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CONTRACTIONS OF POLYGONS IN ABSTRACT POLYTOPES, PART I

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Abstract. There are several well-known constructions of new polytopes from old, such as the pyramid construction. We define a new local construction on abstract polytopes, called digonal contraction, which allows digonal sections to be removed by merging their two edges into a single edge. This contraction also extends to the question of contracting an edge of a polytope by merging its two vertices. Digonal contraction cannot be applied arbitrarily; we present necessary and sufficient conditions for its use. Digonal contraction also has interesting interactions with another new construction, the *k-bubble* construction introduced by Helfand, which we describe.

1. Introduction

An abstract polytope is a combinatorial object generalizing the face lattice of a convex polytope [3, Section 2A]. Specifically, an abstract polytope is a poset. The elements of the set represent faces of the polytope and one face being less than another in the poset represents two faces being incident, such as a vertex contained in an edge. In order for a poset to be an abstract polytope, it must satisfy four conditions - some are very straightforward, such as the requirement that there is exactly one least face and one greatest face, while others are more complicated, such as the *strong connectivity* condition. As such, testing whether an arbitrary poset is an abstract polytope is not simple.

A construction on abstract polytopes is a way to perturb an abstract polytope, whether locally or globally, in order to get another abstract polytope. A well-known construction is the pyramid construction: just as we can take the pyramid over a convex polytope to get a new convex polytope of one higher dimension [1, Section 4.2], we can generalize this construction to apply to abstract polytopes; performing this new process on an abstract polytope always produces an abstract polytope of one higher rank (dimension).

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Helfand introduced a new global construction on abstract polytopes called the *k-bubble* construction [2, Section 2.2], or in this paper, the global Helfand construction (we will define a local version later). This construction is a generalization of the operation of *truncating* a convex polytope. For example, if we truncate a cube, we imagine slicing the cube near each of its vertices, so that there is now a new triangle for every vertex; the resulting convex polytope is the well-known *truncated cube.* This new polytope is not regular like the original polytope was, but is still highly symmetric. The global Helfand construction allows us to apply a similar operation, in various ways, to an abstract polytope, and always produces a new abstract polytope of the same rank.

Digonal contraction is a local construction focused on polygons in an abstract polytope. A polygon is any two-dimensional section of the face poset of a polytope: for example, the cube clearly has square faces, but it also has triangular sections (its vertex figures). A digon is an abstract polygon with just two vertices and two edges, and the presence of digonal sections in an abstract polytope is sometimes viewed as a degenerate condition. Digonal contraction is the operation of taking a digonal section, removing the greatest face of that section altogether, and merging the two edges into a single edge.

Digonal contraction cannot be applied to arbitrary polygons in arbitrary polytopes; it has various conditions that must be satisfied in order for the contraction to produce an abstract polytope, as otherwise the contraction will just produce a poset which does not qualify as an abstract polytope for one reason or another. However, there are specific cases in which digonal contraction is very easy to apply, i.e., the conditions required for it to produce abstract polytopes become much simpler. For example, after applying the Helfand construction to a polytope (or locally to part of a polytope), we find that certain digons introduced by the Helfand construction can be easily contracted.

2. Basic Notions

In this section, we will introduce some well-known concepts relating to abstract polytopes. Important, well-established results will be presented without proof. The reader is referred to Grünbaum [1] for convex polytopes and to McMullen and Schulte [3] for abstract polytopes.

Definition 2.1. *An* abstract polytope *is a poset P with elements called faces such that it has the following properties:*

- (P1) *P has a least face (a face F_{-1} such that for all $F \in P, F_{-1} \leq F$) and greatest face (a face F_n such that for all $F \in P, F \leq F_n$);*
- (P2) *All flags have the same number of faces, i.e., P is ranked;*
- (P3) *P is strongly connected;*
- (P4) *For every pair of faces $F \leq G$ with $rk(G) = rk(F)+2$, the section G/F has exactly two proper faces; this is called the* diamond condition*.*

Remark 2.2: Note that by condition (P3) above, P is *connected*, meaning that we can get from any proper face to any proper face via a finite sequence of successive incidences between proper faces, or that P is rank 1. In fact, P is *strongly connected*, meaning that every section of P is connected.

