NORMALITY OF ONE-MATCHING SEMI-CAYLEY GRAPHS OVER FINITE ABELIAN GROUPS WITH MAXIMUM DEGREE THREE

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Abstract. A graph Γ is said to be a semi-Cayley graph over a group G if it admits G as a semiregular automorphism group with two orbits of equal size. We say that Γ is normal if G is a normal subgroup of Aut(Γ). We prove that every connected intransitive one-matching semi-Cayley graph, with maximum degree three, over a finite abelian group is normal and characterize all such nonnormal graphs.

1. Introduction

Throughout this paper, groups are finite and graphs are finite, connected, simple, and undirected. For the graph-theoretic and group-theoretic terminology not defined here, we refer the reader to [7, 24]. Let G be a permutation group on Ω and α ∈ Ω. Denote by G α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is semiregular on Ω if G α = 1 for every α ∈ Ω and regular if G is transitive and semiregular. Let G be a group and S a subset of G not containing the identity element 1G. The Cayley digraph Γ = Cay(G, S) of G with respect to S has vertex set G and arc set \{(g, sg) | g ∈ G, s ∈ S\}. If S = S⁻¹ then Cay(G, S) can be viewed as an undirected graph, identifying an undirected edge with two directed edges (g, h) and (h, g). This graph is called the Cayley graph of G with...
respect to \( S \). By a theorem of Sabidussi [22], a graph \( \Gamma \) is a Cayley graph over a group \( G \) if and only if there exists a regular subgroup of \( \text{Aut}(\Gamma) \) isomorphic to \( G \).

There is a natural generalization of Sabidussi’s Theorem. A graph \( \Gamma \) is called an \( n \)-Cayley graph over a group \( G \) if there exists an \( n \)-orbit semiregular subgroup of \( \text{Aut}(\Gamma) \) isomorphic to \( G \). Undirected and loop-free 2-Cayley graphs are called semi-Cayley [4, 9], and also bi-Cayley by some authors [25]. \( n \)-Cayley graphs have played an important role in many classical fields of graph theory, such as strongly regular graphs [17, 20, 21, 9], Hamiltonian graphs [23], \( n \)-extendable graphs [11, 19], the spectrum of graphs [3, 2, 12], automorphisms [5, 1, 14, 25], and the connectivity of graphs [8, 18].

A graph \( \Gamma \) is called a semi-Cayley graph over a group \( G \) if \( \text{Aut}(\Gamma) \) admits a semiregular subgroup \( R_G \) isomorphic to \( G \) with two orbits (of equal size). Let \( \Gamma \) be a semi-Cayley graph over a group \( G \). Then there exists subsets \( R, L, \) and \( S \) of \( G \) such that \( R = R^{-1}, L = L^{-1}, \) and \( 1 \notin R \cup L \) such that \( \Gamma \cong \text{SC}(G; R, L, S) \), where \( \text{SC}(G; R, L, S) \) is an undirected graph with vertices \( G \times \{1, 2\} \) and its edge set consists of three sets (see [9, Lemma 2.1]):

\[
\begin{align*}
\{\{(x, 1), (y, 1)\} \mid yx^{-1} \in R\} & \quad \text{(right edges),} \\
\{\{(x, 2), (y, 2)\} \mid yx^{-1} \in L\} & \quad \text{(left edges),} \\
\{\{(x, 1), (y, 2)\} \mid yx^{-1} \in S\} & \quad \text{(spoke edges).}
\end{align*}
\]

Furthermore, \( R_G := \{\rho_g \mid g \in G\} \), where \( \rho_g : G \times \{1, 2\} \to G \times \{1, 2\} \) and \( (x, i)\rho_g = (xg, i), \ i = 1, 2, \) is a semiregular subgroup of \( \text{Aut}(\text{SC}(G; R, L, S)) \) isomorphic to \( G \) with two orbits \( G \times \{1\} \) and \( G \times \{2\} \). A semi-Cayley graph \( \Gamma = \text{SC}(G; R, L, S) \) over a group \( G \) is called normal over \( G \) if \( R_G \) is a normal subgroup of \( \text{Aut}(\Gamma) \) (see [5, p. 42]) and it is called one-matching if \( S = \{1\} \) (see [16, p. 603]). In this paper, we prove:

**Theorem 1.1.** Let \( \Gamma = \text{SC}(G; R, L, \{1\}) \) be a connected one-matching semi-Cayley graph over a finite abelian group \( G \neq 1 \) with \( |R|, |L| \leq 2 \). Then \( \Gamma \) is normal if and only if none of the following are satisfied (even after interchanging \( R \) and \( L \))

