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ON UNIVERSALLY RIGID FRAMEWORKS ON THE LINE

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Abstract. A d-dimensional bar-and-joint framework (G,p) with underlying graph G is called universally rigid if all realizations of G with the same edge lengths, in all dimensions, are congruent to (G,p). We give a complete characterization of universally rigid one-dimensional bar-and-joint frameworks in general position with a complete bipartite underlying graph. We show that the only bipartite graph for which all generic d-dimensional realizations are universally rigid is the complete graph on two vertices, for all $d \geq 1$. We also discuss several open questions concerning generically universally rigid graphs and the universal rigidity of general frameworks on the line.

1. Introduction

A d-dimensional (bar-and-joint) *framework* is a pair (G,p), where $G = (V,E)$ is a graph and p is a *configuration* of the vertices, that is, a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two frameworks (G,p) and (G,q) are *equivalent* if $\|p(u)-p(v)\| = \|q(u)-q(v)\|$ holds for all pairs u,v with $uv \in E$, where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G,p), (G,q) are *congruent* if $\|p(u)-p(v)\| = \|q(u)-q(v)\|$ holds for all pairs u,v with $u,v \in V$. This is the same as saying that (G,q) can be obtained from (G,p) by an isometry of $\mathbb{R}^d$.

Let (G,p) be a d-dimensional framework for some $d \geq 1$. We say that (G,p) is *rigid* in $\mathbb{R}^d$ if there is a neighborhood U_p in the space of configurations in $\mathbb{R}^d$ such that if a d-dimensional framework (G,q) is equivalent to (G,p) and $q \in U_p$, then q is congruent to p. The framework (G,p) is called *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework (G,q) which is equivalent to (G,p) is congruent to (G,p). We obtain an even stronger property by extending this condition to equivalent realizations in any dimension: we say that (G,p) is *universally rigid* if it is a unique realization of G, up to congruence, with the given edge lengths, in all dimensions $\mathbb{R}^{d'}$, $d' \geq 1$.

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It seems to be a hard problem to decide if a given framework is rigid, globally rigid, or universally rigid. Indeed, Abbott [1] verified that recognizing rigid frameworks in the plane is NP-hard and Saxe [17] proved that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. The complexity of the corresponding decision problem for universal rigidity seems to be open, even for $d = 1$.

These problems become more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework. A framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. It is well-known [6] that rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. Global rigidity is also a generic property in $\mathbb{R}^d$, for all d [9, 13]. This property does not hold for universal rigidity, even if $d = 1$, which follows by considering different generic realizations of a four-cycle on the line.

A graph G is called *generically rigid* (resp. *generically globally rigid*, *generically universally rigid*) in $\mathbb{R}^d$ if every d-dimensional generic framework (G, p) is rigid (resp. globally rigid, universally rigid). We shall also use the shorter versions d-GR, d-GGR, and d-GUR, respectively, for these families of graphs. d-GR and d-GGR graphs are well-characterized for $d \leq 2$. It remains an open problem to extend these results to higher dimensions or to characterize d-GUR graphs for any $d \geq 1$. We refer the reader to [18] for more details on the theory of rigid graphs and frameworks.

Let (G, p) be a framework in $\mathbb{R}^d$ with $G = (V, E)$. An *equilibrium stress* (or *stress*, for short) on (G, p) is an assignment of scalars ω_{ij} to the edges $v_i v_j$ such that for each $v_i \in V$ we have

$$\sum_{j | v_i v_j \in E} \omega_{ij}(p(v_i) - p(v_j)) = 0.$$

Given a stress, there is an associated $|V| \times |V|$ symmetric matrix Ω, the *stress matrix* such that for $i \neq j$, the i, j entry of Ω is $-\omega_{ij}$, and the diagonal entries for i, i are $\sum_{j \neq i} \omega_{ij}$. Here we follow the convention that an equilibrium stress can be extended to non-adjacent pairs v_i, v_j by putting $w_{ij} = 0$. Note that all row and column sums are now zero. It is easy to see that the rank of Ω is at most $|V| - d - 1$. We say that Ω is of *full rank* if its rank is equal to $|V| - d - 1$.

Connelly [11] and Gortler and Thurston [14] show that a generic framework (G, p) in $\mathbb{R}^d$ on at least $d+2$ vertices is universally rigid if and only if it has a positive semi-definite (PSD) stress matrix of full rank. The 'if' direction also holds for frameworks in general position by a theorem of Alfakih and Ye [5].

