Contributions to Discrete Mathematics

Volume 12, Number 1, Pages 20–27
ISSN 1715-0868

CONGRUENCES MODULO POWERS OF 2 AND 3 FOR A RESTRICTED BINARY PARTITION FUNCTION A LA ANDREWS AND LEWIS

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ABSTRACT. Let $W(n)$ denote the number of partitions of n into powers of 2 such that for all $i \geq 0$, 2^{2i} and 2^{2i+1} cannot both be parts of a particular partition. Recently, Lan and Sellers proved a number of congruences modulo 2, 3 and 4. In this note, we prove a number of Ramanujan-type congruences modulo powers of 2 and 3.

1. INTRODUCTION

In 2001, Andrews and Lewis [1] utilized an identity of F. H. Jackson to derive several generating functions for new partitions as well as identities involving some of the corresponding partition functions. At the end of their paper, they defined a family of functions $W_1(S_1, S_2; n)$ to be the number of partitions of n into parts from $S_1 \cup S_2$ that do not contain both a_i and b_i as parts (where $S_1 = \{a_1, a_2, a_3, \cdots\}$ and $S_2 = \{b_1, b_2, b_3, \cdots\}$ and $S_1 \cap S_2 = \emptyset$). In this context, let $S_1 = \{2^{2n-2}\}_{n\geq 1}$, $S_2 = \{2^{2n-1}\}_{n\geq 1}$ and $W(n) = W_1(S_1, S_2; n)$. Therefore, $W(n)$ counts the number of partitions of n into powers of 2 such that 1 and 2 cannot both be parts of a particular partition, 4 and 8 cannot both be parts of a particular partition, 16 and 32, and so on. For example, the number of such partitions of $n = 12$ is given by $W(12) = 9$ where the partitions in question are the following:

$$8+2+2, \quad 8+1+1+1+1, \quad 4+4+4, \quad 4+4+2+2,$$
$$4+4+1+1+1+1,\ 4+2+2+2+2,\ 4+1+1+1+1+1+1+1+1,$$
$$2+2+2+2+2+2, \quad 1+1+1+1+1+1+1+1+1+1+1+1.$$

Note that $W(n)$ is a restricted binary partition function. A number of congruences modulo powers of 2 for binary partition function have been discovered, see for example, Churchhouse [2], Gupta [3], Hirschhorn and Loxton [4], and Rodseth and Sellers [6]. Recently, Lan and Sellers [5] proved

Received by the editors (September 16, 2015), and in revised form (September 28, 2015).

2000 *Mathematics Subject Classification.* 05A17, 11P83.

Key words and phrases. restricted binary partition, congruence, generating function.

The authors would like to thank the referee for helpful suggestions. This work was supported by the National Science Foundation of China (11401260).

several elementary recurrence properties satisfied by $W(n)$ and a number of congruences modulo 2, 3, and 4 for $W(n)$. At the end of their paper, Lan and Sellers [5] stated that computational evidence indicates that additional Ramanujan-like congruences are satisfied by $W(n)$ modulo slightly larger moduli (including 8 and 9). In this note, we prove a number of Ramanujan-type congruences modulo powers of 2 and 3 for $W(n)$. The main results of this note can be stated as follows.