1. \( |R| = |L| = 1 \) (so \( G \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_2^2 \)),
2. \( |R| = |L| = 2 \) and \( |R \cap L| = 1 \) (so \( G \cong \mathbb{Z}_2^2 \) or \( \mathbb{Z}_2^3 \)),
3. \( R = L = \{a, a^{-1}\}, \) where \( o(a) = 4 \) (so \( G = \langle a \rangle \cong \mathbb{Z}_4 \)),
4. \( R = \{a, b\}, \ L = \{c, c^{-1}\}, \) where \( o(a) = o(b) = 2, \ o(c) = 4 \) and \( G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4 \),
5. \( R = \{a, a^{-1}\}, \ L = \{b, ba^2\}, \) where \( o(a) = 4, \ o(b) = 2 \) and \( G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \),
(6) \( R = \{a, a^{-1}\}, \ L = \{a^k, a^{-k}\} \), where \( o(a) = n \) and \( (n, k) \) is one of the pairs \((5, 2), (8, 3), (10, 2), (10, 3), (12, 5) \) or \((24, 5) \) (so \( G \cong \mathbb{Z}_n \)),

(7) \( R = \{a, a^{-1}\}, \ L = \{a^3b, a^{-3}b\} \) or \( L = \{a^2b, a^{-2}b\} \), where \( o(a) = 10, \ o(b) = 2 \) and \( G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{10}\times \mathbb{Z}_2 \),

(8) \( R = \{a, a^{-1}\}, \ L = \{ab, a^{-1}b\} \), where \( o(a) = 4, \ o(b) = 2 \) and \( G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_4\times \mathbb{Z}_2 \).

Furthermore, in all of the above cases, \( \Gamma \) is transitive.

For a graph \( \Gamma \), we use \( V(\Gamma), E(\Gamma), A(\Gamma) \) and \( \text{Aut}(\Gamma) \) to denote its vertex set, edge set, arc set and its full automorphism group respectively. For \( v \in V(\Gamma) \), \( N(u) \) is the neighborhood of \( u \) in \( \Gamma \), that is, the set of vertices adjacent to \( u \) in \( \Gamma \). A graph \( \Gamma \) is called transitive if \( \text{Aut}(\Gamma) \) is transitive on \( V(\Gamma) \), otherwise it is called intransitive. Also a graph \( \Gamma \) is said to be edge-transitive and arc-transitive (or symmetric) if \( \text{Aut}(\Gamma) \) acts transitively on \( E(\Gamma) \) and \( A(\Gamma) \), respectively.

2. Preliminaries

Let \( \Gamma = \text{SC}(G; R, L, \{1\}) \) be a one-matching semi-Cayley graph over a finite group \( G \neq 1 \). Let \( \Gamma_0 = \text{SC}(G; L, R, \{1\}) \) the graph obtained from interchanging the left and right edges of \( \Gamma \). Then \( \Gamma \cong \Gamma_0 \). Furthermore, \( \text{Aut}(\Gamma) \cong \text{Aut}(\Gamma_0) \) and also \( R_G \unlhd \text{Aut}(\Gamma) \) if and only if \( A_G \unlhd \text{Aut}(\Gamma_0) \). Hence, in studying the normality of \( \Gamma \), we may assume that \( |L| \leq |R| \). Moreover, since \( \Gamma \) is a normal over a group \( G \) if and if its complement \( \Gamma^c \) is normal over \( G \), we may assume that \( \Gamma \) is connected or equivalently \( G = \langle R \cup L \rangle \).

Let \( \Gamma = \text{SC}(G; R, L, \{1\}) \) be a connected semi-Cayley graph over a finite abelian group \( G \), and let \( A \) and \( V \), be its automorphism group and vertex set, respectively. For each \( \sigma \in \text{Aut}(G) \) we define two maps

\[
\varphi_{\sigma} \ : \ V(\Gamma) \rightarrow V(\Gamma); \ (x, 1)^{\varphi_{\sigma}} = (x^\sigma, 1), \ (x, 2)^{\varphi_{\sigma}} = (x^\sigma, 2),
\]

\[
\psi_{\sigma} \ : \ V(\Gamma) \rightarrow V(\Gamma); \ (x, 1)^{\psi_{\sigma}} = (x^\sigma, 2), \ (x, 2)^{\psi_{\sigma}} = (x^\sigma, 1).
\]

Set

\[
X := \{ \varphi_{\sigma} \mid \sigma \in \text{Aut}(G), R^\sigma = R, L^\sigma = L \},
\]

\[
Y := \{ \psi_{\sigma} \mid \sigma \in \text{Aut}(G), R^\sigma = L, L^\sigma = R \},
\]

and let us denote \( X \cup Y \) by \( \text{Aut}(G; R, L) \). Then \( N_A(R_G) = R_G \rtimes \text{Aut}(G; R, L) \) by [5, Theorem 1]. So \( R_G \leq A \) if and only if \( A = R_G \rtimes \text{Aut}(G; R, L) \) \[5, \text{Proposition 2 (1)}\]. Moreover, if \( R_G \leq A \), then \( A_{(1,1)} = X \) and the converse holds if \( \Gamma \) is intransitive \[5, \text{Proposition 2 (2)}\]. Also if \( R_G \leq A \) then \( \Gamma \) is intransitive if and only if \( A_{(1,1)} = \text{Aut}(G; R, L) \) \[5, \text{Corollary 2.9}\]. Note that if \( Y \neq \emptyset \), then \( \Gamma \) is transitive. So if \( R_G \leq A \)
then $\Gamma$ is transitive if and only if $Y \neq \emptyset$. Also, by the following lemma and above results, if $\Gamma$ is intransitive or $Y \neq \emptyset$, then $\Gamma$ is normal if and only if $A(1,1) = X$. It is easy to see that $A(1,1) \cap N_A(R_G) = A(1,2) \cap N_A(R_G) = X$. In particular, if $R_G \leq A$ then $A(1,1) = A(1,2) = X$. In what follows, unless otherwise stated, we keep the above notations and use the above results without referring them.