2. Complete bipartite graphs

In this section, we give a complete characterization of the universally rigid one-dimensional realizations of complete bipartite graphs. As a corollary we shall deduce that no bipartite graph, other than $K_{1,1}$, is 1-GUR (or d-GUR, for any $d \geq 1$).

We will need the following result due to Alfakih [3, Theorem 6].

Theorem 2.1. [3] *Let (G, p) be a framework on n vertices, where G is not a complete graph. Then the only PSD stress matrix Ω of (G, p) is the zero matrix if and only if (G, p) has an equivalent realization in $\mathbb{R}^{n-1}$, in which the vertices are located at affinely independent points.*

Let (G, p) be a framework on the line with $G = (V, E)$. A pair of vertices $\{u, v\}$, $u, v \in V$ is called *universally linked* in (G, p) if $\|q(u) - q(v)\| = \|p(u) - p(v)\|$ holds for all frameworks (G, q) which are equivalent to (G, p) (in all dimensions). Let C be a cycle of G passing through $v_1, \ldots, v_k$ with $E(C) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$. If $p(v_1) < p(v_2) < \cdots < p(v_k)$ then C is called a *stretched cycle* in (G, p). If C is a stretched cycle in (G, p) then it is not difficult to see that every pair of vertices of C is universally linked in (G, p).

Theorem 2.2. *Let G be a complete bipartite graph on at least three vertices and let (G, p) be a realization of G on the line in which the vertices are located at pairwise distinct points. Then (G, p) is universally rigid if and only if it contains a stretched cycle.*

Proof. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be the vertex sets of the two colour classes of G. For simplicity we shall also use $x_1, \ldots, x_m, y_1, \ldots, y_n$ to denote the coordinates of the corresponding vertices in (G, p).

First suppose that (G, p) contains a stretched cycle. Without loss of generality we may assume that x_1, y_1, x_2, y_2 form a stretched cycle. Then $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are universally linked pairs in (G, p). This implies that the pairwise distances among these four vertices are the same in all realizations of G equivalent to (G, p) and hence (G, p) is universally rigid if and only if (G', p) is universally rigid, where $G' = G + x_1x_2 + y_1y_2$. It remains to observe that (G', p) can be obtained from a framework on a complete graph on four vertices by iteratively attaching vertices of degree two (and adding edges). These operations are known to preserve universal rigidity on the line. Therefore (G', p) and hence (G, p) are universally rigid, as required.

Next consider the case where (G, p) contains no stretched cycle. Observe that there is no streched cycle if and only if the vertices of at least one of the two colour classes are consecutive on the line. Thus we may assume that

$$x_1 < \cdots < x_k < y_1 < \cdots < y_n < x_{k+1} < \cdots < x_m \quad \text{for some } 1 \leq k \leq m$$

(note that when $k = m$ we have $x_1 < \cdots < x_m < y_1 < \cdots < y_n$). If $n = 1$, that is, when colour class Y consists of a single vertex y_1, the framework is not even globally rigid on the line. This follows by observing that y_1 is a cut-vertex of G and hence we may obtain an equivalent but not congruent realization by reflecting any vertex x_i to the other side of y_1. Hence we may assume that $n \geq 2$.

Consider a stress ω on (G,p) for which the corresponding stress matrix Ω is PSD. We shall prove that Ω is the zero matrix. This fact and Theorem 2.1 will imply that (G,p) has a higher dimensional equivalent realization and hence it is not universally rigid.

Let ω_{ij} denote the stress on the edge x_iy_j for $1 \leq i \leq m, 1 \leq j \leq n$. Let

$$A_i = \begin{cases} (y_1 - x_i)\sum_{j=1}^n \omega_{ij} & \text{for } i \leq k, \\ (x_i - y_1)\sum_{j=1}^n \omega_{ij} & \text{for } i > k. \end{cases}$$

Since $\sum_{j=1}^n (x_i - y_j)\omega_{ij} = 0$ for each $i = 1, \ldots, m$ by the equilibrium condition at x_i, we have

$$A_i = \begin{cases} \sum_{j=1}^n (y_1 - y_j)\omega_{ij} & \text{for } i \leq k, \\ \sum_{j=1}^n (y_j - y_1)\omega_{ij} & \text{for } i > k. \end{cases}$$

