independence poset of a generalized hypergraph and that of the incidence graph.

Note that given a bipartite graph B whose vertex set is decomposed into sides $V_1 \amalg V_2$, we may think of $\mathrm{Pind}(B)$ as a subset of $\mathcal{P}(V_1) \times \mathcal{P}(V_2)$ via the natural identification $\mathcal{P}(V_1 \amalg V_2) \cong \mathcal{P}(V_1) \times \mathcal{P}(V_2)$. We denote the poset $\mathrm{Pind}(B) \cap \left(\mathcal{P}(V_1)_+ \times \mathcal{P}(V_2)_+\right)$ by $\mathrm{Pind}(B)_{++}$. That is, $\mathrm{Pind}(B)_{++}$ is the poset of independent sets which have nonempty intersections with both V_1 and V_2.

Theorem 4.10. *Let $H = (P, f_1, f_2)$ be a generalized hypergraph.*

Define maps $f_{i*} \colon \mathcal{P}(V_i) \to P$ by the composite $\mathcal{P}(V_i) \xrightarrow{f_i} \mathcal{P}(P) \xrightarrow{\bigvee} P$, that is, $f_{i*}(\sigma) = \bigvee f(\sigma) = \bigvee_{x \in \sigma} f(x) \in P$. Then we have a commutative diagram of posets

$$\begin{tikzcd}
\mathrm{Pind}(B(H))_{++} \arrow[r, phantom, "\subset"] \arrow[d, "f_{1*}\times f_{2*}"', "\simeq"] & \mathrm{Pind}(B(H)) \arrow[d, "\simeq"', "f_{1*}\times f_{2*}"] \\
D\left(P_{\geq f_1(V_1)} \cap P_{\leq \neg f_2(V_2)}\right) \arrow[r, phantom, "\subset"] \arrow[d, "p_2"', "\simeq"] & D\left(P_{\geq f_1(V_1)}\right) \cup D\left(P_{\leq \neg f_2(V_2)}\right) \arrow[d, "\psi_2"] \\
\neg\left(P - \mathrm{ind}(H)\right) \arrow[r, phantom, "\subset"] & \{0,1\} * \neg\left(P - \mathrm{ind}(H)\right)
\end{tikzcd}$$

where the vertical arrows are homotopy equivalences except the map ψ_2. Furthermore, if $V_i \neq \emptyset$ for $i = 1, 2$, then the map ψ_2 is also a homotopy equivalence.

Proof. Clearly, $f_{i*} \colon \mathcal{P}(V_i) \to P$ has a right adjoint $f_i^* \colon P \to \mathcal{P}(V_i)$ given by $f_i^*(p) = f_i^{-1}(P_{\leq p})$. Since $f_{i*}(\{x\}) = f_i(x)$ for $x \in V_i$, we see that $P_{\geq f_{i*}\left(\mathcal{P}(V_i)_+\right)} = P_{\geq f_i(V_i)}$. Therefore, by Corollary 3.10, we have homotopy equivalences

$$\begin{tikzcd}
D(f_{1*}, f_{2*})_{++} \arrow[r, phantom, "\subset"] \arrow[d, "f_{1*}\times f_{2*}"', "\simeq"] & D(f_{1*}, f_{2*})_{+} \arrow[d, "\simeq"', "f_{1*}\times f_{2*}"] \\
D\left(P_{\geq f_1(V_1)} \cap P_{\leq \neg f_2(V_2)}\right) \arrow[r, phantom, "\subset"] \arrow[d, "p_2"', "\simeq"] & D\left(P_{\geq f_1(V_1)}\right) \cup D\left(P_{\leq \neg f_2(V_2)}\right) \arrow[d, "\psi_2"] \\
P_{\leq \neg f_1(V_1)} \cap P_{\geq f_2(V_2)} \arrow[r, phantom, "\subset"] & \{0,1\} * \left(P_{\leq \neg f_1(V_1)} \cap P_{\geq f_2(V_2)}\right).
\end{tikzcd}$$

If $V_i \neq \emptyset$, then $\mathcal{P}(V_i)$ is nontrivial and the map ψ_2 is also a homotopy equivalence.