**Lemma 2.1.** Let $Y \neq \emptyset$. Then $\Gamma$ is normal if and only if $A(1,1) = X$.

**Proof.** If $\Gamma$ is normal then $A(1,1) = X$. Conversely, suppose that $A(1,1) = X$. Let $\beta \in A$ be arbitrary. We have to show that $\beta \in N_A(R_G)$. Since $G \neq 1$ (and $G \subseteq N_A(R_G)$), we may assume that $(1,1)\beta \in G \times \{1\}$ (if $(1,1)\beta \in G \times \{2\}$, then we replace $\beta$ with $\beta y$ for some $y \in Y$). Then after multiplying by an element of $R_G$, we may assume that $(1,1)^\beta = (1,1)$. So $\beta \in A(1,1) = X \subseteq N_A(R_G)$. □

### 3. Proof of Theorem 1.1

Keeping the notations of previous section, recall that $\Gamma = SC(G; R, L, \{1\})$ is a connected semi-Cayley graph over a finite abelian group $G \neq 1$ with $|L| \leq |R| \leq 2$, and $A$ denotes the automorphism group of $\Gamma$. To prove Theorem 1.1, we consider all the possibilities for the orders of $R$ and $L$ and their intersection.

Let us start with the following lemma:

**Lemma 3.1.** Let $\Gamma$ be edge-transitive. Then it is nonnormal. Also if $\Gamma$ is arc-transitive then $\Gamma$ is nonnormal.

**Proof.** It is enough to note that any element of the normalizer of $R_G$ must map $G$-orbits to $G$-orbits but an element of $A$ that takes a right edge or left edge to a spoke edge does not do this. Since every connected arc-transitive graph is edge-transitive, the second part is clear. □

**Lemma 3.2.** Let $L = \emptyset$, $R \neq \emptyset$. Then $\Gamma$ is intransitive and normal, and

1. if $|R| = 1$ then $G \cong \mathbb{Z}_2$, $\Gamma \cong P_4$, $A \cong \mathbb{Z}_2$,
2. if $|R| = 2$ then $G \cong \mathbb{Z}_n$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, $n \geq 3$, and $A \cong D_{2|G|}$.

**Proof.** (1) It is clear.

(2) Since $\Gamma$ is connected and $L = \emptyset$, we have $G = \langle R \rangle \cong \mathbb{Z}_n$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, for some $n \geq 3$. Hence Cay$(G, R)$ is a $|G|$-cycle. By [5, Lemma 4.1], $A \cong Aut(Cay(G, R)) \cong D_{2|G|}$. □

**Lemma 3.3.** Let $R = \{a\}$ and $L = \{b\}$. Then $\Gamma$ is transitive and nonnormal and one of the following holds:

1. $G \cong \mathbb{Z}_2$, $A \cong D_8$, 

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(2) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $A \cong D_{16}$.

Proof. If $a = b$ then $G \cong \mathbb{Z}_2$ and otherwise $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In both cases, $\Gamma$ is a 2-$|G|$-cycle and so $A \cong D_{4|G|}$. Furthermore, in both cases $A_{(1,1)} \neq X$, which implies that both are nonnormal. □

Lemma 3.4. Let $\Gamma$ be intransitive, $R \cap L = \emptyset$, and $\Gamma_\Omega$ be the quotient graph of $\Gamma$ with respect to the one-matching set $\Omega = \{(g, 1), (g, 2)\} \mid g \in G$. Then $A \leq \text{Aut}(\Gamma_\Omega)$, where $\Gamma_\Omega$ is a Cayley graph of $RG$ with respect to $S = \{\rho_r, \rho_l \mid r \in R, l \in L\}$ of valency $|R| + |L|$. In particular, if $\Gamma_\Omega$ is a normal Cayley graph of $RG$, then $\Gamma$ is a normal semi-Cayley graph of $RG$.

Proof. We consider the action of $A$ on $\Omega$. Let $K$ be the kernel of this action. Since $\Gamma$ is intransitive, it implies that $K = 1$ and so $A \leq \text{Aut}(\Gamma_\Omega)$. Clearly $RG$ acts transitively on $V(\Gamma_\Omega)$. Now suppose that $\rho_h \in RG$ and $\{(g, 1), (g, 2)\}_{\rho_h} = \{(g, 1), (g, 2)\}$. Therefore $(g, 1)^{\rho_h} = (g, 1)$ and $(g, 2)^{\rho_h} = (g, 2)$ and so $(gh, 1) = (g, 1)$. Thus $\rho_h = 1$ and $RG$ acts regularly on $V(\Gamma_\Omega)$ and so $\Gamma_\Omega$ is a Cayley graph on $RG$ with respect to $S$. Also since $R \cap L = \emptyset$, it implies that $\Gamma_\Omega$ has valency $|R| + |L|$. □

Lemma 3.5. If $|R| = 2$ and $|L| = 1$ then $\Gamma$ is normal.