Let $B_j = (x_{k+1} - y_j)\sum_{i=1}^m \omega_{ij}$ for $j = 1, \ldots, n$. Using the equilibrium condition $\sum_{i=1}^m (y_j - x_i)\omega_{ij} = 0$ at y_j we have

$$B_j = \sum_{i=1}^m (x_{k+1} - x_i)\omega_{ij}.$$

Since $\sum_{j=1}^n \omega_{ij}$, for $i = 1, \ldots, m$, and $\sum_{i=1}^m \omega_{ij}$, for $j = 1, \ldots, n$, are diagonal entries of the PSD matrix Ω, we have $\sum_{j=1}^n \omega_{ij} \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m \omega_{ij} \geq 0$ for $j = 1, \ldots, n$. Hence, by definition, $A_i, B_j \geq 0$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Suppose that $k < m$. Then we obtain

$$\begin{aligned} 0 &\leq \sum_{i \leq k}(x_{k+1} - x_i)A_i + \sum_{i > k}(x_i - x_{k+1})A_i \\ &= \sum_{i \leq k}(x_{k+1} - x_i)\sum_j (y_1 - y_j)\omega_{ij} + \sum_{i>k}(x_i - x_{k+1})\sum_j (y_j - y_1)\omega_{ij} \\ &= \sum_j (y_1 - y_j)\Big(\sum_{i \leq k}(x_{k+1} - x_i)\omega_{ij} + \sum_{i > k}(x_{k+1} - x_i)\omega_{ij}\Big) \\ &= \sum_j (y_1 - y_j)B_j \\ &\leq 0. \end{aligned}$$

Hence we must have $A_i = 0$ for all $i \neq k+1$ and $B_j = 0$ for all $j \neq 1$. By applying the same argument to the mirror image of the framework and using that $n \geq 2$ we can conclude that in fact $A_i = 0$ for all $i = 1, \ldots, m$ and $B_j = 0$ for all $j = 1, \ldots, n$. Therefore, all the diagonal entries of the PSD matrix Ω are zeros, which implies that Ω is the zero matrix, as claimed.

It remains to consider the case when $k = m$. This case is even simpler as it suffices to consider the A_i's for which we can deduce that

$$0 \leq \sum_{i=1}^{m} A_i = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_1 - y_j)\omega_{ij} = \sum_{j=1}^{n}(y_1 - y_j)(\sum_{i=1}^{m}\omega_{ij}) \leq 0.$$

Thus $A_i = 0$ must hold for all $1 \leq i \leq m$. Therefore $\sum_{j=1}^{n}\omega_{ij} = 0$ for all $1 \leq i \leq m$. Moreover, $\sum_{j=1}^{n}\sum_{i=1}^{m}\omega_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n}\omega_{ij} = 0$ holds. Hence $\sum_{i=1}^{m}\omega_{ij} = 0$ for all $1 \leq j \leq n$. Therefore Ω is the zero matrix, as required. The theorem follows. □

Theorem 2.2 implies the following observation of Connelly[1].

Corollary 2.3. [10] *The only generically universally rigid bipartite graph in $\mathbb{R}^1$ is the single edge $K_{1,1}$.*

In fact, this corollary extends to all dimensions. To see this first observe that no general position d-dimensional realization of a non-complete graph on at most $d+1$ vertices is rigid. Thus the only d-GR (or d-GUR) bipartite graph on at most $d+1$ vertices is $K_{1,1}$. Next consider a complete bipartite graph G on at least $d+2$ vertices and a d-dimensional generic realization (G,p) with the property that the projected one-dimensional framework (G,p'), obtained by projecting the configuration p to one coordinate axis, contains no streched cycles. A PSD stress matrix of (G,p) is also a PSD stress matrix of (G,p') and hence, by the proof of Theorem 2.2, it must be the zero matrix. Thus, by Theorem 2.1, (G,p) is not universally rigid. Therefore we have the following stronger result.

Theorem 2.4. *The only generically universally rigid bipartite graph in $\mathbb{R}^d$ is the single edge $K_{1,1}$, for every dimension $d \geq 1$.*

We close this section with two questions. Motivated by Theorem 2.2 one may ask:

Question 2.5. *Is it true that the universal rigidity of a general position framework (G,p) in $\mathbb{R}^1$ depends only on the ordering of vertices on the line (and not on the coordinates)?*

Theorem 2.1, which was used in the proof of Theorem 2.2, might be extended in the following sense, at least for generic frameworks.