Recall that we have $\mathrm{Pind}(B(H)) = \left(\mathcal{P}(V) - \mathcal{P}(V)_{\geq E}\right)_+$ where $V = V(B(H))$ and $E = E(B(H))$. We identify

$$\mathcal{P}(V) = \mathcal{P}(V_1 \amalg V_2) \cong \mathcal{P}(V_1) \times \mathcal{P}(V_2)$$

and with this identification, we have

$$E = \left\{(\{x_1\}, \{x_2\}) \in \mathcal{P}(V_1) \times \mathcal{P}(V_2) \;\middle|\; f_1(x_1) \wedge f_2(x_2) \neq \hat{0}\right\}.$$

Since P is a complete Boolean lattice, it is infinitely distributive; hence, for all $(\sigma_1, \sigma_2) \in \mathcal{P}(V_1) \times \mathcal{P}(V_2)$, we have

$$f_{1*}(\sigma_1) \wedge f_{2*}(\sigma_2) = \bigvee_{\substack{x_1 \in \sigma_1 \\ x_2 \in \sigma_2}} (f_1(x_1) \wedge f_2(x_2))$$

and we see that $(\sigma_1, \sigma_2) \in \mathcal{P}(V)_{\geq E}$ if and only if $f_{1*}(\sigma_1) \wedge f_{2*}(\sigma_2) \neq \hat{0}$. In other words, we have

$$\begin{aligned} \mathrm{Pind}(B(H)) &= \left(\mathcal{P}(V) - \mathcal{P}(V)_{\geq E}\right)_+ \\ &= \left\{(\sigma_1, \sigma_2) \in \mathcal{P}(V_1) \times \mathcal{P}(V_2) \;\middle|\; f_{1*}(\sigma_1) \wedge f_{2*}(\sigma_2) = \hat{0}\right\}_+ \\ &= D(f_{1*}, f_{2*})_+, \\ \mathrm{Pind}(B(H))_{++} &= D(f_{1*}, f_{2*})_{++}. \end{aligned}$$

Finally, we see that

$$\begin{aligned} \neg\,(P - \mathrm{ind}(H)) &= \neg\left(P - \left(P - P_{\geq f_1(V_1)}\right) \cup \left(P - P_{\leq \neg f_2(V_2)}\right)\right) \\ &= \neg\left(P_{\geq f_1(V_1)} \cap P_{\leq \neg f_2(V_2)}\right) \\ &= P_{\leq \neg f_1(V_1)} \cap P_{\geq f_2(V_2)}. \end{aligned}$$

□

Recall that, given a simplicial complex K on a finite vertex set V, the combinatorial Alexander dual of K is the simplicial complex given by

$$K^* = \{\sigma \subset V \mid V - \sigma \notin K\}.$$

Clearly, the face poset is given by

$$P(K^*) = (\neg\,(\mathcal{P}(V) - K))_+ = \neg\left((\mathcal{P}(V) - K)_-\right).$$

Let H be a hypergraph with V finite. We have seen in Example 4.8 that $\mathrm{ind}(H)$ is equal to $\mathrm{Ind}(H) \cup \{V\}$ as subposets of $\mathcal{P}(V)$; hence,

$$\begin{aligned} \neg\,(\mathcal{P}(V) - \mathrm{ind}(H)) &= \neg\,(\mathcal{P}(V) - \mathrm{Ind}(H) \cup \{V\}) \\ &= \neg\left((\mathcal{P}(V) - \mathrm{Ind}(H))_-\right) \\ &= P(\mathrm{Ind}(H)^*). \end{aligned}$$

In view of this observation, for a generalized hypergraph $H = (P, f_1, f_2)$, we denote the poset $\neg\,(P - \mathrm{ind}(H))$ by $\mathrm{Pind}^*(H)$ and its order complex by

$\mathfrak{Ind}^*(H)$:

$$\begin{aligned}\mathrm{Pind}^*(H) &= \neg\left(P - \mathrm{ind}(H)\right),\\ \mathfrak{Ind}^*(H) &= \Delta\left(\mathrm{Pind}^*(H)\right).\end{aligned}$$

If H is a hypergraph with finite vertices, $\mathfrak{Ind}^*(H)$ is the barycentric subdivision of $\mathrm{Ind}(H)^*$. We also note that, if V_1 and V_2 are finite, then the incidence graph $B(H)$ is a finite graph and the complexes $\mathfrak{Ind}(B(H))$ and $\mathrm{Ind}(B(H))$ are homeomorphic.