Theorem 1.1. *For $n \geq 0$,*

$$W(64n+10) \equiv W(64n+54) \equiv 0 \pmod{2^3}, \tag{1}$$

$$W(256n+i) \equiv 0 \pmod{2^4}, \tag{2}$$

$$W(1024n+j) \equiv 0 \pmod{2^5}, \tag{3}$$

$$W(1024n+k) \equiv 0 \pmod{2^6}, \tag{4}$$

$$W(4096n+s) \equiv 0 \pmod{2^7}, \tag{5}$$

$$W(16384n+t) \equiv 0 \pmod{2^8}, \tag{6}$$

$$W(32n+12) \equiv W(32n+20) \equiv 0 \pmod{3^2}, \tag{7}$$

$$W(512n+u) \equiv 0 \pmod{3^3}, \tag{8}$$

$$W(8192n+v) \equiv 0 \pmod{3^4}, \tag{9}$$

where

$$\begin{aligned}
i \in &\{18,\ 30,\ 74,\ 118,\ 138,\ 182,\ 226,\ 238\},\\
j \in &\{30,\ 90,\ 102,\ 137,\ 139,\ 150,\ 170,\ 181,\ 183,\ 238,\ 786,\ 841,\\
&\ 843,\ 854,\ 874,\ 885,\ 887,\ 922,\ 934,\ 994\},\\
k \in &\{138,\ 182,\ 842,\ 886\},\\
s \in &\{138,\ 238,\ 842,\ 994,\ 3102,\ 3254,\ 3858,\ 3958\},\\
t \in &\{842, 854, 922, 3102, 3858, 3958, 12426, 12526, 13282, 15462,\\
&\ 15530, 15542\},\\
u \in &\{76,\ 84,\ 108,\ 140,\ 244,\ 268,\ 372,\ 404,\ 428,\ 436\},\\
v \in &\{108, 244, 428, 436, 596, 756, 780, 940, 1676, 1780, 1908, 1964, 2124,\\
&\ 2452, 2636, 2668, 2700, 2932, 2964, 2996, 3148, 3180, 3212, 3444,\\
&\ 3476, 3508, 3692, 4020, 4172, 4500, 4684, 4716, 4748, 4980, 5012,\\
&\ 5044, 5196, 5228, 5260, 5492, 5524, 5556, 5740, 6068, 6228, 6284,\\
&\ 6412, 6516, 7252, 7412, 7436, 7596, 7756, 7764, 7948, 8084\}.
\end{aligned}$$

2. Proof of Theorem 1.1

We just prove

$$W(8192n+780) \equiv 0 \pmod{81}. \tag{10}$$

The rest can be proved similarly.

Lan and Seller [5] found the following generating function for $W(n)$:

$$H(q) := \sum_{n=0}^{\infty} W(n)q^n = \prod_{j=1}^{\infty} \frac{(1-q^{3\times 2^{2j-2}})}{1-q^{2^{j-1}}}. \tag{11}$$

Therefore,

$$H(q) = \sum_{n=0}^{\infty} W(n)q^n = \frac{(1-q^3)}{(1-q)(1-q^2)} H(q^4) = \frac{1+q+q^2}{(1-q^2)} H(q^4). \tag{12}$$

Note that

$$\begin{aligned} \{8192n+780\}_{n=0}^{\infty} &\subset \{4096n+780\}_{n=0}^{\infty} \\ &\subset \{2048n+780\}_{n=0}^{\infty} \subset \{1024n+780\}_{n=0}^{\infty} \\ &\subset \{512n+268\}_{n=0}^{\infty} \subset \{256n+12\}_{n=0}^{\infty} \\ &\subset \{128n+12\}_{n=0}^{\infty} \subset \{64n+12\}_{n=0}^{\infty} \\ &\subset \{32n+12\}_{n=0}^{\infty} \subset \{16n+12\}_{n=0}^{\infty} \subset \{8n+4\}_{n=0}^{\infty} \\ &\subset \{4n\}_{n=0}^{\infty} \subset \{2n\}_{n=0}^{\infty}. \end{aligned} \tag{13}$$

Thus, we are ready to find the generating functions for $W(2n)$, $W(4n)$, $W(8n+4)$, $W(16n+12)$, $W(32n+12)$, $W(64n+12)$, $W(128n+12)$, $W(256n+12)$, $W(512n+268)$, $W(1024n+780)$, $W(2048n+780)$, $W(4096n+780)$ and $W(8192n+780)$.