The face lattice of every convex polytope is an abstract polytope. From here on, if we simply refer to a 'polytope', we mean an abstract polytope.

A *section* G/F of a polytope P, where F and G are faces of P with $F \leq G$, is the poset consisting of the faces H of P having $F \leq H \leq G$ with the same incidence structure. Every section of a polytope is itself a polytope.

A *Hasse diagram* is a representation of a polytope as a graph: the nodes of the diagram represent the faces, and the edges of the diagram represent incidences between faces whose rank differs by 1. The nodes are arranged in rows so that each row contains each face of a given rank and the lower ranks are below the upper ranks. Every abstract polytope is thus completely described by its Hasse diagram.

Consider the following useful note about connectivity: the condition of a poset being *connected* is equivalent to it being *flag-connected*. The poset P is flag-connected if and only if any flag can be joined to any other flag via a finite sequence of successively adjacent flags. Similarly, the condition (P3) of strong connectivity is equivalent to the condition of *strong flag-connectivity*: P is strongly flag-connected if and only if every section is flag-connected, or equivalently, any flag Φ can be joined to any other flag Ψ via a finite sequence of successively adjacent flags such that every flag in the sequence contains $\Phi \cap \Psi$.

The pyramid construction on convex polytopes can be extended to abstract polytopes as follows:

Definition 2.3. *Given an n-polytope P, the* pyramid *over P is the poset P' obtained by taking P as a subposet and creating a new $(i+1)$-face F' for every i-face F in P, with $i \geq -1$, so that every F' contains its corresponding old face F as well as every face G' such that $F > G$ in P.*

Remark 2.4: Note that if P is an n-polytope, the pyramid over P, P', is in fact an $(n+1)$-polytope.

Definition 2.5. *In a polytope, if G is a j-face of an i-face F then the* corank *of G in F is given by $i - j$, and G is called a corank $i - j$ face of F.*

Definition 2.6. *The* 1-skeleton *of an n-polytope P is the poset composed of its vertices and edges.*

For example, the 1-skeleton of a cube consists of all eight vertices and twelve edges of the cube, but not the (-1)-face, the six squares, or the 3-face. Note that the 1-skeleton of any polytope is connected, i.e., a sequence of successively incident proper faces can be found from any vertex or edge of the polytope to any other vertex or edge of the polytope.

Definition 2.7. *The* dual *of an n-polytope P is the poset P^* with the same face set, but with reversed incidence structure. Thus each i-face of $P = F_n/F_{-1}$ is an $(n-i-1)$-face of $P^* = F_{-1}/F_n$.*

Definition 2.8. *A* polytope lattice *is a polytope which, when viewed as a poset, is a lattice (i.e., every two elements have a supremum and an infimum).*

The dual of a polytope is also a polytope, and the dual of a polytope lattice is also a polytope lattice. We will implicitly refer to dual polytopes whenever we reverse the order of two faces: if we have a section G/F of a polytope, then F/G refers to the dual of that section.

3. Preliminary Results for Digonal Contraction

In this section, we present some useful lemmas for digonal contraction.

Lemma 3.1. *Suppose P is an n-polytope and $\mathcal{E}$ a non-empty set of facets of P such that, for every $(n-2)$-face of P, the facets containing the $(n-2)$-face either both belong to $\mathcal{E}$ or both do not belong to $\mathcal{E}$. Then $\mathcal{E}$ is the set of facets of P.*

Proof. Suppose there is a facet J not contained in $\mathcal{E}$. Since P is flag-connected, we must have a sequence of pairwise-adjacent flags taking any flag containing J to any flag containing any $E \in \mathcal{E}$. This means that we must at some point move from a flag with its facet not in $\mathcal{E}$, to an $(n-1)$-adjacent flag with its facet in $\mathcal{E}$. Then those two flags share an $(n-2)$-face, which, by the diamond condition, is contained in exactly one facet from $\mathcal{E}$. However, this contradicts our assumption on $\mathcal{E}$, so no such J can exist. □