With this notation, we restate Theorem 4.10 as follows:

Corollary 4.11. *Let $H = (P, f_1, f_2)$ be a generalized hypergraph. Then we have a commutative diagram*

$$\begin{tikzcd} {|\,\mathrm{Pind}(B(H))_{++}|} \arrow[r, "\simeq"] \arrow[d] & \mathfrak{Ind}^*(H) \arrow[d] \\ \mathfrak{Ind}(B(H)) \arrow[r] & S^0 * \mathfrak{Ind}^*(H) \end{tikzcd}$$

where the top horizontal arrow is a homotopy equivalence.

Furthermore, if $V_i \neq \emptyset$ for $i = 1, 2$, then the bottom horizontal arrow is also a homotopy equivalence:

$$\mathfrak{Ind}(B(H)) \xrightarrow{\simeq} S^0 * \mathfrak{Ind}^*(H).$$

In particular, if $H = (V, f\colon E \to \mathcal{P}(V))$ is a finite hypergraph with $V \neq \emptyset$ and $E \neq \emptyset$, then we have a homotopy equivalence:

$$\mathrm{Ind}(B(H)) \xrightarrow{\simeq} S^0 * \mathrm{Ind}(H)^*.$$

Remark 4.12: Let $G = (V, E)$ be a finite simple graph. Then $B(G)$ is the edge subdivision G_2 of G. By Corollary 4.11, we have a homotopy equivalence:

$$\mathrm{Ind}(G_2) \xrightarrow{\simeq} S^0 * \mathrm{Ind}(G)^*.$$

Thus, Corollary 4.11 is a generalization of a result of Csorba [2, Theorem 6] to the case of (generalized) hypergraphs. Moreover, this homotopy equivalence is given by the following map

$$\psi\colon P(\mathrm{Ind}(G_2)) \to \{0,1\} * P(\mathrm{Ind}(G)^*)$$

between face posets: We have

$$\begin{aligned} P(\mathrm{Ind}(G_2)) &= \left\{ (\tau, \sigma) \in \mathcal{P}(E) \times \mathcal{P}(V) \;\middle|\; \begin{array}{l} (\tau,\sigma) \neq (\emptyset, \emptyset) \text{ and} \\ \qquad \left(\bigcup_{e \in \tau} e \right) \cap \sigma = \emptyset \end{array} \right\}, \\ P(\mathrm{Ind}(G)^*) &= \{\sigma \in \mathcal{P}(V) \mid \sigma \neq \emptyset \text{ and } \sigma \cap e = \emptyset \text{ for some } e \in E\} \end{aligned}$$

and

$$\psi(\tau,\sigma)=\begin{cases}0, & \tau=\emptyset\\ 1, & \sigma=\emptyset\\ \sigma, & \tau\neq\emptyset \text{ and } \sigma\neq\emptyset.\end{cases}$$

Corollary 4.13. *Let $H=(P,f_1,f_2)$ be a generalized hypergraph. Then we have homotopy equivalences*

$$\mathfrak{Ind}^*(H_{V_2})\simeq\mathfrak{Ind}^*(H)\simeq\mathfrak{Ind}^*(H_{V_1})$$

where hypergraphs H_{V_2} and H_{V_1} are defined in Example 4.4.

