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A LOWER BOUND FOR RADIO k-CHROMATIC NUMBER OF AN ARBITRARY GRAPH

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Abstract. Radio k-coloring is a variation of Hale's channel assignment problem, in which one seeks to assign positive integers to the vertices of a graph G, subject to certain constraints involving the distance between the vertices. Specifically, for any simple connected graph G with diameter d and a positive integer k, $1 \leq k \leq d$, a *radio k-coloring* of G is an assignment f of positive integers to the vertices of G such that $|f(u)-f(v)| \geq 1+k-d(u,v)$, where u and v are any two distinct vertices of G and $d(u,v)$ is the distance between u and v. In this paper we give a lower bound for the radio k-chromatic number of an arbitrary graph in terms of k, the total number of vertices n and a positive integer M such that $d(u,v)+d(v,w)+d(u,w) \leq M$ for all $u,v,w \in V(G)$. If M is the triameter we get a better lower bound. We also find the triameter M for several graphs, and show that the lower bound obtained for these graphs is sharp for the case $k = d$.

1. Introduction

The channel assignment problem is the problem of assigning frequencies to the transmitters in some optimal manner and with no interferences; see Hale [7]. Griggs and Yeh [6] adapted this to graphs as follows: For nonnegative integers $p_1, \ldots, p_m$, an $L(p_1, \ldots, p_m)$-coloring of a graph G is a coloring of its vertices by nonnegative integers such that vertices at distance exactly i receive labels that differ by at least p_i. The maximum label assigned to any vertex is called the *span* of the coloring. The goal of the problem is to construct an $L(p_1, \ldots, p_m)$-coloring of the smallest span. Chartrand et al. [3] introduced a variation $L(p_1, \ldots, p_m)$-coloring known as radio k-colorings of graphs. A *radio k-coloring* of a simple connected graph G, $1 \leq k \leq \mathrm{diam}(G)$, is an assignment f of positive integers to the vertices of G such that for any two distinct vertices u and v, $|f(u) - f(v)| \geq 1 + k - d(u,v)$, where $d(u,v)$ is the distance between u and v. The *span* of f, $rc_k(f)$, is $\max\{f(u) : u \in V(G)\}$. The *radio k-chromatic number*, $rc_k(G)$, of G is the minimum of spans of all possible radio k-colorings of G. A radio k-coloring having the span $rc_k(G)$ is called a *minimal radio k-coloring.* Radio k-colorings have

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been studied by many authors [2, 4, 8, 9, 10, 11, 12, 13, 14, 15, 19]. For some specific values of k there are specific names for radio k-colorings as well as the radio k-chromatic number in the literature, which are given below in the table:

k	*Name of coloring*	$rc_k(G)$
1	Usual	Chromatic number, $\chi(G)$
$\text{diam}(G)$	Radio	Radio number, $rn(G)$
$\text{diam}(G)-1$	Antipodal	Antipodal number, $ac(G)$
$\text{diam}(G)-2$	Nearly antipodal	Nearly antipodal number, $ac'(G)$

Finding the radio k-chromatic number of graphs is highly non-trivial and therefore is known for very few graphs. For a path P_n, $rc_k(P_n)$ is known for $k = n-1$ [15], $n-2$ [11], $n-3$ [17], and $n-4$ (n odd) [18]. For a cycle C_n, the radio number was determined by Liu and Zhu [15], and the antipodal number is known only for $n \equiv 1, 2, 3 \pmod 4$ (see [1, 8]). The radio number of square of paths and cycles have been determined in [14] and [13] respectively. For the current status of radio k-chromatic number of graphs see [16]. Prior to this paper, no bound for the radio k-chromatic number of an arbitrary graph was known. In this paper, we give a lower bound for the radio k-chromatic number of an arbitrary graph in terms of k, the total number of vertices n, and a positive integer M with the property $d(u,v)+d(v,w)+d(u,w) \le M$ for all $u, v, w \in V(G)$. The value of M can be taken as a large positive real number, but it is of no use, as the smallest possible value of M gives a better lower bound of the graph. That is, if M takes the triameter ($\max\{d(u,v)+d(v,w)+d(u,w) : u,v,w \in V(G)\}$, see [5] for instance) of G then we get a better lower bound. In this article we find the smallest value of M for several graphs: cycles, prism graphs, and stacked book graphs. Furthermore, we show that the lower bound obtained is sharp with the radio number for many of these graphs. For any two graphs G_1 and G_2, we also give a lower bound for the radio k-chromatic number of their Cartesian product $G_1 \square G_2$ in terms of their respective M values. For any graph G, we find a lower bound for the radio k-chromatic number of G^r, the rth power of G in terms of M.