Proof. Let $R = \{a, b\}$ and $L = \{c\}$. If $c = a$ or $c = b$ then $a^2 = b^2 = 1$, $A = RG \cong \mathbb{Z}_2^2$ and so $\Gamma$ is normal. Hence, we may assume that $c \neq a, b$. Suppose, towards a contradiction, that $\Gamma$ is nonnormal. Then $R \cap L = \emptyset$. Let $\Omega = \{(g, 1), (g, 2)\} \mid g \in G$ and $\Gamma_\Omega$ be the Cayley graph of $RG$ with respect to $S = \{\rho_o, \rho_h, \rho_c\}$. Since $\Gamma$ is nonnormal, Lemma 3.4 and [6, Theorem 1.2] imply that one of the following occurs:

1. $o(a) = 4$, $b = a^{-1}$ and $c = a^2$,
2. $o(a) = 4$, $b = a^{-1}$, $c^2 = 1$ and $c \notin \langle a \rangle$,
3. $o(a) = 6$, $b = a^{-1}$ and $c = a^3$.

In the first case, $A \cong D_8$ and $\Gamma$ is normal, in the second case $A \cong \mathbb{Z}_2 \times D_8$ and $\Gamma$ is normal, and in the last case, $A \cong D_{12}$ and $\Gamma$ is normal. Hence we get a contradiction. □

Lemma 3.6. Let $R = L$, $|R| = 2$. Then $\Gamma$ is transitive and the following are equivalent:

1. $\Gamma$ is normal,
2. $\Gamma$ is not arc-transitive,
3. $R = \{a, a^{-1}\}$, where $a$ is of order $k > 2$ and $k \neq 4$.

Proof. It is easy to see that $\Gamma$ is isomorphic to the $n$-prism graph, the cartesian product of an $n$-cycle with a path with two vertices, where
$n = |G|$, which is isomorphic to a Cayley graph on the dihedral group $D_{2n} = \langle s, t \mid s^n = t^2 = (st)^2 = 1 \rangle$, with respect to $S = \{s, s^{-1}, t\}$. Hence $\Gamma$ is transitive.

By Lemma 3.1, (1) implies (2). Now suppose that (2) holds. If $R = \{b, c\}$, where $b^2 = c^2 = 1$, then $\Gamma$ is isomorphic to the three dimensional hypercube, which is arc-transitive, a contradiction. Hence $R = \{a, a^{-1}\}$, where $a$ is of order $k > 2$. Hence $G = \langle a \rangle \cong \mathbb{Z}_k$. Hence, by [16, Theorem 1.1], $k \neq 4$. Thus (2) implies (3). To complete the proof, it is enough to prove that (3) implies (1). Suppose (3) holds. Then $G \cong \mathbb{Z}_k$ and it is easy to see that $\Gamma$ is isomorphic to the generalized Petersen graph $GP(k, 1)$ (see [10]). Also by [10, Theorems 1 and 2] $GP(k, 1)$ is vertex transitive and $A \cong D_{2k} \times \mathbb{Z}_2$. Hence $\Gamma$ is vertex transitive and so $|A| = |A^{(1,1)}|2k$. This shows that $|A^{(1,1)}| = 2$. Since $R = L = \{a, a^{-1}\}$, $Y \neq \emptyset$ and $|X| \geq 2$. Since $X \leq A^{(1,1)}$, we have $X = A^{(1,1)}$. Hence $\Gamma$ is normal, i.e. (1) holds. This completes the proof.

Lemma 3.7. Let $|R| = |L| = 2$, $|R \cap L| = 1$. Then $\Gamma$ is transitive and nonnormal. Also one of the following holds:

(1) $G = \langle a, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \{a, b\}$ and $L = \{ab, b\}$,

(2) $G = \langle a, b, c \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \{a, b\}$ and $L = \{b, c\}$.

Proof. Since $R = R^{-1}$ and $L = L^{-1}$, both $R$ and $L$ consist of two involutions. Assume that $R = \{a, b\}$ and $L = \{b, c\}$. Since $G = \langle a, b, c \rangle$, if $c = ab$, $G = \langle a, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, otherwise $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In the former case,

$\sigma = ((ab, 1), (b, 2))((a, 2), (ab, 2))((a, 1), (1, 2)) \in A^{(1,1)}$

but $\sigma \notin X$. Therefore $\Gamma$ is not normal. In the latter,

$\sigma = ((a, 1), (1, 2))((a, 2), (c, 2))((ab, 1), (b, 2))$

$((ab, 2), (bc, 2))((ac, 2), (c, 1)) \in A^{(1,1)}$

but $\sigma \notin X$. So $\Gamma$ is not normal. Also in both cases we see that $\Gamma$ is transitive.