Proof. As we have noted in Example 4.4, we have identifications $B(H_{V_2})=B(H)=B(H_{V_1})$. Clearly, these identifications give identifications of posets $\mathrm{Pind}(B(H_{V_2}))_{++}=\mathrm{Pind}(B(H))_{++}=\mathrm{Pind}(B(H_{V_1}))_{++}$ and the result follows. □

Remark 4.14: In fact, we can construct adjoint pairs giving these homotopy equivalences. Recall that we have

$$\begin{aligned}\mathrm{Pind}^*(H)&=P_{\leq\neg f_1(V_1)}\cap P_{\geq f_2(V_2)},\\ \mathrm{Pind}^*(H_{V_2})&=\mathcal{P}(V_2)_{\leq\neg f_{V_1}(V_1)}\cap\mathcal{P}(V_2)_{\geq V_2},\\ \mathrm{Pind}^*(H_{V_1})&=\mathcal{P}(V_1)_{\leq\neg f_{V_2}(V_2)}\cap\mathcal{P}(V_1)_{\geq V_1}.\end{aligned}$$

Consider the following adjoint pairs:

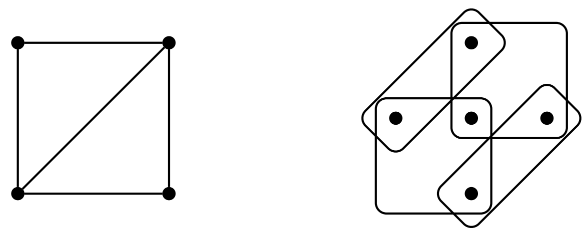
$$\mathcal{P}(V_2)\underset{f_{2*}}{\overset{f_2^*}{\rightleftarrows}}P\underset{\neg}{\overset{\neg}{\rightleftarrows}}P^o\underset{f_1^{*o}}{\overset{f_{1*}^o}{\rightleftarrows}}\mathcal{P}(V_1)^o.$$

We apply Lemma 3.7(2) to the adjoint pair $\mathcal{P}(V_2)\underset{f_{2*}}{\overset{f_2^*}{\rightleftarrows}}P$ and subsets $V_2\subset\mathcal{P}(V_2)$, $\neg f_1(V_1)\subset P$. Recall that the map f_2^* is given by $f_2^*(p)=f_2^{-1}(P_{\leq p})$. A straightforward calculation shows that $f_{V_1}=\neg f_2^*\neg f_1\colon V_1\to\mathcal{P}(V_2)$, whence $f_2^*(\neg f_1(V_1))=\neg f_{V_1}(V_1)$. Since $f_{2*}(V_2)=f_2(V_2)$, we obtain the following adjoint pair:

$$\begin{tikzcd}\mathrm{Pind}^*(H_{V_2}) \arrow[d, equal] & \mathrm{Pind}^*(H) \arrow[d, equal]\\ \mathcal{P}(V_2)_{\leq\neg f_{V_1}(V_1)}\cap\mathcal{P}(V_2)_{\geq V_2} \arrow[r, shift right, "f_{2*}"'] & P_{\leq\neg f_1(V_1)}\cap P_{\geq f_2(V_2)}. \arrow[l, shift right, "f_2^*"']\end{tikzcd}$$

Applying the same argument to f_1, we obtain the desired adjoint pairs:

$$\mathrm{Pind}^*(H_{V_2})\underset{f_{2*}}{\overset{f_2^*}{\rightleftarrows}}\mathrm{Pind}^*(H)\underset{\neg}{\overset{\neg}{\rightleftarrows}}(\neg\,\mathrm{Pind}^*(H))^o\underset{f_1^{*o}}{\overset{f_{1*}^o}{\rightleftarrows}}\mathrm{Pind}^*(H_{V_1})^o.$$



G $\qquad\qquad$ $G^{\vee}$

FIGURE 1. $\mathrm{Ind}(G) \not\simeq \mathrm{Ind}(G^{\vee})$

One can show that $f_2^* \neg f_{1*}^o = \neg f_{V_1 *}^o$ by direct calculation or using the equality $f_{V_1} = \neg f_2^* \neg f_1$ and the fact that the map $\neg f_2^* \neg$ preserves supremum. Therefore, the composite of the maps above yields:

$$\mathrm{Pind}^*(H_{V_2}) \underset{\neg f_{V_2 *}}{\overset{\neg f_{V_1 *}^o}{\leftrightarrows}} \mathrm{Pind}^*(H_{V_1})^o \ .$$

