Contributions to Discrete Mathematics

Volume 16, Number 1, Pages 150–159
ISSN 1715-0868

H-ABSORBENCE AND H-INDEPENDENCE IN 3-QUASI-TRANSITIVE H-COLOURED DIGRAPHS.

HORTENSIA GALEANA-SÁNCHEZ AND EUGENIA O'REILLY-REGUEIRO

Abstract. In this paper we prove that if D is a loopless asymmetric 3-quasi-transitive arc-coloured digraph having its arcs coloured with the vertices of a given digraph H, and if in D every C_4 is an H-cycle and every C_3 is a quasi-H-cycle, then D has an H-kernel.

1. Introduction

The concepts of independence, absorbing set, kernel, and colouring of digraphs have been studied for quite some time (see Section 2 for definitions). For example, a digraph always has a set K of pairwise nonadjacent vertices such that any other vertex has directed distance at most two to at least one vertex in K (the *semikernel*), but it need not have a set of pairwise nonadjacent vertices such that any other vertex has directed distance one to at least one vertex in it (the *kernel*), that is, replacing distance two by distance one. There are many applications of kernels in different topics of mathematics (see, for instance, [4, 5, 10, 11]) and they have been studied by several authors. Interesting surveys of kernels in digraphs can be found in [8, 12]. This has been generalised in many ways by colouring the arcs and asking for each vertex not in K to have a directed path with some specified properties to some vertex in K. See, among others [1, 9]. This paper is concerned with colouring the arcs of a digraph D with vertices of another digraph H and showing that D has a kernel when the paths in question are obtained from H and D is 3-quasi-transitive.

2. Preliminaries

For general concepts and notation, we refer the reader to [3, 6]. A digraph D has a vertex set $V(D)$ and an arc set $A(D)$. An *arc* is an element of $(V(D))^2$ and is written (u, v); loops $(u = v)$ are allowed. All paths and cycles in digraphs in this paper are directed and cycles are elementary. A

Received by the editors October 23, 2014, and in revised form December 9, 2020.

The first author was supported by UNAM-DGAPA-PAPIIT with grant number IN102320, and by CONACyT-FORDECyT-PRONACES/39570/2020. The second author was supported by UNAM-DGAPA-PAPIIT with grant number IN106811.

path (cycle) of length n (n a positive integer), $n \geq 1$ for paths, ($n \geq 2$ for cycles) will be $P_n = (u_0, \dots, u_n)$ ($C_n = (u_0, \dots, u_n = u_0)$, so the length is the number of arcs on it; all arcs and vertices are distinct. An arc $(u, v) \in A(D)$ is *asymmetric* if $u \neq v$ and $(v, u) \notin A(D)$, and it is *symmetric* if both (u, v) and (v, u) are in $A(D)$. A *walk* in D is an alternating sequence of vertices and arcs $(u_0, e_1, u_2, e_2, \dots e_n, u_n)$ with $e_i = (u_{i-1}, u_i)$; this definition is needed in particular if D is a multidigraph, i.e. a digraph where multiple arcs between the same pair of vertices may exist. Paths and cycles are particular walks. Our digraphs may have loops but no multiple arcs, unless otherwise stated. A digraph is *asymmetric* if all of its arcs are asymmetric. Note that in such a D, every walk of length three with distinct endpoints is a path. A digraph D is 3-*quasi-transitive* if whenever $(x, y), (y, w), (w, z) \in A(D)$, x, y, w, z pairwise distinct, then one of $(x, z), (z, x)$ is in $A(D)$. A set $N \subseteq V(D)$ is *absorbent* if from any vertex not in N there is a path to some vertex in N. It is *independent* is there are no arcs between the vertices of N. A set which is both absorbent and independent is called a *kernel*. Kernels have found many uses (as stated in Section 1, see [4, 5, 10, 11]). These ideas can be generalised. For example, the arcs of D can be coloured, the paths can be required to be monochromatic and the set deemed independent if there are no monochromatic paths between them (this was done by Sands, Sauer, and Woodrow in [2]) and the conjecture, attributed to Erdős, that if k colours are used, than the kernel size is bounded by a function of k was only proved recently by Thomassé et al. (see [14]). We explore another way of defining a kernel.