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[1]Connelly's argument is as follows: map the bipartite graph $G = K_{m,n}$ onto the unit interval on the line. This framework has a realization as a subframework of a unit-length simplex (S,p) in $\mathbb{R}^d$, where $d = m+n-1$. Then perturb the realization on the line to a generic one and follow it with a modified realization of the simplex in $\mathbb{R}^d$. The inverse function theorem can be used to verify the construction. (In detail, consider the rigidity map f_G on the d-dimensional realizations of G which assigns the edge lengths to the realizations. Since the simplex is minimally infinitesimally rigid in $\mathbb{R}^d$, p is a regular point of f_S. By the inverse function theorem, we can choose an open neibourhood U_p of p and an open neighbourhood W of $f_S(p)$ such that f_S maps U_p diffeomorphically onto W. Thus there is a realization of S for which the edge lengths of the complete bipartite subframework are consistent with the edge lengths of the perturbed one-dimensional framework.)

Question 2.6. *Let G be a graph on n vertices and let $k \le n-2$. Is it true that a framework (G,p) in generic position has a PSD stress matrix Ω of rank at least k if and only if (G,p) has no equivalent realization (G,q) in $\mathbb{R}^{n-i}$ which contains $n-i+1$ affinely independent points, for all $1 \le i \le k$?*

The answer to the "only if" direction is affirmative even for frameworks in general position. This can be verified as follows. Suppose that Ω is a PSD stress matrix of (G,p) of rank at least k and let (G,q) be a framework equivalent to (G,p) in $\mathbb{R}^d$. By another result of Alfakih [3, Theorem 5] Ω is a stress matrix for (G,q) as well. Thus $\operatorname{rank}\Omega \le |V(G)| - d - 1$, which implies that $d \le |V(G)| - k - 1$.

The answer to the "if" direction is negative if we replace generic position by general position. A counterexample, due to Connelly and Whiteley, can be found e.g. in [4, Example 2.1].

3. Generic universal rigidity on the line

In this section we consider generic frameworks in $\mathbb{R}^1$ and list a few questions and observations concerning the family of 1-GUR graphs. As we noted earlier, the complexity of recognizing these graphs is still an open question.

First we recall a conjectured inductive construction of 1-GUR graphs.

Conjecture 3.1. [10] *A graph G on at least three vertices is 1-GUR if and only if G can be obtained from K_3 by the following operations:*
(i) add an edge,
(ii) choose two graphs G_1, G_2 built by these operations, choose two sets $U_1 \subseteq V(G_1)$, $U_2 \subseteq V(G_2)$ with $|U_1| = |U_2| \ge 2$, delete all edges joining vertices of U_1 in G_1, then glue the two graphs together along the vertices in U_1 and U_2.

The "if" direction of Conjecture 3.1 follows from a recent result of Ratmanski [16]. Note that the graphs built up from a triangle by operations (i) and (ii) must contain a triangle. Thus finding triangle-free 1-GUR graphs would be interesting, c.f. Section 4. Furthermore, Conjecture 3.1, if true, does not seem to provide a good characterization of 1-GUR graphs since it is not clear how to test whether G can be constructed from a triangle by the above operations.

This leads us to *minimally 1-GUR* graphs, for which the deletion of any edge makes them not 1-GUR. These graphs may be sparse and may have small vertex separations, along which they may be decomposable by the inverse operation of glueing (as in Conjecture 3.1(ii) above).

Question 3.2. *Let $G = (V,E)$ be a minimally 1-GUR graph. Is there an upper bound on $|E|$ as a linear function of $|V|$?*

We remark here that there is no constant k for which the k-vertex-connectivity of G would imply that G is 1-GUR, and there exist dense not 1-GUR graphs, for example, the complete bipartite graphs (c.f. Corollary 2.3). However, the end of the proof of Theorem 2.2 shows that by adding an edge to a complete bipartite graph we obtain a 1-GUR graph which contains a sparse 1-GUR spanning subgraph.