2. Lower Bound for $rc_k(G)$

In this section, we give the main result of the paper, i.e., a lower bound for radio k-chromatic number of an arbitrary graph G. The following definition may be found in [11].

Definition 2.1. *For any radio k-coloring f of a simple connected graph G on n vertices and an ordering $x_1, x_2, \ldots, x_n$ of vertices of G with $f(x_i) \le f(x_{i+1})$, $1 \le i \le n-1$, we define ϵ_i (or ϵ_i^f if we wish to specify the coloring f) as $\epsilon_i = (f(x_i) - f(x_{i-1})) - (1 + k - d(x_i, x_{i-1}))$, $2 \le i \le n$.*

It is clear from the definition of radio k-coloring that $\epsilon_i \ge 0$ for all i.

Lemma 2.2. *For any radio k-coloring f of G,*

$$rc_k(f) = (n-1)(1+k) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f + 1,$$

where the x_i's are as taken in Definition 2.1.

Proof. Let f be a radio k-coloring of G, then

$$\begin{aligned} f(x_n) - f(x_1) &= \sum_{i=2}^{n} [f(x_i) - f(x_{i-1})] = \sum_{i=2}^{n} [1 + k - d(x_i, x_{i-1}) + \epsilon_i^f] \\ &= (n-1)(1+k) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f. \end{aligned}$$

Since $f(x_1) = 1$, we get,

$$f(x_n) = (n-1)(1+k) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f + 1. \tag{2.1}$$

□

Remark: From Lemma 2.2, we observe that for any graph G, $rc_k(G)$ is the minimum value of a function which consists of a constant term (i.e., $(n-1)(k+1)+1$) and two variable terms; the distance sum $\sum_{i=2}^{n} d(x_i, x_{i-1})$, and the ϵ-sum $\sum_{i=2}^{n} \epsilon_i$. Therefore any radio k-coloring f, if it exists, must have $\sum_{i=2}^{n} \epsilon_i = 0$ and maximum distance sum $\sum_{i=2}^{n} d(x_i, x_{i-1})$ among all possible radio k-colorings to be minimal.

Theorem 2.4. *In a simple connected graph G on n vertices, if $d(x,z) \leq M - (d(x,y) + d(y,z))$ for a fixed positive real number M and any three vertices $x, y,$ and z, if a_k denotes the quantity*

$$\begin{cases} \frac{(n-1)(3(k+1)-M)}{4} + 1, & \\ & \text{if } n \text{ odd}, M \not\equiv k \pmod 2; \\ \frac{(n-1)(3(k+1)-(M-1))}{4} + 1, & \\ & \text{if } n \text{ odd}, M \equiv k \pmod 2; \\ \frac{(n-2)(3(k+1)-M)}{4} + k + 2 - \operatorname{diam}(G), & \\ & \text{if } n \text{ even}, M \not\equiv k \pmod 2; \\ \frac{(n-2)(3(k+1)-(M-1))}{4} + k + 2 - \operatorname{diam}(G), & \\ & \text{if } n \text{ even}, M \equiv k \pmod 2; \end{cases}$$

we have $rc_k(G) \geq a_k$.