In the rest of the paper, we assume that D is a loopless asymmetric 3-quasi-transitive digraph and that H is a digraph possibly with loops. We say that D is H-*arc-coloured* if there is a mapping $c : A(D) \longrightarrow V(H)$. We will forgo the notation (D, H, c) but will assume them fixed once given and most of the time we will speak of "colours" when we mean the vertices of H. An H-walk in D is a path $(u_0, \dots, u_n)$ such that $c(u_0), \dots, c(u_n)$ is a walk in H. This was introduced by Linek and Sands in [13]. Building on this, Arpin and Linek defined an H-*walk absorbing* set as a set N such that from any vertex not in N there is an H-walk to some vertex in N. Similarly, a set N is H-*walk independent* if there are no H-walks between the vertices of N. As expected, an H-*walk kernel* is a set that both H-walk absorbing and H-walk independent.

Similarly, we can define an H-*cycle* in D and an H-*uv path*. If C is a cycle $(u_0, \dots, u_n)$ in D such that $(c(u_0), \dots, c(u_n))$ is a closed walk in H then C is an H-cycle in D. If $(u = u_0, \dots, u_n = v)$ is a uv-path in D such that $(c(u_0), \dots, c(u_n))$ is a walk in H it is an H-uv-path. We do not require that the walk have distinct endpoints. It particular, if H has no arcs other than loops, all H-cycles and H-paths will be *monochromatic*. If a cycle C of length n in D contains an H-path of length at least $n - 1$, it will be called a *quasi-H-cycle* (note an H-cycle is one).

We finish this section with two more definitions. Given an H-coloured digraph D, its *H-closure* $\mathcal{C}(D)$ is the multidigraph on the vertex set $V(D)$ with $A(\mathcal{C}(D)) = A(D) \cup \{(u,v)$: there is a $H - uv$-path in $D\}$. Recall that a *tournament* is a complete oriented graph, that is, a digraph any two of whose vertices are connected by exactly one arc (so a tournament is an asymmetric digraph).

3. Results

We will use the following theorem (in which more than one arc between the same pair of vertices is allowed):

Theorem 3.1 (Berge–Duchet [7])**.** *Let D be a digraph. If every directed cycle of D has at least one symmetric arc, then D has a kernel.*

The following lemma is essential for the proof of our main result:

Lemma 3.2. *If D is an H-arc-coloured digraph such that every cycle γ in $\mathcal{C}(D)$ has a symmetric arc, then D has an H-kernel.*

Proof. Let D be an H-arc-coloured digraph such that every cycle in $\mathcal{C}(D)$ has a symmetric arc. By Theorem 3.1 this implies $\mathcal{C}(D)$ has a kernel, which in turn implies D has an H-kernel. □

We give a well-known result in digraphs:

Lemma 3.3. *Let D be a digraph and $x,\ y \in V(D)$. Then every xy-walk in D contains an xy-path in D.*

Lemma 3.4. *Let D be an asymmetric 3-quasi-transitive digraph, and $u, v \in V(D)$ such that there is a uv-path P of length n and no vu-path. Then one of the following holds:*

1. *$(u,v) \in A(D)$ (when n is odd), or*
2. *there is a vertex $w \in D$ such that (u,w) and (w,v) are arcs in D (when n is even).*

Proof. Let D be an asymmetric 3-quasi-transitive digraph, and $u, v \in V(D)$ such that there is a path $P = (u = w_0, w_1, \ldots, w_n = v)$ and no vu-path. We have two cases:

Case 1: n odd.

We will prove, by induction on n, that $(u,v) \in A(D)$. If $n = 3$ then there are vertices w_1 and w_2 in $V(D)$ such that $P = (u, w_1, w_2, v)$ (and all these vertices are distinct). Since D is 3-quasi-transitive, either (u,v) or (v,u) is in $A(D)$. Since we assumed $(v,u) \notin A(D)$, we conclude $(u,v) \in A(D)$, which proves the basis of our induction.

Now suppose the result is true for all paths in D of length m such that $3 \le m < n = 2k+1$, and let $u, v \in V(D)$ such that there is a path $P = (u = w_0, w_1, \ldots w_n = v)$ of length $n = 2k+1$ and there is no vu-path in D. Since D is 3-quasi-transitive, either (w_0, w_3) or (w_3, w_0) is in $A(D)$.