Corollary 4.15. *Let V and E be finite sets,*

$$H = (\mathcal{P}(V), f_E \colon E \to \mathcal{P}(V), V \subset \mathcal{P}(V)),$$
$$H^{\vee} = (\mathcal{P}(E), f_V \colon V \to \mathcal{P}(E), E \subset \mathcal{P}(E))$$

a hypergraph and its dual (See Example 4.3). Then we have a homotopy equivalence:

$$\mathrm{Ind}(H)^* \simeq \mathrm{Ind}(H^{\vee})^*.$$

Remark 4.16: (1) In general, $\mathrm{Ind}(H)$ and $\mathrm{Ind}(H^{\vee})$ are not homotopy equivalent. As an example, consider the graph G in Figure 1. One sees that $\mathrm{Ind}(G) \simeq S^0 \vee S^0$ and $\mathrm{Ind}(G^{\vee}) \simeq S^1 \vee S^1$.

(2) If V or E is empty, then $\mathrm{Ind}(H)^*$ and $\mathrm{Ind}(H^{\vee})^*$ are either empty or degenerate simplicial complexes.

Recall that, given a finite simple graph $G = (V, E)$, a subset $\tau \subset E$ is called an edge cover if $\bigcup_{e \in \tau} e = V$. A simplicial complex $\mathrm{EC}(G)$ called the edge covering complex is defined by

$$\mathrm{EC}(G) = \{\tau \subset E \mid E - \tau \text{ is an edge cover}\}.$$

Kawamura [4] showed that $S^0 * (\mathrm{Ind}(G)^*) \simeq S^0 * (\mathrm{EC}(G)^*)$. Applying Corollary 4.15 to a finite simple graph, we have the following result which refines (de-suspends) this:

Corollary 4.17. *Let $G = (V, E)$ be a finite simple graph and*

$$G^{\vee} = (\mathcal{P}(E), f_V \colon V \to \mathcal{P}(E), E \subset \mathcal{P}(E))$$

its dual hypergraph. Then we have $\mathrm{Ind}(G^{\vee}) = \mathrm{EC}(G)$. In particular, we have a homotopy equivalence:

$$\mathrm{Ind}(G)^* \simeq \mathrm{EC}(G)^*.$$

Proof. Recall that the map f_V is defined by $f_V(v) = \{e \in E \mid v \in e\}$. Hence, a subset $\tau \subset E$ is transversal with respect to $f_V(V)$ if and only if τ is an edge cover. Since

$$\mathrm{Ind}(G^\vee) = \{\tau \subset E \mid E - \tau \text{ is transversal}\},$$

we have $\mathrm{Ind}(G^\vee) = \mathrm{EC}(G)$. □

Remark 4.18: The maps

$$P(\mathrm{Ind}(G)^*) \underset{\neg f_{V*}}{\overset{\neg f^o_{E*}}{\rightleftarrows}} P(\mathrm{EC}(G)^*)^o$$

in Remark 4.14 giving homotopy equivalences are given as follows: We have

$$P(\mathrm{Ind}(G)^*) = \{\sigma \subset V \mid \sigma \neq \emptyset \text{ and } \sigma \cap e = \emptyset \text{ for some } e \in E\},$$

$$P(\mathrm{EC}(G)^*) = \left\{\tau \subset E \,\middle|\, \tau \neq \emptyset \text{ and } \bigcup_{e \in \tau} e \neq V\right\}$$

and

$$\neg f_{V*}(\sigma) = \{e \in E \mid \sigma \cap e = \emptyset\}, \quad \neg f_{E*}(\tau) = V - \bigcup_{e \in \tau} e.$$

5. Bipartite graphs

It was observed by Nagel and Reiner [5] and Jonsson [3] that the independence complexes of bipartite graphs have the homotopy types of suspensions. (They also showed a converse. See Corollary 5.4.) In view of Corollary 4.11, we can give a certain natural interpretation of this phenomenon.