It follows from (12) that

$$\sum_{n=0}^{\infty} W(2n)q^n = \frac{1+q}{1-q} H(q^2) = \frac{1+2q+q^2}{1-q^2} H(q^2), \tag{14}$$

which yields

$$\sum_{n=0}^{\infty} W(4n)q^n = \frac{1+q}{1-q} H(q). \tag{15}$$

Combining (12) and (15), we get

$$\begin{aligned} \sum_{n=0}^{\infty} W(4n)q^n &= \frac{(1+q)(1+q+q^2)}{(1-q)(1-q^2)} H(q^4) \\ &= \frac{(1+q)^2(1+q+q^2)}{(1-q^2)^2} H(q^4) \\ &= \frac{1+3q+4q^2+3q^3+q^4}{(1-q^2)^2} H(q^4). \end{aligned} \tag{16}$$

Therefore,

$$\sum_{n=0}^{\infty} W(8n+4)q^n = \frac{3(1+q)}{(1-q)^2} H(q^2). \tag{17}$$

We can rewrite (17) as

$$\sum_{n=0}^{\infty} W(8n+4)q^n = \frac{3(1+q)^3}{(1-q^2)^2}H(q^2) = \frac{3(1+3q+3q^2+q^3)}{(1-q^2)^2}H(q^2). \tag{18}$$

The following identity follows from (18):

$$\sum_{n=0}^{\infty} W(16n+12)q^n = \frac{3(3+q)}{(1-q)^2}H(q). \tag{19}$$

By (12) and (19),

$$\begin{aligned} \sum_{n=0}^{\infty} W(16n+12)q^n &= \frac{3(3+q)(1+q+q^2)}{(1-q)^2(1-q^2)}H(q^4) \\ &= \frac{3(3+q)(1+q)^2(1+q+q^2)}{(1-q^2)^3}H(q^4) \\ &= \frac{3(3+10q+15q^2+13q^3+6q^4+q^5)}{(1-q^2)^3}H(q^4), \end{aligned} \tag{20}$$

which implies

$$\sum_{n=0}^{\infty} W(32n+12)q^n = \frac{9(1+5q+2q^2)}{(1-q)^3}H(q^2). \tag{21}$$

We can also rewrite (21) as

$$\begin{aligned} \sum_{n=0}^{\infty} W(32n+12)q^n &= \frac{9(1+5q+2q^2)(1+q)^3}{(1-q^2)^3}H(q^2) \\ &= \frac{9(1+8q+20q^2+22q^3+11q^4+2q^5)}{(1-q^2)^3}H(q^2), \end{aligned} \tag{22}$$

which yields

$$\sum_{n=0}^{\infty} W(64n+12)q^n = \frac{9(1+20q+11q^2)}{(1-q)^3}H(q). \tag{23}$$

Thanks to (12) and (23),

$$\begin{aligned} &\sum_{n=0}^{\infty} W(64n+12)q^n \\ &= \frac{9(1+20q+11q^2)(1+q+q^2)(1+q)^3}{(1-q^2)^4}H(q^4) \\ &= \frac{9(1+24q+98q^2+191q^3+221q^4+158q^5+64q^6+11q^7)}{(1-q^2)^4}H(q^4). \end{aligned} \tag{24}$$

It follows from (24) that

$$\sum_{n=0}^{\infty} W(128n+12)q^n = \frac{9(1+98q+221q^2+64q^3)}{(1-q)^4}H(q^2). \tag{25}$$

Thus,

(26)
$$\sum_{n=0}^{\infty} W(128n+12)q^n$$
$$= \frac{9(1+98q+221q^2+64q^3)(1+q)^4}{(1-q^2)^4} H(q^2)$$
$$= \frac{9(1+102q+619q^2+1540q^3+1975q^4+1366q^5+477q^6+64q^7)}{(1-q^2)^4} H(q^2),$$

which yields

$$\sum_{n=0}^{\infty} W(256n+12)q^n = \frac{9(1+619q+1975q^2+477q^3)}{(1-q)^4} H(q). \quad (27)$$

In view of (12) and (27),

$$\sum_{n=0}^{\infty} W(256n+12)q^n \quad (28)$$
$$= \frac{9(1+619q+1975q^2+477q^3)(1+q+q^2)(1+q)^4}{(1-q^2)^5} H(q^4)$$
$$= \frac{9P_1(q)}{(1-q^2)^5} H(q^4),$$

where

$$P_1(q) = 1+624q+5081q^2+17175q^3+32787q^4+39711q^5 + 31499q^6+15741q^7+4360q^8+477q^9.$$