If $(w_0, w_3) \in A(D)$, then there is a path $P' = (w_0, w_3, \dots, w_n)$ in D with length $2k-1$, and no $w_n w_0$-path, so by the induction hypothesis, $(w_0 = u, v = w_n) \in A(D)$ and we are done.
Now suppose $(w_3, w_0) \in A(D)$, and consider the path (which is part of P) $Q = (w_2, \dots, w_n)$, of length $2k-1$. If there is a path from w_n to w_2, then there is a directed walk

$$(w_n, \dots, w_2, w_3, w_0),$$

which by Lemma 3.3 contains a $w_n w_0$-path, contradicting our assumption. So there is no path from w_n to w_2, and by our induction hypothesis, $(w_2, w_n) \in A(D)$, hence there is a path (w_0, w_1, w_2, w_n) in D, and since there is no path from w_n to w_0 and D is 3-quasi-transitive, we conclude $(w_0, w_n) \in A(D)$.

Case 2: n is even.
As above, we will prove the result by induction on n, so first let $n = 2$, and the proof is immediate. For the sake of clarity, now let $n = 4$, so $P = (w_0, w_1, w_2, w_3, w_4)$. Since P is a path, these vertices are all distinct, and D 3-quasi-transitive implies either $(w_0, w_3) \in A(D)$ or $(w_3, w_0) \in A(D)$, and either $(w_1, w_4) \in A(D)$ or $(w_4, w_1) \in A(D)$. If both (w_3, w_0) and (w_4, w_1) are in $A(D)$, then $(w_4, w_1, w_2, w_3, w_0)$ is a path from w_4 to w_0, a contradiction, so at least one of $(w_0, w_3) \in A(D)$ or $(w_1, w_4) \in A(D)$ holds. Then (w_0, w_3, w_4) and (resp. or) (w_0, w_1, w_4) are paths (resp. is a path) in D, which proves the lemma for $n = 4$.
For the induction hypothesis, suppose the result is true for all paths in D of length m such that $6 \le m \le n-2$ and let $P = (u = w_0, w_1, \dots, w_{n-2}, w_{n-1}, w_n = v)$ be a path in D such that there is no $w_n w_0$-path in D.
Since P is a path, $w_0, \dots, w_n$ are all distinct, and since D is 3-quasi-transitive, for every w_i, w_{i+3} with $0 \le i \le n-3$ either $(w_i, w_{i+3}) \in A(D)$ or $(w_{i+3}, w_i) \in A(D)$.
If $(w_{i+3}, w_i) \in A(D)$ $\forall i = 0, \dots, n-3$ then

$$(w_n, w_{n-3}, w_{n-2}, w_{n-1}, w_{n-4}, w_{n-3}, w_{n-2}, \dots, w_1, w_2, w_3, w_0)$$

is a directed walk from w_n to w_0, which by Lemma 3.3 contains a $w_n w_0$-path, a contradiction.
Therefore there is an $i \in \{0, \dots, n-3\}$ such that $(w_i, w_{i+3}) \in A(D)$, the path $P' = (w_0, \dots, w_i, w_{i+3}, \dots, w_n)$ has length $n-2$ and there is no path from w_n to w_0, so by our induction hypothesis there is $w \in V(D)$ such that (w_0, w) and (w, w_n) are in $A(D)$, and the proof is complete.

□

Lemma 3.5. *Let H be any digraph, D an H-arc-coloured asymmetric 3-quasi-transitive digraph such that every C_4 in D is an H-cycle, and $u, v \in V(D)$ such that there is an H-uv-path P of length n and no H-vu-path. Then either:*

(1) $(u, v) \in A(D)$ *(when n is odd), or*

(2) *there exists $w \in V(D)$ such that (u, w) and (w, v) are arcs in D (when n is even).*

Proof. Let D be an H-arc-coloured asymmetric 3-quasi-transitive digraph with every C_4 an H-cycle, and suppose $u, v \in V(D)$ are such that there is an H-uv-path P and there is no H-vu-path. Let $P = (u = w_0, w_1, \ldots, w_n = v)$ be an $H - uv$-path of length n. If $n = 1$ or 2, then we are done.

If $n = 3$ then D being 3-quasi-transitive implies either $(w_0, w_3) \in A(D)$ or $(w_3, w_0) \in A(D)$, the latter is an H-vu-path, a contradiction, so $(w_0, w_3) \in A(D)$, proving the lemma for $n = 3$. Suppose then that there is an $H - uv$-path of length at least 4, and let $P = (u = w_0, w_1, \ldots, w_n = v)$ be such a path of minimum length $n \geq 4$, (and there is no $H - vu$-path).

First we will prove by induction on i that either the lemma holds, or for every w_i, w_j with $0 \leq i < j \leq n$ and $j - i \geq 2$, $(w_i, w_j) \notin A(D)$, that is, if there is an arc between w_i and w_j then it is (w_j, w_i), i.e., it goes "backwards".