Lemma 3.2. *Suppose we have a polytope P with an i-face F containing $(i-1)$-faces H and H' and a common corank-2 face of H and H'. If all common corank-2 faces R of H and H' are such that the 1-sections H/R and H'/R of P have the same proper faces, then F contains no $(i-1)$-faces other than H and H'.*

Proof. Let $\mathcal{R}$ denote the set of all common corank-2 faces of H and H'. Suppose all faces R in $\mathcal{R}$ are such that H/R and H'/R have the same proper faces and suppose that F contains an $(i-1)$-face I distinct from H and H'.

Suppose there is an $(i-2)$-face A contained in both I and either H or H': WLOG let $A < H$. Then we cannot have $A < H'$, since this would violate the diamond condition between F and A. Now consider the section H/F_{-1} and let $\mathcal{E}$ be the set of $(i-2)$-faces in H/F_{-1} containing a face in $\mathcal{R}$; clearly this is a nonempty set, as $\mathcal{R}$ is nonempty and every face in $\mathcal{R}$ is contained in H. Thus every face E in $\mathcal{E}$ contains a face R in $\mathcal{R}$. Since $H > R$, $H' > R$, and the two sections H/R and H'/R have the same proper faces, we have $E < H'$. Now let B be any $(i-3)$-face in H/F_{-1} with $B < E$ for some E in $\mathcal{E}$. Then we have $B < H$ and $B < H'$ and hence $B \in \mathcal{R}$; thus both proper faces of H/B are in $\mathcal{E}$. Therefore $\mathcal{E}$ is a nonempty set of $(i-2)$-faces of the $(i-1)$-polytope H/F_{-1} such that every $(i-3)$-face B containing any face in $\mathcal{E}$ must contain another face in $\mathcal{E}$, so by Lemma 3.1, A must also be in $\mathcal{E}$ and hence contain a face in $\mathcal{R}$; however, this is impossible since we established that we do not have $A < H'$, which gives us a contradiction.

Hence, in P, there can be no $(i-2)$-face A contained in both I and either H or H'. Then, since I here is just any $(i-1)$-face distinct from H and H', we see that in fact any $(i-2)$-face A contained in one of H or H' must also be contained in the other; by (P4) we know that F/A must have two proper faces, but if one proper face is H then the other proper face cannot be any face other than H', and similarly with H and H' interchanged.

Now consider the section F/F_{-1}. Let $\mathcal{H}$ be the set of $(i-1)$-faces $\{H, H'\}$. We know that any $(i-2)$-face contained in any face in $\mathcal{H}$ must be contained in another face in $\mathcal{H}$, so by Lemma 3.1 applied to $\mathcal{H}$, I must be in $\mathcal{H}$; however, this is impossible since by assumption I is distinct from H and H', which gives us a contradiction. Therefore F contains no $(i-1)$-faces other than H and H'. □

Lemma 3.3. *Suppose we have a polytope P with an i-face F containing $(i-1)$-faces H and H' and a common corank-2 face of H and H'. If all common corank-2 faces R of H and H' have the property that the 1-sections H/R and H'/R of P have the same proper faces, then each face contained in both H and H' is also contained in a common corank-1 face of H and H'.*

Proof. Suppose all common corank-2 faces R of H and H' have the property that the 1-sections H/R and H'/R have the same proper faces. Then by Lemma 3.2 F contains no $(i-1)$-faces other than H and H'. Let A be any face such that $A < H, H'$. Then $A < F$ and A has rank at most $i-2$. Find an $(i-2)$-face B, possibly equal to A, such that $A \leq B < F$. By the diamond condition between F and B, there must be two $(i-1)$-faces containing B and contained in F; by Lemma 3.2, these faces must be exactly H and H'. Hence, $B < H, H'$, so A is contained in B, which is a common corank-1 face of H and H'. □