Lemma 3.8. Let $|R| = |L| = 2$, $R \cap L = \emptyset$. If $R = \{a, b\}$, where $a^2 = b^2 = 1$, then one of the following holds:

(1) $L = \{ab, c\}$, where $c^2 = 1$. In this case $G \cong \mathbb{Z}_2^3$, $\Gamma$ is intransitive and normal.

(2) $L = \{c, d\}$, where $c^2 = d^2 = 1$. In this case $G \cong \mathbb{Z}_2^4$ and $\Gamma$ is transitive and normal.
(3) \( L = \{c, c^{-1}\} \), where \( c \) is of order \( n > 2 \). In this case, \( G \cong \mathbb{Z}_2^3 \times \mathbb{Z}_n \), and \( \Gamma \) is normal if and only if \( \Gamma \) is intransitive if and only if \( n \neq 4 \).

(4) \( L = \{c, c^{-1}\} \), where \( o(c) = n > 2 \) is even, and \( b = c^{n/2} \). In this case, \( G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \), and \( \Gamma \) is normal and intransitive.

(5) \( L = \{c, c^{-1}\} \), where \( o(c) = n > 2 \) is even, and \( b = ac^{n/2} \). In this case, \( G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \) and \( \Gamma \) is normal if and only if \( \Gamma \) is intransitive if and only if \( n \neq 4 \).

**Proof.** It is obvious that the possibilities of \( L \) are exactly the same given in (1)–(5).

(1) In this case, \( G = \langle a, b, c \rangle \cong \mathbb{Z}_2^3 \). Then, by GAP [13], \( \Gamma \) is intransitive and normal.

(2) In this case, \( G \cong \mathbb{Z}_2^4 \) and by GAP, \( A \cong (D_8 \times D_8) \rtimes \mathbb{Z}_2 \), \( \Gamma \) is transitive and normal.

(3) Suppose that \( L = \{c, c^{-1}\} \), where \( c \) is an element of order \( n > 2 \). Then \( G \cong \mathbb{Z}_2^3 \times \mathbb{Z}_n \). We prove that \( \Gamma \) is normal if and only if it is intransitive if and only if \( n \neq 4 \).

If \( n = 4 \) then by GAP, \( \Gamma \) is transitive. Conversely, suppose that \( \Gamma \) is transitive. Then there exists \( \alpha \in \text{Aut}(\Gamma) \) such that \( (1, 1)^\alpha = (1, 2) \). Then \( \alpha \) maps the 4-cycle

\[(1, 1), (b, 1), (ab, 1), (a, 1), (1, 1)\]

to a 4-cycle including the point \((1, 2)\). Since \( R \cap L = \emptyset \), we have \((a, 1)^\alpha, (b, 1)^\alpha \neq (1, 1) \). Hence \((a, 1)^\alpha, (b, 1)^\alpha \in \{(c, 2), (c^{-1}, 2)\} \), which implies that \((ab, 1)^\alpha = (c^2, 2) = (c^{-2}, 2) \). This means that \( n = 4 \).

Let \( \Gamma \) be normal. Then, since \( Y = \emptyset \), \( \Gamma \) is intransitive. Conversely, suppose that \( \Gamma \) is intransitive. So \( n \neq 4 \), by the above discussion. Now [6, Theorem 1.2] and Lemma 3.4, imply that \( \Gamma \) is normal.

(4) In this case \( G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \). If \( n = 4 \), then by GAP, \( \Gamma \) is intransitive and normal. Hence, we may assume that \( n \neq 4 \). So, by a similar argument of the previous case, \( \Gamma \) is intransitive. Suppose, towards a contradiction, that \( \Gamma \) is nonnormal. Then, by Lemma 3.4 and [6, Theorem 1.2], \( n = 6 \). Now, by GAP, \( A \cong \mathbb{Z}_2^3 \rtimes S_3 \) which implies that \( \Gamma \) is normal, a contradiction.

(5) In this case \( G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \). If \( n = 4 \) then \( \Gamma \) is transitive and nonnormal. Let \( n \neq 4 \). Then by a similar argument of the case (3), \( \Gamma \) is intransitive. By the same argument in case (4), if \( \Gamma \) is nonnormal, then \( n = 6 \), which implies that, by GAP, \( A \cong D_8 \times \mathbb{Z}_3 \) and \( \Gamma \) is normal. \( \square \)
Let $S$ be an inverse-closed subset of a group $G$ not containing the identity element of $G$. Recall that a permutation $o$ of $G$ is a colour-preserving automorphism of $\text{Cay}(G, S)$ if and only if we have $(xs)^o \in \{x^{\sigma} s^{\pm 1}\}$ for each $x \in G$ and $s \in S$ [15, p. 190].

**Lemma 3.9.** Let $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$, $o(a), o(b) \geq 3$, and $R \cap L = \emptyset$. If $\Gamma$ is intransitive then it is normal.