Let $G = (V, E)$ be a graph. A pair (G_1, G_2), where G_1, G_2 are subgraphs of G, is called a *k-separation* of G if $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, and $|V(G_1) \cap V(G_2)| = k$ hold. For a subset $X \subseteq V$ let $G + K(X)$ denote the supergraph of G obtained by adding all edges connecting pairs of vertices of X (which are non-adjacent in G). The first observation about separations is as follows.

Lemma 3.3. *Let G be a 1-GUR graph and let (G_1, G_2) be a k-separation of G with $X = V(G_1) \cap V(G_2)$. Then $G_i + K(X)$ is 1-GUR for $i = 1, 2$.*

Proof. Suppose that $\overline{G}_1 = G_1 + K(X)$ is not 1-GUR. Then there exists a generic realization $(\overline{G}_1, p_1)$ of $\overline{G}_1$ in $\mathbb{R}^1$ which is not UR and hence there exists a realization $(\overline{G}_1, p_1')$ equivalent but not congruent to $(\overline{G}_1, p_1)$. We can assume that $p_1'(v) = p_1(v)$ for every v in X. Extend p_1 to a generic realization p of G in $\mathbb{R}^1$. Let

$$p'(v) = \begin{cases} p_1'(v), & v \in V(G_1) \\ p(v), & v \in V(G_2) \end{cases}$$

Then (G, p') is equivalent but not congruent to (G, p), which means that G is not 1-GUR, a contradiction. □

Lemma 3.3 implies that we can cut a 1-GUR graph along a separating vertex pair u, v into two smaller 1-GUR graphs if we add the edge uv to both pieces[2]. What if we are not allowed to add the edge? In this context the following statement may help.

A pair of vertices $\{u, v\}$ in graph G is called *universally linked* in $\mathbb{R}^d$ if $\{u, v\}$ is universally linked in all d-dimensional generic realizations of G.

Conjecture 3.4. *Suppose that $\{u, v\}$ is not universally linked in G in $\mathbb{R}^1$, for some pair $u, v \in V$. Then there exist generic 1-dimensional realizations $(G, p), (G, q)$ of G with the property that there exist a realization (G, p') equivalent to (G, p) and a realization (G, q') equivalent to (G, q), such that $\|p'(u) - p'(v)\| > \|p(u) - p(v)\|$ and $\|q'(u) - q'(v)\| < \|q(u) - q(v)\|$.*

The truth of this conjecture may imply:

Conjecture 3.5. *Let G be a 1-GUR graph and let (G_1, G_2) be a 2-separation in G with $V(G_1) \cap V(G_2) = \{x, y\}$. Then G_1 or G_2 is 1-GUR.*

By assuming the truth of Conjecture 3.4 we can show that if G_1 and G_2 are not 1-GUR then there exists a general position realization (G, p) on the line which is not universally rigid. We believe that p can be made generic, too (c.f. Question 2.5), which would imply Conjecture 3.5.

Our argument is as follows. We may assume that $\{x, y\}$ is not universally linked in G_1 and G_2. Thus there is a generic realization (G_1, p) in $\mathbb{R}^1$ and an

[2]It is easy to see that every 1-GUR graph (in fact, every 1-GGR graph) is 2-connected. Thus we may begin the study of small separators with the 2-separations.

equivalent realization (G_1, q) such that the distance between $p(x)$ and $p(y)$ is, say, stricly smaller than the distance between $q(x)$ and $q(y)$. By assuming the truth of Conjecture 3.4 we can find a generic realization (G_2, p') in $\mathbb{R}^1$ and an equivalent realization (G_2, q') such that the distance between $p'(x)$ and $p'(y)$ is, say, stricly smaller than the distance between $q'(x)$ and $q'(y)$. By carefully choosing the generic realization (G_2, p') and rescaling, if necessary, we may assume that $\|p(x) - p(y)\| = \|p'(x) - p'(y)\|$. Now we can use a result of Bezdek and Connelly[3] to obtain a pair of realizations (G_1, r) and (G_2, r') for which $\|r(x) - r(y)\| = \|r'(x) - r'(y)\| > \|p(x) - p(y)\|$ and such that (G_1, r) is equivalent to (G_1, p) and (G_2, r') is equivalent to (G_2, p'). By glueing together (G_1, p) and (G_2, p') as well as (G_1, r) and (G_2, r') along the pair x, y we obtain two equivalent but not congruent realizations of G, where the former realization is one-dimensional.