Proposition 3.4. *Given a polytope P with a 1-section L whose greatest face, of rank $i \geq 2$ (in P), is called F and whose intermediate faces are called H and H', we can produce a poset P' from P by removing F and every other i-face containing both H and H' (we call the set of all such faces $\mathcal{F}$, with F one of them), and combining H and H' into a single face. In P', the combined face inherits all incidences of H and H' in P, except for those involving the removed faces in $\mathcal{F}$, while all other incidences are unchanged. In particular, P' can only be a polytope if for every corank-1 face G of the least face of L, the 2-section F/G in P is a digon; equivalently, every F'/G is a digon for every F' in $\mathcal{F}$. If P' is indeed a polytope, we call the operation of moving from P to P' a* digonal contraction*.*

Proof. Suppose we have some corank-1 face G of the least face A of L such that F/G is not a digon. Then there exist faces B and C of P such that A and B lie between H and G, and A and C lie between H' and G. We must have $B \neq C$, since we would otherwise have a digon F/G. Then in P', we have three distinct faces A, B, and C between H and G, violating (P4), so P' is not a polytope. $\square$

Note that removing only some of the faces in $\mathcal{F}$ and not others will never result in P' being a polytope, since (P4) will be violated between every non-removed face in $\mathcal{F}$ and the merged face H (in fact, there will be exactly one face between them in P').

4. The Digonal Contraction Theorem

Theorem 4.1 (Digonal Contraction Theorem)**.** *Suppose P is an n-polytope with a digon D whose greatest face F is of rank i, and whose $(i-1)$-faces are called H and H', with the set of all i-faces containing both H and H' called $\mathcal{F}$ (so $F \in \mathcal{F}$). Suppose that we have produced a poset P' by performing digon contraction on D as in Proposition 3.4. Then P' is a polytope if and only if the following conditions hold in P:*

1. *No two of the i-faces in $\mathcal{F}$ are contained in a common $(i+1)$-face;*
2. *All faces containing both H and H' also contain some face in $\mathcal{F}$;*
3. *All common corank-2 faces R of H and H' are such that F'/R is a digon for every F' in $\mathcal{F}$;*
4. *For every $(i+2)$-face I incident to both H and H' in P, the rank-2 section I/H is connected in P'.*

In particular, if P' is a polytope, it is rank-$(n-1)$ or rank-n, depending on if F is the greatest face of P or not.

Note that Condition (4) is a condition on P' and not P: this is for simplicity, as the corresponding equivalent condition on P is nontrivial to state. We examine Condition (4) in more detail after the proof of this theorem.

An example of digonal contraction can be seen in Figure 1: on the left, we have a polyhedron P which is identical to a cube, except that it has a digon added in where one of the edges of the cube would be; on the right, we have the cube P', the result of the digonal contraction.

Proof. In the first part of this proof, we will show that for P' to be a polytope, all four conditions must hold. We show this by proving first that P' is not a polytope if Condition (1) does not hold, then that P' is not a polytope if Condition (1) holds but Condition (2) does not hold, then that P' is not a polytope if Conditions (1) and (2) hold but Condition (3) does not hold, and finally that P' is not a polytope if all of the conditions hold except (4). In the second part of the proof, we will show that the conditions are sufficient for P' to be a polytope, i.e., if all four of the conditions hold then P' is a polytope.

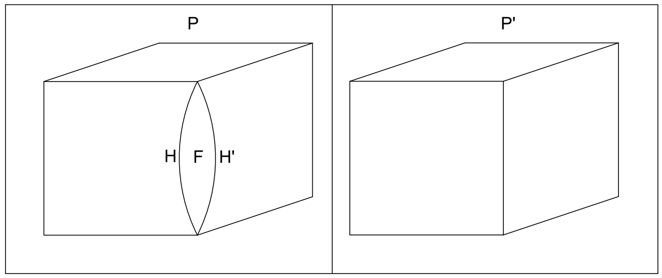

FIGURE 1. Digonal contraction on a cube-like polytope.