Congruence (28) implies the following identity:

$$\sum_{n=0}^{\infty} W(512n+268)q^n \quad (29)$$
$$= \frac{27(208+5725q+13237q^2+5247q^3+159q^4)}{(1-q)^5} H(q^2).$$

Therefore,

$$\sum_{n=0}^{\infty} W(512n+268)q^n \quad (30)$$
$$= \frac{27(208+5725q+13237q^2+5247q^3+159q^4)(1+q)^5}{(1-q^2)^5} H(q^2)$$
$$= \frac{27P_2(q)}{(1-q^2)^5} H(q^2),$$

where

$$\begin{aligned}P_2(q) = {} & 208 + 6765q + 43942q^2 + 130762q^3 + 217054q^4 \\ & + 214468q^5 + 125970q^6 + 41062q^7 + 6042q^8 + 159q^9.\end{aligned}$$

It follows from (30) that

$$\begin{aligned}&\sum_{n=0}^{\infty} W(1024n+780)q^n \\ &= \frac{27(6765 + 130762q + 214468q^2 + 41062q^3 + 159q^4)}{(1-q)^5} H(q).\end{aligned} \tag{31}$$

We can rewrite (31) as follows

$$\sum_{n=0}^{\infty} W(1024n+780)q^n = \frac{27P_3(q)}{(1-q^2)^6} H(q^4), \tag{32}$$

where

$$\begin{aligned}P_3(q) &= (6765 + 130762q + 214468q^2 \\ &\quad + 41062q^3 + 159q^4)(1+q+q^2)(1+q)^5 \\ &= 6765 + 171352q + 1107280q^2 + 3589187q^3 + 7116194q^4 \\ &\quad + 9396936q^5 + 8523576q^6 + 5253350q^7 + 2078537q^8 \\ &\quad + 463384q^9 + 42016q^{10} + 159q^{11}.\end{aligned}$$

By (32),

$$\sum_{n=0}^{\infty} W(2048n+780)q^n = \frac{27P_4(q)}{(1-q)^6} H(q^2) = \frac{27P_4(q)(1+q)^6}{(1-q^2)^6} H(q^2), \tag{33}$$

where

$$P_4(q) = 6765 + 1107280q + 7116194q^2 + 8523576q^3 + 2078537q^4 + 42016q^5.$$

It is a routine to verify that

$$\begin{aligned}P_4(q)(1+q)^6 = {} & 6765 + 1147870q + 13861349q^2 \\ & + 67965240q^3 + 182209978q^4 + 299340548q^5 \\ & + 315295026q^6 + 213859064q^7 + 90276025q^8 \\ & + 21625038q^9 + 2330633q^{10} + 42016q^{11}.\end{aligned} \tag{34}$$

It follows from (33) and (34) that

$$\sum_{n=0}^{\infty} W(4096n+780)q^n = \frac{27P_5(q)}{(1-q)^6} H(q). \tag{35}$$

where

$$P_5(q) = 6765 + 13861349q + 182209978q^2 + 315295026q^3 + 90276025q^4 + 2330633q^5.$$

Combining (12) and (35),

$$\sum_{n=0}^{\infty} W(4096n+780)q^n = \frac{27P_5(q)(1+q)^6(1+q+q^2)}{(1-q^2)^7} H(q^4). \tag{36}$$

It is easy to check that

$$\begin{aligned} &P_5(q)(1+q)^6(1+q+q^2) \\ &= 6765 + 13908704q + 279388251q^2 + 1895991915q^3 + 6874614282q^4 \\ &\quad + 15734707293q^5 + 24608446086q^6 + 27292948382q^7 \\ &\quad + 21642108993q^8 + 12043670442q^9 + 4470903663q^{10} \\ &\quad + 998501127q^{11} + 106590456q^{12} + 2330633q^{13}. \end{aligned} \tag{37}$$

Thus,

$$\sum_{n=0}^{\infty} W(8192n+780)q^n = \frac{81P_6(q)}{(1-q)^7} H(q^2), \tag{38}$$

where

$$P_6(q) = 2255 + 93129417q + 2291538094q^2 + 8202815362q^3 + 7214036331q^4 + 1490301221q^5 + 35530152q^6. \tag{39}$$

Congruence (10) follows from (38). □

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