Conjecture 3.5 would imply Conjecture 3.1 by induction in the case when there is a 2-separation.

We close this section with the following question.

Question 3.6. *Let $G = (V, E)$ be 1-GUR. Does this imply that*
(a) $|E| \geq 2|V| - 3$ holds?
(b) G is 2-GR?

Note that the truth of Conjecture 3.1 would imply an affirmative answer to (b), and hence also to (a), since both operations preserve generic rigidity in $\mathbb{R}^2$.

4. Cover graphs and universal rigidity

Since it is probably difficult to characterize 1-GUR graphs, special families of 1-GUR (or not 1-GUR) graphs may be of interest. In this context we offer the study of the following family of graphs as a candidate for being not 1-GUR.

Let $G = (V, E)$ be a graph and let $\vec{G}$ be an acyclic orientation of G. An edge e of G is *dependent* if the reversal of e in $\vec{G}$ creates a directed cycle. An acyclic orientation without dependent edges is called *strongly acyclic.* We say that G is a *cover graph* if G has a strongly acyclic orientation. (It is known that G is a cover graph if and only if it is the Hasse diagram of some partially ordered set on V.) Note that all bipartite graphs are cover graphs: orient all edges from one colour class to the other. Also note that cover graphs are triangle-free. We should also remark that it is NP-hard to test whether a given graph is a cover graph [8, 15].

Question 4.1. *Is it true that no cover graph is 1-GUR (except $K_{1,1}$)?*

[3]Bezdek and Connelly [7] proved that if $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ are two configurations in $\mathbb{R}^d$ then there is a continuous motion $p(t)$ in $\mathbb{R}^{2d}$, that is analytic in t, such that $p(0) = p$, $p(1) = q$ and for $0 \leq t \leq 1$, $\|p_i(t) - p_j(t)\|$ is monotone for all $1 \leq i < j \leq n$.

It is also known that triangle-free planar graphs (and more generally, triangle-free 3-colorable graphs) are cover graphs [12]. (Recall that by a theorem of Grötzsch, every triangle-free planar graph is 3-colorable.) These special cases would also be interesting:

Question 4.2. *Is it true that no triangle-free planar graph (or even triangle-free* 3*-colorable graph) is* 1*-GUR (except* $K_{1,1}$*)?*

We may also ask whether all non-cover graphs are 1-GUR. An interesting graph to analyse is the Grötzsch graph, which is triangle-free and 4-chromatic, see Figure 1. This graph is not a cover graph [12]. Is it 1-GUR? Since it is triangle-free, an affirmative answer to this question would disprove Conjecture 3.1.

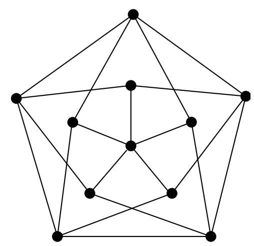

FIGURE 1. The Grötzsch graph.

5. Further observations on cover graphs

This section contains some further questions and observations about cover graphs, loosely related to (universal) rigidity of graphs. Let $G = (V, E)$ be a graph. We say that G is $(2,4)$*-sparse* (resp. $(2,3)$*-sparse*) if for all subsets $X \subseteq V$ with $|X| \geq 3$ (resp. $|X| \geq 2$) the subgraph induced by X has at most $2|X| - 4$ (resp. $2|X| - 3$) edges. It follows that every $(2,4)$-sparse graph is triangle-free. It can also be seen that triangle-free planar graphs are $(2,4)$-sparse and minimally 2-GR graphs are $(2,3)$-sparse. Perhaps the following families also consist of cover graphs.

Question 5.1. *Is every* $(2,4)$*-sparse graph a cover graph?*

Question 5.2. *Is every triangle-free* $(2,3)$*-sparse graph a cover graph?*

A well-known inductive construction of $(2,3)$-sparse graphs is based on the following operations. The 0*-extension* operation adds a new vertex v to a graph G and two new edges vx, vy connecting v to different vertices of G. The 1*-extension* operation deletes an edge xy of G, adds a new vertex v, and three new edges vx, vy, vz, for some vertex $z \neq x, y$. These operations are also called *Henneberg operations*. The next lemmas show that an affirmative answer to the above questions may be obtained by finding a similar inductive construction for triangle-free $(2,3)$-sparse graphs.