Proof. Let f be a radio k-coloring of G and $x_1, x_2, \ldots, x_n$ be an ordering of vertices of G such that $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$. For any three vertices x_i, x_{i+1}, and x_{i+2}, we have

$$f(x_{i+1}) - f(x_i) = 1 + k - d(x_i, x_{i+1}) + \epsilon_{i+1}^f \tag{2.2}$$

and

$$f(x_{i+2}) - f(x_{i+1}) = 1 + k - d(x_{i+1}, x_{i+2}) + \epsilon^f_{i+2}. \tag{2.3}$$

By adding (2.2) and (2.3), we get

$$f(x_{i+2}) - f(x_i) = 2(1+k) - [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})] + \epsilon^f_{i+1} + \epsilon^f_{i+2}. \tag{2.4}$$

Since f is a radio k-coloring, we have

$$f(x_{i+2}) - f(x_i) \geq 1 + k - d(x_i, x_{i+2}). \tag{2.5}$$

From (2.4) and (2.5),

$$2(1+k) - [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})] + \epsilon^f_{i+1} + \epsilon^f_{i+2} \geq 1 + k - d(x_i, x_{i+2}).$$

Since $d(x, z) \leq M - (d(x, y) + d(y, z))$, for every three vertices x, y, and z in G, we have

$$\begin{aligned} 2(1+k) - [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})] + \epsilon^f_{i+1} + \epsilon^f_{i+2} \\ \geq 1 + k - [M - (d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}))]. \end{aligned}$$

That is,

$$d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) \leq \frac{M+k+1}{2} + \frac{\epsilon^f_{i+1} + \epsilon^f_{i+2}}{2}. \tag{2.6}$$

Case 1: $n \not\equiv 0 (\mathrm{mod}\, 2)$:

Subcase(ii): $M \not\equiv (\mathrm{mod}\, 2)$**:** **Sub-subcase** (a)**:** Suppose that

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) \leq \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right).$$

Then by equation (2.1), we have

$$\begin{aligned} f(x_n) &= (n-1)(k+1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon^f_i + 1 \\ &\geq (n-1)(k+1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + 1 \\ &\geq (n-1)(k+1) - \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + 1 \\ &= (n-1)\left(\frac{3(k+1)-M}{4}\right) + 1. \end{aligned}$$

Sub-subcase (b)**:** Here we suppose that

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) > \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right).$$

Let

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) = \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + N,$$

where N is a positive integer. Since $\sum_{i=2}^{n} d(x_i, x_{i-1})$ contains exactly $(n-1)/2$ disjoint consecutive pairs, we get $d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) > (M+k+1)/2$ for some triplets x_i, x_{i+1}, and x_{i+2}.
Suppose that $d(x_{i_j}, x_{i_j+1}) + d(x_{i_j+1}, x_{i_j+2}) > (M+k+1)/2$, for $j = 1, 2, \ldots, l$, where i_j is odd for each j. Let $d(x_{i_j}, x_{i_j+1}) + d(x_{i_j+1}, x_{i_j+2}) = (M+k+1)/2 + m_j$, where m_j is a positive integer, $1 \leq j \leq l$.
Now from equation (2.6),

$$\frac{M+k+1}{2} + \frac{\epsilon^f_{i_j+1} + \epsilon^f_{i_j+2}}{2} \geq \frac{M+k+1}{2} + m_j, \quad 1 \leq j \leq l$$

implies

$$\epsilon^f_{i_j+1} + \epsilon^f_{i_j+2} \geq 2m_j, 1 \leq j \leq l. \tag{2.7}$$

Since

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) = \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + N,$$

we have

$$m_1 + m_2 + \cdots + m_l \geq N. \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$\sum_{i=2}^{n} \epsilon^f_i \geq 2(m_1 + m_2 + \cdots + m_l) \geq 2N.$$

From equation (2.1), we have

$$f(x_n) = (n-1)(k+1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon^f_i + 1.$$

Using this equality we find that:

$$\begin{aligned}
&(n-1)(k+1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon^f_i + 1 \\
&= (n-1)(k+1) - \left[\left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + N\right] + \sum_{i=2}^{n} \epsilon^f_i + 1 \\
&\geq (n-1)(k+1) - \left[\left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + N\right] + 2N + 1 \\
&= (n-1)(k+1) - \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + N + 1 \\
&> (n-1)(k+1) - \left(\frac{n-1}{2}\right)\left(\frac{M+k+1}{2}\right) + 1 \\
&= (n-1)\left(\frac{3(k+1)-M}{4}\right) + 1.
\end{aligned}$$

Since f is an arbitrary radio k-coloring of G,

$$rc_k(G) \geq (n-1)\left(\frac{3(k+1)-M}{4}\right)+1$$

in this case. Notice that in subcase (i) we are not using the assumption that $(M+k+1)/2$ is an integer; the result here is also true when $(M+k+1)/2$ is not an integer. However, since we can improve the lower bound in the latter situation, we have the subcase (ii) below.