We will do the basis of our induction for both w_0 and w_1. Consider the set of vertices $\{w_j\}$ of P such that there is an arc between w_0 and w_j, with $j \geq 2$, and note w_3 is one such vertex since D is 3-quasi-transitive, so this set is not empty. If for all such w_j the arc (w_j, w_0) is in $A(D)$, then we are done, so suppose there is at least one vertex w_j such that $(w_0, w_j) \in A(D)$, and let j be the maximum subscript such that $(w_0, w_j) \in A(D)$. If $j = n$ then $(w_0, w_n) \in A(D)$ and the lemma is proved. Similarly, if $j = n - 1$ then (w_0, w_{n-1}) and (w_{n-1}, w_n) are in $A(D)$, also proving the lemma, so suppose $j \leq n - 2$.

Since D is 3-quasi-transitive, $(w_0, w_j) \in A(D)$ implies there is an arc between w_0 and w_{j+2}, and j being the maximum subscript such that the arc forces $(w_{j+2}, w_0) \in A(D)$, i.e., it goes "forwards" (so $j \leq n - 3$ since there is no $H - vu$-path). Then $(w_0, w_j, w_{j+1}, w_{j+2})$ is a C_4 in D, and is therefore an H-cycle, which makes $(w_0, w_j, \ldots, w_n)$ an H-uv-path of length at least 4 (since $j \leq n-3$) and shorter than n, a contradiction. We conclude that for every $w_j \in P$ such that there is an arc between w_0 and w_j and $j \geq 2$, $(w_j, w_0) \in A(D)$.

We now prove the result for w_1 in the same way. If for every $w_j \in P$ (with $j > 2$) such that there is an arc between w_1 and w_j the arc $(w_j, w_1) \in A(D)$, then we are done. So consider w_j to be the vertex furthest from w_1 such that $(w_1, w_j) \in A(D)$ and suppose $j > 2$. As above, if $j = n$ then the lemma follows. If $j = n-1$ then since D is 3-quasi-transitive, there is an arc between w_0 and w_n. If $(w_0, w_n) \in A(D)$ we are done, and if $(w_n, w_0) \in A(D)$ then there is an H-vu-path in D, a contradiction. Therefore $2 < j \leq n-2$. Since D is 3-quasi-transitive, there is an arc between w_0 and w_{j+1}, and from the previous paragraph we conclude $(w_{j+1}, w_0) \in A(D)$ so $(w_0, w_1, w_j, w_{j+1}, w_0)$ is a C_4 in D and therefore an H-cycle, so as above, $(w_0, w_1, w_j, \ldots, w_n)$ is an H-uv-path in D of length shorter than n and greater than 3, a contradiction.

Now for our induction hypothesis suppose that for every $i < k$, if there is an arc between w_i and w_j and $j > i+1$, then $(w_j, w_i) \in A(D)$, and consider w_k. If $k = n$ or $n-1$ then we have nothing to prove, so $1 < k < n-1$. Suppose also there is $j > k+1$ such that $(w_k, w_j) \in A(D)$, and let m be the maximum of these subscripts j.

If $m = n$ then $(w_k, w_n) \in A(D)$, also (w_{k-2}, w_{k-1}) and (w_{k-1}, w_k) are in $A(D)$, and since D is 3-quasi-transitive, there is an arc between w_{k-2} and w_n, which by our induction hypothesis must be (w_n, w_{k-2}).

Then $(w_{k-2}, w_{k-1}, w_k, w_n)$ is a C_4 in D which must therefore be an H-cycle, which implies $(w_0, \ldots, w_k, w_n)$ is an H-uv-path. If $k = 2$ then this path has length 3, in which case $(w_0, w_n) \in A(D)$ and the lemma holds. Otherwise this path is of length at least 4 and shorter than n, a contradiction.

If $m = n-1$ then (w_{k-1}, w_k), (w_k, w_{n-1}), and (w_{n-1}, w_n) are arcs in D which is 3-quasi-transitive, this implies there is an arc between w_{k-1} and w_n, which by induction hypothesis must be (w_n, w_{k-1}). This implies the cycle $(w_{k-1}, w_k, w_{n-1}, w_n)$ is a C_4 in D, which is an H-cycle, so $(w_0, \ldots, w_k, w_{n-1}, w_n)$ is an H-uv-path of length shorter than n and greater than 3, a contradiction.