**Proof.** Suppose, towards a contradiction, that there exists $\alpha \in A$ that does not normalize $R_G$. Since $\Gamma$ is intransitive, there is a permutation $\sigma$ of $G$ such that $(g, i)^\alpha = (g^\sigma, i)$ for all $g \in G$ and $i = 1, 2$. There is a natural colouring of $\text{Cay}(G, \{a^\pm 1, b^\pm 1\})$ with two colours, where $a$-edges have one colour and $b$-edges have the other colour. Then $\sigma$ is a colour-preserving automorphism of $\text{Cay}(G, \{a^\pm 1, b^\pm 1\})$ because $\alpha$ is an automorphism of $\Gamma$, which means that $(ga)^\sigma \in \{ga^\pm 1\}$ and $(gb)^\sigma \in \{gb^\pm 1\}$.

Since $\alpha$ does not normalize $R_G$ (and $G$ is 2-generated), we know from [15, Proposition 4.1] that $G$ has a direct factor that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$. So $o(a)$ and $o(b)$ are even. Therefore $o(a), o(b) \neq 3$ and so $o(a), o(b) \geq 4$. If $o(a) = o(b) = 4$ then, by GAP, $\Gamma$ is a transitive graph which is a contradiction. So we may assume that $o(b) > 4$.

By composing with a translation, we may assume that $\sigma$ fixes 1. We may also assume that $\sigma$ fixes $a$ by composing with inversion if necessary. Then $(a^k)^\sigma = a^k$ for all $k$.

We claim that we may assume $b^\sigma = b$. Suppose $b^\sigma \neq b$, so $b^\sigma = b^{-1}$. Then $\sigma$ is the identity on $\langle a \rangle$ but inverts $\langle b \rangle$, which implies that $|\langle a \rangle \cap \langle b \rangle| \leq 2$. Therefore there is an automorphism of $G$ agrees with $\sigma$ on $\langle a \rangle \cup \langle b \rangle$. By composing with this automorphism, we have $b^\sigma = b$ as desired.

Since $\sigma$ does not normalize $R_G$, we know that $\sigma$ is not the identity permutation and so there is some minimal $k > 0$ such that $(a^k b)^\sigma = a^k b^{-1}$. Since $a^{k-1} b$ is adjacent to $a^k b$ via an $a$-edge, we have $a^{k-1} ba = a^k b^{-1}$ or $a^{k-1} ba^{-1} = a^k b^{-1}$. The first implies that $b^2 = 1$ which contradicts the fact that $o(b) \geq 4$. The second implies $a^2 = b^2$. Since $|\langle a \rangle \cap \langle b \rangle| \leq 2$, we have $o(b) \leq 4$ which contradicts the fact $o(b) > 4$. \square

**Corollary 3.10.** Let $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$, where $o(a) \neq o(b)$ and $R \cap L = \emptyset$, then $\Gamma$ is normal if and only if $\Gamma$ is intransitive.

**Proof.** One direction is clear by Lemma 3.9. Let $\Gamma$ be normal and suppose, towards a contradiction, that $\Gamma$ is transitive. Then there exists $\alpha \in A$ such that $(1, 1)^\alpha = (1, 2)$. Since $\Gamma$ is normal, there exists $\sigma \in \text{Aut}(G)$ such that $\alpha = \psi_\sigma$, $R^\sigma = L$ and $L^\sigma = R$, which implies that $o(a) = o(b)$, a contradiction. \square
Lemma 3.11. Let $|R| = |L| = 2$ and $R \cap L = \emptyset$. If $R = \{a, a^{-1}\}$, where $o(a) = n \geq 3$, then, perhaps after interchanging $R$ and $L$, one of the following holds:

1. $L = \{a^{n/2}, b\}$, where $n$ is even, $b^2 = 1$ and $b \notin \langle a \rangle$. In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, $\Gamma$ is normal and intransitive.
2. $L = \{b, ba^{n/2}\}$, where $n$ is even, $b^2 = 1$ and $b \notin \langle a \rangle$. In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, if $n = 4$ then $\Gamma$ is transitive and nonnormal and otherwise $\Gamma$ is normal and intransitive.
3. $L = \{b, c\}$, where $b^2 = c^2 = 1$, $\langle a \rangle \cap \langle b, c \rangle = 1$ and $b, c \notin \langle a \rangle$. In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, if $n = 4$ then $\Gamma$ is transitive and nonnormal and otherwise $\Gamma$ is normal and intransitive.
4. $L = \{a^k, a^{-k}\}$, for some $k \geq 2$. In this case, $G \cong \mathbb{Z}_n$. Furthermore, $\Gamma$ is nonnormal if and only if $(n, k) = (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)$. Also if $\Gamma$ is nonnormal then $\Gamma$ is transitive.
5. $L = \{b, b^{-1}\}$, $b \notin \langle a \rangle$, and $\langle b \rangle \cap \langle a \rangle \neq 1$. In this case, $\Gamma$ is nonnormal if and only if $L = \{a^3y, a^{-3}y\}$ or $L = \{a^2y, a^{-2}y\}$, where $o(a) = 10$, $o(y) = 2$, and $G = \langle a \rangle \times \langle y \rangle \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$, or $L = \{ay, a^{-1}y\}$, where $o(a) = 4$, $o(y) = 2$, and $G = \langle a \rangle \times \langle y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Also if $\Gamma$ is nonnormal then $\Gamma$ is transitive.
6. $L = \{b, b^{-1}\}$, and $\langle b \rangle \cap \langle a \rangle = 1$. In this case, $G \cong \mathbb{Z}_n \times \mathbb{Z}_l$, where $l = o(b)$ and $\Gamma$ is normal. Furthermore, $\Gamma$ is transitive if and only if $o(a) = o(b)$.