First, suppose that Condition (1) does not hold, i.e., some $(i+1)$-face G of P contains more than one face in $\mathcal{F}$; say it contains both F' and F''. Consider the section G/F_{-1} of P, and compare it to the section G/F_{-1} of P'; note that if G does not contain F_{-1} in P', then of course P' is not a polytope, so we assume that it does. We note that it is impossible for G in P' to contain the merged face H, since this would mean that we have some i-face I in P' such that $H < I < G$, while in P we would have $I < G$ and either $H < I$ or $H' < I$, (but not both, because otherwise I would have been removed), which violates the diamond condition between H and G or H' and G in P', respectively, since we already have F' and F'' between H and G and between H' and G. Hence, in G/F_{-1} in P', the face H does not occur. Therefore, we see that G/F_{-1} in P' consists of the section G/F_{-1} of P with some i-faces removed. However, this means that G/F_{-1} is not a polytope in P', so P' itself is not a polytope: given any polytope, removing faces from it while preserving the rank will necessarily produce a poset that is not a polytope – by removing faces we remove flags, and by removing flags we necessarily deprive some (not removed) flag of its m-adjacent flag for some m, which violates connectivity. Thus Condition (1) is necessary for P' to be a polytope.

Now suppose that Condition (1) holds but Condition (2) does not hold, so in particular H and H' are both contained in some face G, necessarily of rank at least $i+1$, which does not contain any face in $\mathcal{F}$. Let A be a common corank-1 face of H and H'. We may assume that every proper face of G/A in P contains at most one of H or H' (otherwise we can replace G by a proper face of G containing both H and H'; since none of these proper faces contain any of the faces from $\mathcal{F}$, which are the only i-faces containing both H and H' by definition, we will eventually find a face G of rank at least $i+1$ containing both H and H', with each of its proper faces containing at most one of them). Suppose G has rank $i+1$. Then the proper faces of G/H in P are both distinct from the proper faces of G/H', since G contains no face from $\mathcal{F}$; however, then in P' there are four proper faces of the 1-section G/H, violating (P4). Hence, G must have rank at least $i+2$.

If P' is a polytope, then in the section G/H of P' we can find a sequence of proper incident faces taking us from any proper face to any other proper face (note that G/H is at least of rank 2); in particular, we can take a corresponding sequence from a face of P containing H and not H' to a face of P containing H' and not H (in P', the faces H and H' are identified, so all faces containing the merged face H in P' contain H or H', or both, in P). Since G/H is a section of G/A in P', every proper face of G/H in P' must contain exactly one of H or H' in P, as we established earlier that every proper face of G/A in P contains at most one of H or H'. However, if we have a sequence of incident proper faces of G/H in P' taking us from a face containing only H to a face containing only H', then at some point we must pass through a face containing both: if some face B contains H and some face C contains H', and the two faces are incident, say B contains C, then one of them must contain both H and H' (in this case B would contain both). This gives us a contradiction. Hence, P' is not a polytope and therefore Condition (2) is necessary for P' to be a polytope.

Now suppose that Conditions (1) and (2) hold, but Condition (3) does not hold, i.e., we have a common corank-2 face R of H and H' such that F'/R is not a digon for some F' in $\mathcal{F}$. Then in the 2-section F'/R of P, the subsections H/R and H'/R do not share both of their proper faces with each other, so we have at least three distinct faces between H and R in P', violating the diamond condition. Hence Condition (3) is necessary for P' to be a polytope.

Finally, Condition (4) is clearly necessary for P' to be a polytope: if we have a disconnected section I/H in P', then (P3) does not hold for P', and so P' is not a polytope.

Thus, all four conditions are necessary for P' to be a polytope; it remains to show that the conditions are sufficient. Moreover, note that Condition (3) implies, by Lemma 3.2, that each face in $\mathcal{F}$ contains only two $(i-1)$-faces: H and H'.

We now check the four defining conditions of a polytope on P' given the four conditions of this theorem.