Finally, if $m < n-1$ we consider the following arcs: (w_{k-1}, w_k), (w_k, w_m), and (w_m, w_{m+1}). Since D is 3-quasi-transitive, there is an arc between w_{k-1} and w_{m+1}. From the definition of k, $(w_{m+1}, w_{k-1}) \in A(D)$, and the cycle $(w_{m+1}, w_{k-1}, w_k, w_m, w_{m+1})$ is a C_4 in D, so it is an H-cycle. This implies $(w_0, \ldots, w_k, w_m, \ldots, w_n)$ is an H-uv-path in D of length shorter than n and greater than 3, again, a contradiction, with which the proof or our claim is complete.

Since D is 3-quasi-transitive, for every $i = 0, \ldots, n-3$ either (w_i, w_{i+3}) or (w_{i+3}, w_i) is in $A(D)$, and by the claim we have just proved, $(w_{i+3}, w_i) \in A(D)$. This implies that for each $i = 0, \ldots, n-3$, there is a C_4 in D, namely $(w_i, w_{i+1}, w_{i+2}, w_{i+3})$, which is an H-cycle.

This yields an H-vu-walk in D, namely

$$(w_n, w_{n-3}, w_{n-2}, w_{n-1}, w_{n-4}, \ldots, w_1, w_2, w_3, w_0),$$

which by Lemma 3.3 contains an H-vu-path, a contradiction, so the conclusions of the lemma hold. □

Lemma 3.6. *Let H be a digraph and D an H-arc-coloured asymmetric 3-quasi-transitive digraph such that every C_4 in D is an H-cycle, and every C_3 in D is a quasi-H-cycle. Suppose there is an asymmetric cycle γ in $\mathcal{C}(D)$. Then the length of γ is at least 4.*

Proof. If γ is an asymmetric cycle in $\mathcal{C}(D)$, then it has length at least 3, so suppose it is 3 and $\gamma = (x, y, z)$ is asymmetric. Then by Lemma 3.5 there are xy-, yz-, and zx-paths with length 1 or 2, so we have four cases:
Case 1: They all have length 1, and (x, y, z) is a directed triangle in D.

In this case, since all directed triangles in D are quasi-H-cycles, there is an H-path of length at least two, say, (x, y, z) that is, there is an H-xz-path, which induces the arc (x, z) in $\mathcal{C}(D)$, this is a contradiction as we assumed the cycle γ asymmetric.

CASE 2: One of the paths, say, xy has length 2 in D, and the others have length 1, that is, there is a vertex x_0 in D such that (x, x_0, y, z) is a cycle in D.

This is a C_4, so it must be an H-cycle, hence there is an H-path from any vertex to any other vertex, that is, γ is symmetric, a contradiction.

CASE 3: Two of the paths, say xy and yz are of length 2, the other is of length 1 in D.

There are vertices x_0 and y_0 in D such that (x, x_0, y, y_0, z) is a cycle in D. Since D is 3-quasi-transitive and (y, y_0, z, x) is a path in D, either (x, y) or (y, x) is in $A(D)$. If $(y, x) \in A(D)$ then this is a symmetric arc in γ, a contradiction. If, on the other hand, $(x, y) \in A(D)$ then (y, y_0, z, x) is a C_4 in D, and must therefore be an H-cycle, which implies the arc (y, x) is in γ contradicting the assumption of γ being asymmetric.

CASE 4: The three paths have length 2 in D, so there are vertices x_0, y_0, and z_0 in D such that (x, x_0, y, y_0, z, z_0) is a cycle in D.

Since D is 3-quasi-transitive, either (x, y_0) or (y_0, x) is in $A(D)$. Suppose first that $(x, y_0) \in A(D)$, so (x, y_0, z, z_0) is a C_4 in D, and hence an H-cycle. Similarly, either (z, x_0) or (x_0, z) is in $A(D)$. If $(z, x_0) \in A(D)$ then (z, x_0, y, y_0) is a C_4 in D, and therefore also an H-cycle. Given the overlap of these two cycles, the path (x_0, y, y_0, z, z_0, x) is is an H-path, so the arc $(y, x) \in \gamma$, a contradiction. Suppose now that $(x_0, z) \in A(D)$. Then (x_0, z, z_0, x) is a C_4 in D, and hence an H-cycle. Again, the overlap of these two cycles implies the path (y_0, z, z_0, x, x_0) is an H-path. Also, either (y, z_0) or (z_0, y) is in $A(D)$. Following the same reasoning as above, if $(y, z_0) \in A(D)$ then there is an H- path in D from z to y, so in γ the arc (y, z) is symmetric, a contradiction. If, however, $(z_0, y) \in A(D)$ then in D there is a H-path from y to x, so the arc (x, y) in γ is symmetric, again, a contradiction.