Proof. It is easy to see that the only possibilities of $L$ are the cases (1)–(6). Since $SC(G; R, L, \{1\}) \cong SC(G; L, R, \{1\})$, by the last three cases of Lemma 3.8, cases (1), (2), and (3) are clear.

(4) In this case, $\Gamma$ is isomorphic to the generalized Petersen graph $GP(n, k)$. Let $\Gamma$ is nonnormal and suppose, for the contrary,

$$\{n, k\} \notin \{(5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$ 

Then $\Gamma$ is not edge-transitive by [10, Lemma 3 and Theorem 2]. Hence $|A| = 4n$ [10, Theorem 1 and Theorem 2]. Since $\Gamma$ is nonnormal, Lemma 3.9 and [7, p. 105] imply that $k^2 \equiv \pm 1 \pmod n$. Hence $(k, n) = 1$. Let $\sigma_1, \sigma_2, \sigma_3 : G \to G$ be the maps by the rules $(a^i)^{\sigma_1} = a^{ik}$, $(a^i)^{\sigma_2} = a^{-i}$, and $(a^i)^{\sigma_3} = a^{-ik}$. Then these three maps are automorphisms of $G$. Furthermore, $\psi_{\sigma_1}, \psi_{\sigma_2}, \varphi_{\sigma_3} \in Aut(G; R, L)$. So $|Aut(G; R, L)| \geq 4$, which implies that $A = N_A(R_G)$ i.e., $\Gamma$ is normal, a contradiction.

Conversely, suppose that

$$\{n, k\} \in \{(5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$
Then $\Gamma$ is arc-transitive by [7, p. 105] and so it is nonnormal by Lemma 3.1. If $\Gamma$ is nonnormal then it is transitive by Lemma 3.9.

(5) Let $L = \{b, b^{-1}\}$, $b \notin \langle a \rangle$, $a \notin \langle b \rangle$, and $\langle b \rangle \cap \langle a \rangle \neq 1$. If $\Gamma$ is intransitive, then by Lemma 3.9, $\Gamma$ is normal. Hence, we may assume that $\Gamma$ is transitive. If $\Gamma$ is arc-transitive then by [26, Proposition 5.1(2)], it is the unique arc-transitive cubic graph of order 40, denoted by F040A in the Foster Census, $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$, and we may assume that $a = x$ and $b \in \{x^3 y, x^2 y\}$. Then $\Gamma$ is nonnormal by Lemma 3.1. So we may now assume that $\Gamma$ is not arc-transitive. Then by [26, Theorem 1.1], up to isomorphism, one of the following occurs:

(1) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_{mk} \times \mathbb{Z}_m$, $k \geq 3, m \geq 1$, where $(m, k, t) = (1, 10, 2)$ or $(t, mk) = 1$ and $t^2 \equiv -1 \pmod{k}$ and we may assume that $a = x$, $b = x^t y$. Clearly $m = 1$ is impossible, because $b \notin \langle a \rangle$. Also $\text{Aut}(\Gamma) \cong R_G \times \mathbb{Z}_4$ [26, Theorem 5.5(3)], which implies that $\Gamma$ is normal.

(2) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_{mk} \times \mathbb{Z}_m$, $km \geq 3$, and $m \geq 1$, where $(t, mk) = 1$, $t^2 \equiv 1 \pmod{k}$ and $\Gamma$ is a Cayley graph over $G \rtimes \langle z \rangle$ for some involution $z$, [26, Theorem 5.2(5)]. Furthermore, we may assume that $a = x$, $b = x^t y$. Clearly $m = 1$ is impossible because $b \notin \langle a \rangle$.

Since $\Gamma$ is connected and transitive but not edge-transitive, every automorphism of $\Gamma$ maps $G$-orbits to $G$-orbits. If $(m, k) = (2, 2)$, then $L = \{ay, a^{-1} y\}$, $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and $\Gamma$ is nonnormal over $G$ by GAP. Hence, we may assume that $(m, k) \neq (2, 2)$. Then we claim that $\Gamma$ is normal. Suppose, towards a contradiction, that $\Gamma$ is not normal. So, there exists a colour-preserving automorphism $\sigma$ of $\Gamma_0 = \text{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ which fixes 1 but is not a group automorphism of $G$ (see [15, Remark 2.1]). Since the map $x \mapsto x^{-1}$ is an automorphism of $G$, we may assume that $a^\sigma = a$.