Lemma 5.3. *Let G be a triangle-free graph obtained from a graph H by a 0-extension operation. Then G is a cover graph if and only if H is a cover graph.*

Proof. Since H is a subgraph of G, necessity is obvious. To see the other direction consider a strongly acyclic orientation $\vec{H}$ of H. Suppose that $G = H + vx + vy$. Since $\vec{H}$ is acyclic, we cannot have an (x, y)-directed path and a (y, x)-directed path in $\vec{H}$ simultaneously. Thus, by relabelling x and y, if necessary, we have two cases to consider: either there is an (x, y)-directed path in $\vec{H}$ (which must have at least two edges, since G is triangle-free), or there is neither an (x, y)-directed path nor a (y, x)-directed path in $\vec{H}$.

In the former case we extend $\vec{H}$ to an orientation of G by orienting vx from x to v and vy from v to y. In the latter case we orient vx from v to x and vy from v to y. In both cases it is easy to check that the extended orientation gives rise to a strongly acyclic orientation of G. □

We can use similar arguments to deal with the 1-extension operation.

Lemma 5.4. *Let G be a triangle-free graph obtained from a cover graph H by a 1-extension operation. Then G is also a cover graph.*

Proof. Consider a strongly acyclic orientation $\vec{H}$ of H. Suppose that $G = H - xy + vx + vy + vz$. We may assume, without loss of generality, that xy is oriented from x to y in $\vec{H}$. Note that this implies that every path from y to x has at least two backward edges.

As above, we extend $\vec{H}$ to an orientation of G by considering several cases. In all cases it will be easy to check that the extended orientation is a strongly acyclic orientation of G. We omit the simple arguments but note that it is useful to observe that if there is an (a, b)-directed path in $\vec{H}$ then every (b, a)-path has at least two backward edges, and that each path connecting two neighbours of v must have at least two edges since G is triangle-free.

Case 1: There is an (x, z)-directed path P in $\vec{H} - xy$.

Then P has at least two edges and there is no (z, y)-directed path in $\vec{H} - xy$. If there is a (y, z)-directed path in $\vec{H} - xy$ (*Case 1.1*), which must have at least two edges, then we orient vx from x to v, vy from y to v and vz from v to z. If there is no (y, z)-directed path in $\vec{H} - xy$ (*Case 1.2*) then we orient vx from x to v, vy from v to y and vz from v to z. See Figure 2.

Case 2: There is a (z, x)-directed path Q in $\vec{H} - xy$.

Then Q has at least two edges and every (y, z)-path in $\vec{H} - xy$ has at least two backward edges. We orient vx from v to x, vy from v to y and vz from z to v. See Figure 3.

Case 3: There is neither an (x, z)-directed path nor a (z, x)-directed path in $\vec{H} - xy$.

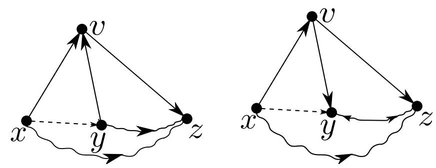

FIGURE 2. Cases 1.1 and 1.2

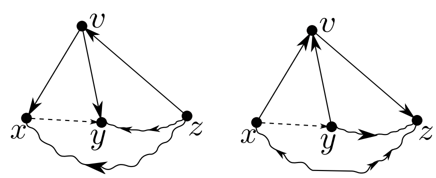

FIGURE 3. Case 2 and Case 3.1

If there is a (y, z)-directed path in $\vec{H} - xy$ (*Case 3.1*) then every (x, z)-path in $\vec{H} - xy$ has at least two forward edges. Then we orient vx from x to v, vy from y to v and vz from v to z. See Figure 3. If there is a (z, y)-directed path in $\vec{H} - xy$ (*Case 3.2*) then we orient vx from x to v, vy from v to y and vz from z to v.

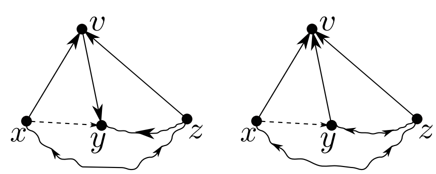

FIGURE 4. Cases 3.2 and 3.3

Finally, if there is neither a (y, z)-directed path nor a (z, y)-directed path in $\vec{H} - xy$ (*Case 3.3*) then we orient vx from x to v, vy from y to v and vz from z to v. See Figure 4. $\square$

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