Now suppose $(y_0, x) \in A(D)$. The proof is analogous, due to symmetry.

$\square$

Lemma 3.7. *Let H be any digraph and D an H-arc-coloured asymmetric 3-quasi-transitive digraph such that every C_4 in D is an H-cycle, every C_3 in D is a quasi-H-cycle, and let $\mathcal{C}(D)$ be the closure of D. Suppose there is an asymmetric cycle γ in $\mathcal{C}(D)$ and consider γ' to be the corresponding closed directed walk in D, that is, the vertices u, v and arcs (u, v) of γ when $(u, v) \in A(D)$ plus the vertices w and arcs (u, w) and (w, v) in D when $(u, v) \in A(\mathcal{C}(D)) \setminus A(D)$.*

If the vertices of the closed directed walk γ' are $x_0, x_1, \ldots, x_n$, then $(x_0, x_{2k+1}) \in A(D)$ for every k such that $3 \leq 2k+1 < n$ and for any $x_0 \in \gamma$.

Proof. Let $x_0, x_1, \ldots, x_n$ be the vertices in γ'. Since D is 3-quasi-transitive there is an arc between x_0 and x_3. Suppose $(x_3, x_0) \in A(D)$. Then (x_0, x_1, x_2, x_3) is a C_4 in D (all vertices are distinct) and is therefore an H-cycle. If $x_1 \in \gamma$ then there is an H-path from x_1 to x_0, contradicting the assumption that γ is asymmetric. If $x_1 \notin \gamma$, then $x_2 \in \gamma$ and the same reasoning applies. Therefore $(x_0, x_3) \in A(D)$.

If $n = 4$ or 5 then we are done, so suppose $n > 5$. Consider x_5 so $x_5 \neq x_0$ and note $x_5 \neq x_4$. Also, D asymmetric implies $x_5 \neq x_3$. If $x_5 = x_1$ then $(x_0, x_5) \in A(D)$. If $x_5 = x_2$ then this vertex is not in γ, which forces x_1, x_3, and x_4 to be all in γ. Since D is 3-quasi-transitive and (x_0, x_3), (x_3, x_4), and (x_4, x_2) are all arcs in D, there must be an arc between x_0 and x_2. If $(x_2, x_0) \in A(D)$ then (x_0, x_3, x_4, x_2) is a C_4 in D, and so it must be an H-cycle, which makes the arc $(x_3, x_4) \in \gamma$ symmetric, a contradiction.

Finally if x_5 is none of the previous vertices since $(x_0, x_3) \in A(D)$ and D is 3-quasi-transitive then either $(x_0, x_5) \in A(D)$ or $(x_5, x_0) \in A(D)$. If $(x_5, x_0) \in A(D)$ then (x_0, x_3, x_4, x_5) is a C_4 in D, hence an H-cycle, which implies an arc in γ is symmetric, a contradiction. Therefore $(x_0, x_5) \in A(D)$.

Following the same reasoning, by induction, suppose the lemma is not true and let j be the first subscript such that $(x_0, x_{2j+1}) \notin A(D)$ (with $2j + 1 < n$). Since $(x_0, x_{2j-1}) \in A(D)$ and D is 3-quasi-transitive, we conclude $(x_{2j+1}, x_0) \in A(D)$. We observe $x_0 \neq x_{2j-1}, x_{2j}$, and x_{2j+1}. Also, $x_{2j} \neq x_{2j-1}$ and x_{2j+1}, and since D is asymmetric $x_{2j-1} \neq x_{2j+1}$. That is, all four vertices are distinct. Then $(x_0, x_{2j-1}, x_{2j}, x_{2j+1})$ is a C_4 in D, hence an H-cycle. If $x_{2j-1} \in \gamma$ then so is at least one of x_{2j} and x_{2j+1}, this implies there is a symmetric arc in γ, a contradiction. If $x_{2j-1} \notin \gamma$ then $x_{2j} \in \gamma$. If $x_{2j+1} \in \gamma$ then again we get a symmetric arc in γ, a contradiction. Now suppose neither x_{2j-1} nor x_{2j+1} are in γ (and $x_{2j} \in \gamma$).