Case ii: $M \equiv k (\bmod 2)$:

Subcase (ii)**:** $M \equiv k$ (**mod** 2)**. Sub-subcase** (c)**:** Suppose that

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) \leq \left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right).$$

Then by equation (2.1), we have

$$\begin{aligned} f(x_n) &= (n-1)(k+1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f + 1 \\ &\geq (n-1)\left(\frac{3(k+1)-(M-1)}{4}\right)+1. \end{aligned}$$

Sub-subcase (d)**:** Here we suppose that

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) > \left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right).$$

Let

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) = \left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right)+S,$$

where S is a positive integer. Since $\sum_{i=2}^{n} d(x_i, x_{i-1})$ contains exactly $(n-1)/2$ disjoint consecutive pairs, we get $d(x_i, x_{i+1})+d(x_{i+1}, x_{i+2}) > (M+k)/2$ for some triplet x_i, x_{i+1}, x_{i+2}. Suppose that $d(x_{i_j}, x_{i_j+1})+d(x_{i_j+1}, x_{i_j+2}) > M+k/2$ for $j = 1, 2, \ldots, l$, where i_j is odd for each j. Let $d(x_{i_j}, x_{i_j+1}) + d(x_{i_j+1}, x_{i_j+2}) = M+k/2+m_j$, where m_j is a positive integer$1 \leq j \leq l$. Now from equation (2.6),

$$\frac{M+k+1}{2} + \frac{\epsilon_{i_j+1}^f + \epsilon_{i_j+2}^f}{2} \geq \frac{M+k}{2} + m_j, \quad 1 \leq j \leq l$$

implies

$$\epsilon_{i_j+1}^f + \epsilon_{i_j+2}^f \geq 2m_j - 1, \quad 1 \leq j \leq l. \tag{2.9}$$

Since

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) = \left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right)+S,$$

$$m_1 + m_2 + \cdots + m_l \geq S. \tag{2.10}$$

We assume $l \leq S$ as equation (2.10) remains true, since each m_j, $j = 1, 2, \ldots, l$, is a positive integer (if $l > S$, we neglect $S - l$ triplets).
From equations (2.9) and (2.10), we have

$$\sum_{i=2}^{n} \epsilon_i^f \geq 2m_1 - 1 + 2m_2 - 1 + \cdots + 2m_l - 1 \geq 2S - l \geq S.$$

From equation (2.1), we have

$$\begin{aligned}
f(x_n) &= (n-1)(k+1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f + 1 \\
&= (n-1)(k+1) - \left[\left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right) + S\right] + \sum_{i=2}^{n} \epsilon_i^f + 1 \\
&\geq (n-1)(k+1) - \left[\left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right) + S\right] + S + 1 \\
&= (n-1)(k+1) - \left(\frac{n-1}{2}\right)\left(\frac{M+k}{2}\right) + 1 \\
&= (n-1)\left(\frac{3(k+1) - (M-1)}{4}\right) + 1.
\end{aligned}$$

Since f is an arbitrary radio k-coloring of G,

$$rc_k(G) \geq (n-1)\left(\frac{3(k+1) - (M-1)}{4}\right) + 1.$$

The distance sum $\sum_{i=2}^{n} d(x_i, x_{i-1})$ consists of $(n-2)/2$ disjoint pairs and one single term. Similar to the n odd case, in sub-subcases (b) and (d), we collect pairs of distances among these disjoint pairs such that the sum of distances in each pair is strictly greater than $(M + k + 1)/2$, which yields the equations (2.7) and (2.9). Since any distance term is less than or equal to $\operatorname{diam}(G)$, we proceed similarly to the n odd case, replacing $((n-1)/2)((M+k+1)/2))$ by $((n-2)/2)((M+k+1/2)) + \operatorname{diam}(G)$ in sub-subcases (a) and (b) and then replacing $((n-1)/2)((M+k)/2))$ by $((n-2)/2))((M+k)/2)) + \operatorname{diam}(G)$ in sub-subcases (c) and (d) to get the following:

$$rc_k(G) \geq \begin{cases} (n-2)\left[\frac{3(k+1)-M}{4}\right] + k + 2 - \operatorname{diam}(G), & \text{if } \frac{M+k+1}{2} \in \mathbb{Z}; \\ k + 2 - \operatorname{diam}(G) + (n-2)\left[\frac{3(k+1)-(M-1)}{4}\right], & \text{if } \frac{M+k+1}{2} \notin \mathbb{Z}. \end{cases}$$

□

Remark: In the above theorem, we can see that a smaller value of M will give a better lower bound for $rc_k(G)$. Suppose that

$$M^0 = \inf\{M : d(x,z) \leq M - [d(x,y) + d(y,z)], \forall x, y, \text{ and } z \in G\}.$$

It is clear that $M^0 \leq 3\operatorname{diam}(G)$. Since we can have three vertices u, v, and w such that $d(u,v) + d(v,w) = 2\operatorname{diam}(G) - 1$, we get $1 \leq d(u,w) \leq M -$

$[d(u,v)+d(v,w)] = M - [2\operatorname{diam}(G)-1]$, which implies $2\operatorname{diam}(G) \leq M$. Therefore we have $2\operatorname{diam}(G) \leq M^0 \leq 3\operatorname{diam}(G)$.

3. Sharp Lower Bound for Radio k-chromatic Number of some Graphs

In this section, we give a lower bound for radio k-chromatic number of cycles C_n, stacked book graphs, and prism graphs before proceeding to prove that for some of these graphs, the bound agrees with their radio number.

3.1. **Cycles.**

Lemma 3.1. *For any three vertices $x, y,$ and z in a cycle C_n, $d(x,z) \leq n - [d(x,y)+d(y,z)]$.*

Proof. Let $x, y,$ and z be any three vertices in C_n. If $d(x,y)+d(y,z) \leq n/2$, then we are done. Suppose that $d(x,y)+d(y,z) > n/2$. Then we get that $d(x,z) \leq n - [d(x,y)+d(y,z)]$ if z lies on the shortest path between x and y, and $d(x,z) = n - [d(x,y)+d(y,z)]$ otherwise. □

From Lemma 3.1 and Theorem 2.4, we get the following.

Theorem 3.2. *For any cycle C_n,*

$$rc_k(C_n) \geq \begin{cases} (n-1)\left(\frac{3(k+1)-n}{4}\right)+1, & \text{if } n \text{ is odd and } k \text{ is even,} \\ (n-1)\left(\frac{3(k+1)-(n-1)}{4}\right)+1, & \text{if both } n \text{ and } k \text{ are odd,} \\ (n-2)\left(\frac{3(k+1)-n}{4}\right)+\frac{2k-n+4}{2}, & \text{if } n \text{ is even and } k \text{ is odd,} \\ (n-2)\left(\frac{3(k+1)-(n-1)}{4}\right)+\frac{2k-n+4}{2}, & \text{if both } n \text{ and } k \text{ are even.} \end{cases}$$

One can verify that the lower bound of $rc_k(C_n)$ obtained in Theorem 3.2 for $k = \operatorname{diam}(C_n)$, agrees with the radio number of C_n determined in [15].

In the subsection below we show that the triameter for the Cartesian product of two graphs is the sum of the triameters of the graphs in the product. By using this, we determine the triameters of stacked book graphs, grid graphs and prism graphs. From Theorem 2.4, one observes that calculating the triameter of a graph gives a lower bound for the radio k-chromatic number of the graph.