(P1): We see that P' has exactly one least face, i.e., the least face of P. This is because we can still trace a path of incident faces descending in rank from any face of P' to the least face of P since any $(i+1)$-face of P containing any face in $\mathcal{F}$ has at least one other corank-1 face by the diamond condition, and this face cannot be in $\mathcal{F}$ by Condition (1). Suppose F is the greatest face of P. Then P' has exactly one greatest face, i.e., the merged face H, since now P only has two rank-$(n-1)$ faces. Suppose otherwise: then by the diamond condition at rank i and by Condition (1), any $(i-1)$-face contained in a face in $\mathcal{F}$ is also contained in at least one face not in $\mathcal{F}$, so since (P1) holds for P, it also holds for P', with the same greatest face.

(P2): Suppose F is the greatest face of P. Then since P' has exactly one greatest face, i.e., the merged face H, it has property (P2) and is of rank $n-1$ since P has property (P2) and is of rank n. Suppose otherwise. Then every $(i+1)$-face of P containing a face from $\mathcal{F}$ has at least one corank-1 face not in $\mathcal{F}$ by the diamond condition and Condition (1). Furthermore, in P every corank-1 face of every face in $\mathcal{F}$ in P is contained in at least one i-face not in $\mathcal{F}$, again by the diamond condition and (1), so since P has property (P2) and is of rank n, P' also has property (P2) and is of rank n.

(P4): Consider a pair of faces A and B in P' such that $A < B$ and $\text{rank}(A) + 2 = \text{rank}(B)$. The arguments of the proof depend on $\text{rank}(A)$.

Suppose that $\text{rank}(A) > i$. Clearly A and B have exactly two faces between them in P', since no faces of rank greater than i have been changed via our operation.

Now suppose that $\text{rank}(A) = i$: then A and B still have exactly two faces between them, since the only way the i-faces changed via our operation was in the removal of the faces in $\mathcal{F}$, so A must correspond to some face in P which is not in $\mathcal{F}$. Thus we have the diamond condition for this case as well.

Suppose $\text{rank}(A) = i-1$. If A does not lie in a face of $\mathcal{F}$ in P, then A is not in D and the set of i-faces containing A in P' is the same as the set of i-faces containing A in P, so we again have the diamond condition. If $A < F'$ in P for some F' in $\mathcal{F}$, then Condition (3) with Lemma 3.2 implies that $A = H$ or $A = H'$. Suppose in P the $(i+1)$-face B contains some face of $\mathcal{F}$. Then by Condition (1) we have a unique i-face $F_H \notin \mathcal{F}$ such that $H < F_H < B$, as well as a unique face $F_{H'} \notin \mathcal{F}$ such that $H' < F_{H'} < B$. Thus $F_H \neq F_{H'}$, since the only faces having both H and H' as corank-1 faces are in $\mathcal{F}$. Then in P', the faces H and H' are identified, and so we find that F_H and $F_{H'}$ are exactly the faces of P' between A and B, so the diamond condition is satisfied in this case. Now if B does not contain any face from $\mathcal{F}$ in P, and $A = H$, say, then by the diamond condition we have exactly two i-faces between H and B in P, and neither can contain H' since the faces in $\mathcal{F}$ are the only i-faces containing both H and H'. Moreover, by Condition (2), B cannot contain H', so in P' there are exactly two faces between A and B, giving us the diamond condition. The argument for $A = H'$ is similar. This covers all cases for $\text{rank}(A) = i-1$.

Now suppose $rank(A) = i-2$ and first suppose A is in D. If both faces between A and B in P are outside of D, then clearly there are still exactly two faces between A and B in P'. If exactly one face between A and B in P is in D, then we clearly still have exactly two faces between A and B in P'. In particular, if we have $A < H < B$ in P, we also know $A < H'$, so we cannot have $H' < B$ in P; thus in P' the section B/A is unchanged. Finally, it is impossible that both faces between A and B in P are in D, since the only i-faces containing both H and H' are in $\mathcal{F}$ and so we would have $B \in \mathcal{F}$; this is impossible since the faces in $\mathcal{F}$ are not in P'. Now suppose A is not in D. Again, by similar arguments, we see that there are exactly two faces between A and B in P', and hence the diamond condition is satisfied in this case.