We may assume that $\sigma$ is not the identity. Then there is some $i$ such that $(a b)^i = a b^{-1}$. By composing with a translation, we may assume that $i = 0$ and $b^\sigma = b^{-1}$. Then we have $(b^m)^\sigma = b^{-m}$. But $b^m \in \langle a \rangle$ and $\sigma$ is the identity on $\langle a \rangle$. So $b^m$ must have order two which means that $b^m = a^{km/2}$. So $tm \equiv km/2 \pmod{km}$ which means that $t \equiv k/2 \pmod{k}$. Since $t^2 \equiv 1 \pmod{k}$ this implies that $k = 2$. So, by [15, Corollary 4.2], $|G| = 2m^2$ is divisible by 8. Thus $m$ is even and $m \geq 4$ because $(m, k) \neq (2, 2)$. Since $\langle a \rangle \cap \langle b \rangle \neq 1$ and $(b^m)^{-1} = b^m$, the map $\varphi: a b^i \mapsto a b^{-i}$ is a well-defined automorphism of $G$ that is also an automorphism of $\Gamma_0$. Furthermore, $\psi = \sigma \varphi$ is a colour-preserving
automorphism of $\Gamma_0$ which fixes all powers of $a$ (including 1) and $b$, but is not a group automorphism of $G$.

Since $mk, m \geq 4$, it is easy to see that for all $g \in G$, $g$ and $gab$ are the only common neighbours of $ga$ and $gb$ in $\Gamma_0$. Putting $g = 1$, we get $(ab)^\psi = ab$. Now putting $g = a$ we get $(a^2b)^\psi = a^2b$. By continuing this procedure we get $(a^i b)^\psi = a^i b$ for all $i$. Since $m \geq 4$, we have $b^2 \neq 1$. So, for all $i$ we have $(a^i b^2)^\psi = a^i b^2$. This implies that $(a^i b^3)^\psi = a^i b^3$ for all $i$. By continuing this procedure, we get $(a^i b^j)^\psi = a^i b^j$ for all $i, j$. This means that $\psi$ is the trivial automorphism of $G$. Hence $\sigma$ is an automorphism of group $G$, a contradiction.

(6) Let $L = \{b, b^{-1}\}$, where $\langle a \rangle \cap \langle b \rangle = 1$. We claim that $\Gamma$ is normal. If $\Gamma$ is intransitive, then by Lemma 3.9, $\Gamma$ is normal. Hence, we may assume that $\Gamma$ is transitive. Then, by [26, Theorem 1.1], $G = \langle x \rangle \times \langle y \rangle$, $o(x) = mk$, $o(y) = m$, for some $m, k \geq 1$, where $mk \geq 3$. Furthermore $a = x$ and $b = a^t y$ for some integer $t$ with $(t, mk) = 1$ and $t^2 \equiv 1 \pmod{k}$, or $(m, k, t) = (1, 10, 2)$, or $(t, mk) = 1$ and $t^2 \equiv -1 \pmod{k}$. Clearly $(m, k, t) = (1, 10, 2)$ is impossible, because $\langle a \rangle \cap \langle b \rangle = 1$. So we have $b^m = a^m \in \langle a \rangle \cap \langle b \rangle = 1$. Thus $o(b)$ divides $m$, and $k$ divides $t$. The latter implies that $k = t = 1$. Thus $b = ay$ and $o(b) = o(a) = m$.

Since $\Gamma$ is connected, [26, Proposition 5.1] implies that $\Gamma$ is not edge-transitive. So every automorphism of $\Gamma$ maps $G$-orbits to $G$-orbits. Suppose, towards a contradiction, that $\Gamma$ is not normal. Similar to the previous case, there exists a colour-preserving automorphism $\sigma$ of $\Gamma_0 = \text{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ which fixes 1 but is not a group automorphism of $G$ and we may assume that $a^\sigma = a$ and $b^\sigma = b^{-1}$. Then [15, Theorem 1.3(ii)] implies that 8 divides $|G| = m^2$. So 4 divides $m$. Since $\langle a \rangle \cap \langle b \rangle = 1$, $\psi : a^ib^j \mapsto a^ib^{-j}$ is a well-defined automorphism of $G$ that is also an automorphism of $\Gamma_0$. Again, by the same argument in the last paragraph of the proof of previous case, we get $\sigma$ is an automorphism of $G$ which is a contradiction. So we have proved that $\Gamma$ is normal.

As we saw above, if $\Gamma$ is transitive, then $o(a) = o(b)$. Conversely, suppose that $o(a) = o(b)$. Then $\sigma : a^ib^j \mapsto a^ib^j$ is a group automorphism of $G$ and $\langle R_G, \psi_\sigma \rangle$, where $\psi$ is defined by the rule $(g, 1)^\psi = (g^\sigma, 2), (g, 2)^\psi = (g^\sigma, 1)$ for all $g \in G$, is a transitive subgroup of $\text{Aut}(\Gamma)$. This completes the proof. 

Proof of Theorem 1.1. It is a direct consequence of Lemmas 3.2, 3.3, 3.5–3.8, and 3.11. \qed
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