3.2. **Cartesian Product of Graphs.**

Definition 3.3. *The cartesian product of any two graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ and any two vertices (u,u') and (v,v') are adjacent if and only if either $u = v$ and u' is adjacent with v' or $u' = v'$ and u is adjacent with v.*

Theorem 3.4. *If $d(u,v)+d(v,w)+d(u,w) \leq M_1$ and $d(u',v')+d(v',w')+d(u',w') \leq M_2$ for every three vertices u,v,w in G_1 and u',v',w' in G_2, then $d(x,y)+d(y,z)+d(x,z) \leq M_1+M_2$ for every three vertices x,y,z in $G_1 \Box G_2$.*

Proof. Let $V(G_1) = \{u_1,u_2,\dots,u_n\}$ and $V(G_2) = \{v_1,v_2,\dots,v_m\}$. From the definition of Cartesian product of graphs, one sees that $G_1 \Box G_2$ can be obtained by taking n copies of G_2: exactly one copy $G_2^1, G_2^2, \dots, G_2^n$ for each vertex of G_1, and making adjacent the corresponding vertices of G_2^i and G_2^j whenever u_i is adjacent to u_j in G_1. Let $x=(u_{i_1},v_{j_1}), y=(u_{i_2},v_{j_2})$, and $z=(u_{i_3},v_{j_3})$ be any three vertices of $G_1 \Box G_2$. Now

$$\begin{aligned} d(x,y)+d(y,z)+d(x,z) &= d(u_{i_1},u_{i_2})+d(v_{j_1},v_{j_2})+d(u_{i_2},u_{i_3}) \\ &\quad + d(v_{j_2},v_{j_3})+d(u_{i_1},u_{i_3})+d(v_{j_1},v_{j_3}) \\ &= d(u_{i_1},u_{i_2})+d(u_{i_2},u_{i_3})+d(u_{i_1},u_{i_3}) \\ &\quad + d(v_{j_1},v_{j_2})+d(v_{j_2},v_{j_3})+d(v_{j_1},v_{j_3}) \\ &\leq M_1+M_2. \end{aligned}$$

□

Proposition 3.5. *For any three vertices x,y and z in a path P_n, $d(x,y)+d(y,z)+d(x,z) \leq 2(n-1)$.*

Proof. Let x,y, and z be any three vertices in P_n. We know that $d(x,z)$ is equal to either $d(x,y)+d(y,z)$ or $|d(x,y)-d(y,z)|$. If $d(x,z)=d(x,y)+d(y,z)$ then $d(x,y)+d(y,z)+d(x,z)=2d(x,z) \leq 2(n-1)$. Without loss of generality let $d(x,z)=d(x,y)-d(y,z)$. Since $2(n-1)-d(x,y) \geq d(x,y)$, we get $2(n-1)-[d(x,y)+d(y,z)] \geq d(x,y)-d(y,z)=d(x,z)$. □

A star S_m with m vertices is the complete bipartite graph $K_{1,m-1}$.

Proposition 3.6. *For any three vertices x,y, and z in a star graph S_m, $d(x,y)+d(y,z)+d(x,z) \leq 6$.*

Proof. Since the diameter of S_m is 2, we get $d(x,y)+d(y,z)+d(x,z) \leq 6$. □

From Propositions 3.5, 3.6 and Theorem 3.4, we get the results below.

Theorem 3.7. *For any three vertices x,y, and z in a stacked book graph $B_{m,n}=S_{m+1}\Box P_n$, $d(x,y)+d(y,z)+d(x,z) \leq 2n+4$.*

Theorem 3.8. *For any three vertices x,y, and z in a grid graph $P_m \Box P_n$, $d(x,y)+d(y,z)+d(x,z) \leq 2(m+n-2)$.*

Theorem 3.9. *For any three vertices x,y, and z in a prism graph $C_n \Box P_m$, $d(x,y)+d(y,z)+d(x,z) \leq n+2(m-1)$.*

From Theorem 3.9 and Theorem 2.4, we get the following bound for $rc_k(C_n \Box P_m)$.