We go back to x_{2j-3} and note that $(x_0, x_{2j-3}) \in A(D)$ (our first two steps of induction allow us to do this). We also note $x_{2j-2} \in \gamma$. Since D is 3-quasi-transitive, and $(x_{2j+1}, x_0), (x_0, x_{2j-3})$, and (x_{2j-3}, x_{2j-2}) are all in $A(D)$, there must be an arc between x_{2j+1} and x_{2j-2} (and these are distinct vertices as one is in γ and the other one is not). If $(x_{2j+1}, x_{2j-2}) \in A(D)$ then $(x_{2j-2}, x_{2j-1}, x_{2j}, x_{2j+1})$ is a C_4 in D, and hence an H-cycle. This makes the arc (x_{2j-2}, x_{2j}) in γ symmetric, a contradiction.

If, on the other hand, $(x_{2j-2}, x_{2j+1}) \in A(D)$, then $(x_0, x_{2j-3}, x_{2j-2}, x_{2j+1})$ is a C_4 in D, so it is an H-cycle, and as it intersects $(x_0, x_{2j-1}, x_{2j}, x_{2j+1})$ in the arc (x_{2j+1}, x_0), the arcs in these two C_4s all have an H-colouring. This yields an H-path from x_{2j} to x_{2j-2} which makes the arc $(x_{2j-2}, x_{2j}) \in \gamma$ symmetric, a contradiction with which our proof is now complete. □

Now we prove our main result.

Theorem 3.8. *Let H be any digraph and D an H-arc-coloured asymmetric 3-quasi-transitive digraph such that every C_4 in D is an H-cycle and every C_3 in D is a quasi-H-cycle. Then D has an H-kernel.*

Proof. Let H be any digraph and D be an H-arc-coloured asymmetric 3-quasi-transitive digraph such that every C_4 is an H-cycle and every C_3 is a quasi-H-cycle, and consider $\mathcal{C}(D)$, the closure of D. If every cycle in $\mathcal{C}(D)$ has a symmetric arc, then by Theorem 3.1 $\mathcal{C}(D)$ has a kernel, which by Lemma 3.2 implies D has an H-kernel, and we are done. So suppose in $\mathcal{C}(D)$ there is an asymmetric cycle, and let γ be such a cycle of minimum length, which, by Lemma 3.6 is at least 4.

We consider $\gamma' = (x_0, \dots, x_n)$ the corresponding closed directed walk in D, that is, the vertices u, v and arcs (u, v) of γ when $(u, v) \in A(D)$ plus the vertices w and arcs (u, w) and (w, v) in D when $(u, v) \in A(\mathcal{C}(D)) \setminus A(D)$. We can assume w.l.o.g. that $x_0 \in V(\gamma)$, and by Lemma 3.7 $(x_0, x_{2j+1}) \in A(D)$ for every j such that $1 \leq 2j + 1 < n$. We now consider two cases, according to the parity of $n \geq 4$.

First suppose n is odd, and consider the vertices x_0, x_n, x_{n-1}, and x_{n-2}. Note x_0 is different from any of the other vertices, otherwise the length of γ would be shorter. Also, $x_n \neq x_{n-1} \neq x_{n-2}$ since they are adjacent, and $x_n \neq x_{n-2}$ because D is asymmetric. Therefore $(x_0, x_{n-2}, x_{n-1}, x_n)$ is a C_4 and so it is an H-cycle. Since $x_0 \in \gamma$ then at least one of x_n and x_{n-1} is in γ. If $x_n \in \gamma$ then the arc $(x_n, x_0) \in \gamma$ is symmetric, and if $x_n \notin \gamma$ then the arc $(x_{n-1}, x_n) \in \gamma$ is symmetric, in both cases we have a contradiction.

Now suppose n is even. As above, all the vertices x_0, x_n, x_{n-1}, and x_{n-2} are distinct. Since D is 3-quasi-transitive, there is an arc between x_0 and x_{n-2}. If $(x_0, x_{n-2}) \in A(D)$ then as in the previous paragraph there is a symmetric arc in γ, a contradiction. Therefore $(x_{n-2}, x_0) \in A(D)$. Since $(x_0, x_{n-1}) \in A(D)$, if $x_n \notin \gamma$ then $x_{n-1} \in \gamma$ and γ has a symmetric arc, a contradiction which implies $x_n \in \gamma$. Also, the vertices (x_0, x_{n-1}, x_n) form a C_3, which is a quasi-H-cycle. If the arcs (x_0, x_{n-1}) and (x_{n-1}, x_n) form an H-path, then there is an H-path in D from x_0 to x_n, which are consecutive vertices in γ, so in γ there is a symmetric arc, a contradiction.