Recall that any finite simple bipartite graph is regarded as the incidence graph of a hypergraph constructed as follows.

Definition 5.1. *Given a finite simple bipartite graph $G = (V, E)$ with nonempty sides $V = V_1 \amalg V_2$, we define hypergraphs $H(G, V_1)$, $H(G, V_2)$ as follows:*

$$H(G, V_1) = (\mathcal{P}(V_1), N_G \colon V_2 \to \mathcal{P}(V_1), V_1 \subset \mathcal{P}(V_1)),$$
$$H(G, V_2) = (\mathcal{P}(V_2), N_G \colon V_1 \to \mathcal{P}(V_2), V_2 \subset \mathcal{P}(V_2))$$

where $N_G(v) = \{u \in V \mid u \text{ and } v \text{ are adjacent}\}$ is the neighbourhood of v. (Note that we allow hypergraphs to have the empty set as an edge.)

Clearly, $H(G, V_1) = H(G, V_2)^\vee$ and G is the incidence graph of these hypergraphs:

$$G = B(H(G, V_1)) = B(H(G, V_2)).$$

Remark 5.2: Let $H = (P, f_1, f_2)$ be a generalized hypergraph. Then we have

$$H_{V_2} = H(B(H), V_2), \quad H_{V_1} = H(B(H), V_1).$$

Thus, we have the following:

Corollary 5.3. *If G is a finite simple bipartite graph with nonempty sides V_1 and V_2, then there exist homotopy equivalences:*

$$\mathrm{Ind}(G) \simeq S^0 * (\mathrm{Ind}(H(G, V_1))^*) \simeq S^0 * (\mathrm{Ind}(H(G, V_2))^*)\,.$$

Jonsson [3] defined a simplicial complex associated to a bipartite graph with nonempty sides V_1 and V_2 as follows

$$\Gamma_{G,V_1} = \{\sigma \subset V_1 \mid \sigma \cup \{u\} \in \mathrm{Ind}(G) \text{ for some } u \in V_2\}$$

and showed that $\mathrm{Ind}(G) \simeq S^0 * \Gamma_{G,V_1}$. Under the notation we have $\Gamma_{G,V_1} = \mathrm{Ind}(H(G, V_1))^*$ and Corollary 5.3 reproves a result of Jonsson [3]. We also note that the space Nagel and Reiner [5, Proposition 6.2] considered (which they denote by $T \cap f^{-1}(1/2)$) is essentially the same as $|\,\mathrm{Pind}(G)_{++}|$.

We give a proof of a converse, that is, the suspension of any finite simplicial complex has the homotopy type of the independence complex of a bipartite graph. The proof is essentially the same as those of [5] and [3], while we can clarify the nature of the desired bipartite graph with the help of Corollary 4.11.

Corollary 5.4. *Let Δ be a finite simplicial complex. Then there exists a finite hypergraph H such that $\Delta = \mathrm{Ind}(H)^*$. In particular, the suspension of Δ is homotopy equivalent to the independence complex of the bipartite graph $B(H)$: $\mathrm{Ind}(B(H)) \simeq S^0 * \Delta$.*

Proof. We consider the case when Δ is not empty. Note that, for a finite hypergraph $H = (\mathcal{P}(V), f\colon E \to \mathcal{P}(V), V \subset \mathcal{P}(V))$, we have

$$\begin{aligned}\mathrm{Ind}(H)^* &= \{\sigma \subset V \mid V - \sigma \notin \mathrm{Ind}(H)\}\\ &= \{\sigma \subset V \mid \sigma \subset V - f(e) \text{ for some } e \in E\}\,.\end{aligned}$$

Let V be the vertex set of Δ and E the set of maximal faces of Δ. Define a map $f\colon E \to \mathcal{P}(V)$ by $f(\tau) = V - \tau$, then we have $\mathrm{Ind}(H)^* = \Delta$. In particular, by Corollary 4.11, we have:

$$S^0 * \Delta = S^0 * (\mathrm{Ind}(H)^*) \simeq \mathrm{Ind}(B(H)).$$

□

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