Theorem 3.10. *For a prism graph $C_n \square P_m$, let a_k be defined as*

$$a_k = \begin{cases} (mn-1)\left\lceil \frac{3(k+1)-(n+2(m-1))}{4} \right\rceil + 1, & \\ \qquad \text{if } mn \text{ is odd and } k \text{ is even;} & \\ (mn-1)\left\lceil \frac{3(k+1)-(n+2m-3)}{4} \right\rceil + 1, & \\ \qquad \text{if } mn \text{ is odd and } k \text{ is odd;} & \\ (mn-2)\left\lceil \frac{3(k+1)-(n+2(m-1))}{4} \right\rceil + k + 2 - \left(\left\lfloor \frac{n}{2} \right\rfloor + (m-1)\right), & \\ \qquad \text{if } mn \text{ is even}, n \not\equiv k \pmod 2, & \\ (mn-2)\left\lceil \frac{3(k+1)-(n+2m-3)}{4} \right\rceil + k + 2 - \left(\left\lfloor \frac{n}{2} \right\rfloor + (m-1)\right), & \\ \qquad \text{if } mn \text{ is even}, n \equiv k \pmod 2. & \end{cases}$$

Then $rc_k(C_n \square P_m) \geq a_k$.

We now prove that the lower bound for $C_n \square P_2$ agrees with its radio number for the cases $n \equiv 1 (\text{mod } 4)$, $n \equiv 6 (\text{mod } 8)$ and $n \equiv 1 (\text{mod } 8)$ respectively.

An upper bound for radio number of $C_n \square P_2$ (or $G(n,1)$). Here we shall use the notation $G(n,1)$ for $C_n \square P_2$. The $G(n,1)$, consists of two cycles $C_1 : u_0 u_1 \dots u_{n-1} u_0$ (say the outer cycle) and $C_2 : v_0 v_1 \dots v_{n-1} v_0$ (say the inner cycle), as well as the edges $\{u_i, v_i\}$, $0 \leq i \leq n-1$. For any two vertices x and x' in $G(n,1)$, if $d(x,x') = \text{diam}(G(n,1))$, then x and x' are called antipodal vertices. If n is even, then every vertex in $G(n,1)$ has a unique antipodal vertex, while if n is odd every vertex in $G(n,1)$ has exactly two antipodal vertices.

Lemma 3.11. *Let x and x' be antipodal vertices in $G(n,1)$, n even. Then for every vertex y in $G(n,1)$, $d(x,x') = d(x,y) + d(y,x')$.*

Lemma 3.12. *If x' and x'' are the two antipodal vertices of x in $G(n,1)$, n odd, then for every vertex y in $G(n,1)$ we have either $d(x,x') = d(x,y) + d(y,x')$ or $d(x,x'') = d(x,y) + d(y,x'')$.*

Next we give a radio coloring to $G(n,1)$ for the cases $n \equiv 1 \pmod 4$ and $n \equiv 6 (\text{mod } 8)$ respectively.

$G(n,1)$**,** $n \equiv 1$ (**mod** 4)**.** Since the greatest common divisor of n and $(n-1)/4$ is 1, we can have a sequence $x_1, x_3, \dots, x_{2n-1}$ of vertices on the outer cycle such that the corresponding distance sequence $d(x_1,x_3)$, $d(x_3,x_5)$, $\dots$, $d(x_{2n-3}, x_{2n-1})$ is a constant sequence of $(n-1)/4$. Since every vertex on the outer cycle has a vertex on the inner cycle of distance equal to the diameter apart, we can suppose that $x_2, x_4, \dots, x_{2n}$ are the vertices on the inner cycle which are at diameter distance from $x_1, x_3, \dots, x_{2n-1}$. Now, from Lemma 3.12, the sequence $x_1, x_2, \dots, x_{2n}$ is such that the corresponding distance sequence is an alternating sequence of $(n+1)/2$ and $(n+3)/4$. Next the following coloring f is a radio coloring of $G(n,1)$:

$$\begin{aligned} f(x_1) &= 1 \\ f(x_{i+1}) &= f(x_i) + 1, && i = 1, 3, 5, \ldots, 2n-1, \\ f(x_{i+1}) &= f(x_i) + \frac{n+3}{4}, && i = 2, 4, 6, \ldots, 2n-2. \end{aligned}$$

By using Lemma 2.2, the span of f is

$$f(x_{2n}) = 1 + (n-1)\left(\frac{n+3}{4}\right) + n = \frac{n^2+6n+1}{4}.$$

Since $d(x_i, x_{i+1}) = (n+1)/2 = \operatorname{diam}(G(n,1))$ and $d(x_{i+1}, x_{i+2}) = (n+3)/4 = (n-1)/4 + 1 = d(x_i, x_{i+2}) + 1$ for every triplet x_i, x_{i+1}, and x_{i+2} with $1 \le i \le 2n-2$ for which i is odd, by Theorem 3.10, f is the minimal radio coloring.