Now suppose there is an H-path from x_n to x_{n-1}. This forces $x_{n-1} \notin \gamma$, otherwise (x_{n-1}, x_n) would be a symmetric arc in γ, a contradiction. Now $x_{n-1} \notin \gamma$, forces $x_{n-2} \in \gamma$. Since $x_{n-2} \neq x_0, x_n$, and x_{n-1}, and D is 3-quasi-transitive, there is an arc between x_{n-2} and x_n (because of the path $(x_{n-2}, x_0, x_{n-1}, x_n)$). If $(x_n, x_{n-2}) \in A(D)$ then it is a symmetric arc in γ, and if $(x_{n-2}, x_n) \in A(D)$ then in D these two vertices are at distance 1, so the existence of x_{n-1} in the cycle γ' is a contradiction. We conclude the arcs (x_{n-1}, x_n) and (x_n, x_0) form an H-path (and $x_{n-2} \in \gamma$).

We have proved that if n is even and we consider a vertex in γ' which is also in γ, then the preceding vertex is also in γ and the arc in γ' of which this vertex is an endpoint forms an H-path with the preceding arc. Going backwards by induction, we conclude every vertex of γ' is in γ, and $\gamma'(= \gamma)$ is an H-cycle, and therefore it has a symmetric arc, a contradiction.

We have proved every directed cycle in $\mathcal{C}(D)$ has a symmetric arc. This, by Theorem 3.1 implies $\mathcal{C}(D)$ has a kernel, which in turn by Lemma 3.2 implies D has an H-kernel. $\square$

Acknowledgements

The authors are very grateful for the suggestions and comments generously provided by the anonymous referee. They greatly improved the paper, especially the first two sections. We are also grateful to Professor Robert Woodrow for his patience, attention, and correspondence.

References

1. P. Arpin and V. Linek, *Reachability problems in edge-coloured digraphs*, Discrete Math. **307** (2007), 2276–2289.
2. N. Sauer B. Sands and R. Woodrow, *On monochromatic paths in edge-coloured digraphs*, J. Combin. Theory Ser. B **33** (1982), 271–275.
3. J. Bang-Jensen and G. Gutin, *Digraphs: Theory, algorithms and applications*, Springer, London, 2001.
4. J. M. Le Bars, *Counterexample of the 0-1 law for fragments of existential second-order logic; an overview*, Bull. Symbolic Logic **9** (2000), 67–82.
5. ———, *The 0-1 law fails for frame satisfiability of propositional model logic*, Proceedings of the 17th Symposium on Logic in Computer Science (2002), 225–234.
6. C. Berge, *Graphs*, North-Holland, Amsterdam, 1985.
7. C. Berge and P. Duchet, *Recent problems and results about kernels in directed graphs*, Discrete Math. **86** (1990), 27–31.
8. E. Boros and V. Gurvich, *Perfect graphs, kernels and cores of cooperative games*, Discrete Math. **306** (2006), 2336–2354.
9. P. Delgado-Escalante and H. Galeana-Sánchez, *Restricted domination in arc-coloured digraphs*, AKCE Int. J. of Graphs Comb. **11** (2014), no. 1, 95–104.
10. P. Duchet, *Kernels in directed graphs: a poison game*, Discrete Math. **115** (1993), 273–276.
11. A. S. Fraenkel, *Combinatorial game theory foundations applied to digraph kernels*, E-JC **4** (1997), 17.
12. ———, *Combinatorial games: selected bibliography with a succinct gourmet introduction*, E-JC **#DS2** (2012), 1–109.
13. V. Linek and B. Sands, *A note on paths in edge-coloured tournaments*, Ars Combin. **44** (1996), 225–228.
14. W. Lochet N. Bousquet and S. Thomassé, *A proof of the Erdős-Sans-Sauer-Woodrow conjecture*, J. Combin. Theory Ser. B. **137** (2019), 316–319.

Instituto de Matemáticas
UNAM
Ciudad Universitaria
Coyoacán 04510
CDMX, MEXICO
E-mail address: hgaleana@matem.unam.mx

Instituto de Matemáticas
UNAM
Ciudad Universitaria
Coyoacán 04510
CDMX, MEXICO
E-mail address: eugenia@im.unam.mx