Example 3.13. *See Figure 1*

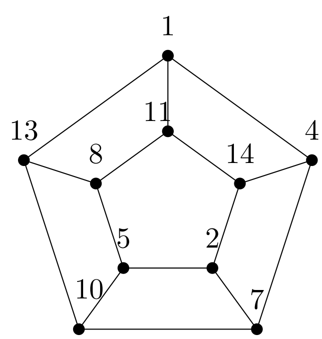

FIGURE 1. Minimal radio coloring of $G(5,1)$.

$G(n,1)$, $n \equiv 6$ (mod 8). Since there exist two integers $-(n-6)/8$ and $(n-4)/2$ such that

$$n\left(-\frac{n-6}{8}\right) + \frac{n-2}{4}\left(\frac{n-4}{2}\right) = 1,$$

n and $(n-2)/4$ are relatively prime. We can then form a sequence x_1, $x_3, \ldots, x_{2n-1}$ of vertices on the outer cycle such that the corresponding distance sequence $d(x_1, x_3)$, $d(x_3, x_5)$, $\ldots$, $d(x_{2n-3}, x_{2n-1})$ is a constant sequence of $(n-2)/4$. Let $x_2, x_4, \ldots, x_{2n}$ be the vertices on the inner cycle which are at a distance equal to the diameter from $x_1, x_3, \ldots, x_{2n-1}$. Now, from Lemma 3.11, the sequence $x_1, x_2, \ldots, x_{2n}$ provides us with a corresponding distance sequence alternating between $(n+2)/2$ and $(n+6)/4$.

Thus the following coloring f is a radio coloring of $G(n,1)$:

$$\begin{aligned} f(x_1) &= 1, \\ f(x_{i+1}) &= f(x_i)+1, && i=1,3,5,\ldots,2n-1, \\ f(x_{i+1}) &= f(x_i)+\frac{n+6}{4}, && i=2,4,6,\ldots,2n-2. \end{aligned}$$

Using Lemma 2.2, the span of f is

$$f(x_{2n}) = 1+(n-1)\left(\frac{n+6}{4}\right)+n = \frac{n^2+9n-2}{4}.$$

Since $d(x_i,x_{i+1}) = (n+2)/2 = \operatorname{diam}(G(n,1))$ and $d(x_{i+1},x_{i+2}) = (n+6)/4 = (n-2)/4+2 = d(x_i,x_{i+2})+2$ with $\epsilon_{i+2}=1$ for every triplet x_i, x_{i+1}, and x_{i+2} with $1 \le i \le 2n-2$ for which i is odd, by Theorem 3.10, f is the minimal radio coloring.

Example 3.14. *See Figure 2.*

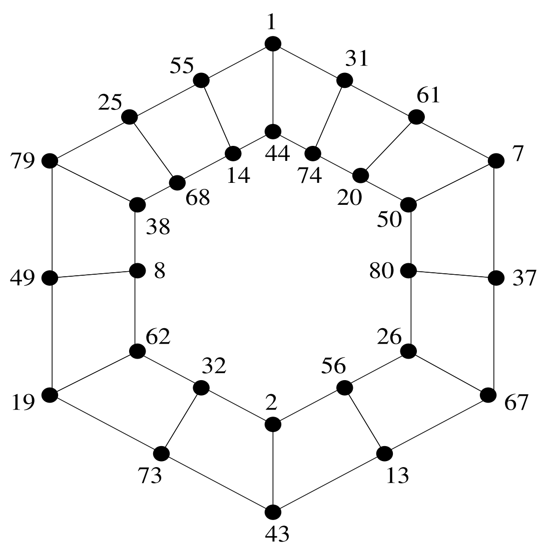

FIGURE 2. Minimal radio coloring of $G(14,1)$.

4. Conclusion and Future Scope

It is not very difficult to find the triameter for several types of graphs. Furthermore, we observe that our lower bound is sharp for at least the radio number of a graph which contains more pairs of antipodal vertices.

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