Next, suppose $\mathrm{rank}(A) = i-3$: the only case we need to consider is when B is in D, i.e., when $B = H$ or $B = H'$, since otherwise the relevant faces and incidences are identical for P and P'. WLOG say $B = H$. If A is not incident to H', then clearly the faces in B/A in P' are exactly the faces in B/A in P, so the diamond condition holds. If A is incident to H', it is contained in both H and H', so by Condition (3) we find that the two faces between A and H in P are the same as the two faces between A and H' in P. Therefore we have exactly two faces between A and B in P', so again the diamond condition holds.

Finally, if $\mathrm{rank}(A) < i - 3$, then clearly the diamond condition holds in P' since it holds in P and no relevant faces or incidences are modified. Therefore, the diamond condition holds for all cases.

(P3): We wish to show that P' is strongly connected. We will first show that H/F_{-1} is strongly flag-connected in P', so consider a section of H/F_{-1} in P'. If this section does not contain H, it is identical to a section of P, and is therefore a polytope and hence strongly connected. If the section does contain H, call it H/A and say we want to get from some proper face X to some proper face Y of H/A in P' via a sequence of proper successively incident faces. If in P both X and Y are contained in H (or both X and Y are contained in H', respectively), then since H/A (or H'/A) is a polytope in P, we can get a sequence of proper successively incident faces of H/A in P' taking X to Y. If this is not the case, say in P, the face X is only contained in H and Y is only contained in H'. Then by Condition (3) and Lemma 3.3, we have some common corank-1 face B of H and H' such that $A < B$. Then we can find a sequence of proper successively incident faces taking X to B in H/A in P, and we can then find a sequence of proper successively incident faces taking B to Y in H'/A in P; concatenating these two sequences, we get a sequence of proper successively incident faces taking X to Y in H/A in P'. Hence, H/F_{-1} is strongly connected in P', and is therefore strongly flag-connected.

We will now show that given any face $C > H$ in P', the section C/H is connected. Suppose in P the face C contains only one of the faces H or H', say H (it must contain at least one of the faces H or H' in P since $C > H$ in P'). Then C/H in P' is identical to C/H in P, so clearly it is connected; we can make a similar argument if $C > H'$ in P. Now suppose otherwise, i.e., that C is incident to both H and H' in P'. If $rk(C) < i+2$, then C/H is trivially connected in P', and if $rk(C) = i + 2$, then C/H is connected in P' by Condition (4).

Now suppose $rk(C) > i + 2$ and let $rk(C) = k$. It is sufficient to show that the 1-skeleton of H/C is connected in $(P')^*$, the dual of P'; if this is the case, then we can move between any two proper faces X and Y of C/H in P' by moving from X up to a $(k-1)$-face, then moving along the 1-skeleton of H/C, i.e., along the $(k-1)$-faces and $(k-2)$-faces of C/H in P', to a

$(k-1)$-face containing Y, and then moving down to Y. To this end, we will determine which faces are in the 1-skeleton of H/C in $(P')^*$. Let E be a face in the 1-skeleton of H/C in P^*, i.e. $H < E < C$ in P and $rk(E) = k-2$ or $rk(E) = k-1$. Since $k > i+2$, we know that $rk(E) > i$. Therefore, $E \notin \mathcal{F}$, and so E occurs in P'. Moreover, by Condition (2), E contains some face $F' \in \mathcal{F}$. Let J be an $(i+1)$-face in P such that $F' < J \leq E$. By (P4) on the 1-section J/H, we have an $(i+1)$-face $\bar{F} \neq F$ such that $H < \bar{F} < J$, and by Condition (1), $\bar{F} \notin \mathcal{F}$; if we had $\bar{F} \in \mathcal{F}$, then J would contain two i-faces in $\mathcal{F}$, violating Condition (1). Therefore, in P', $E \geq J > \bar{F} > H$, and so in particular, E is in the 1-skeleton of H/C in $(P')^*$. Hence, every face in the 1-skeleton of H/C in P^* also occurs in the 1-skeleton of H/C in $(P')^*$, and we see by similar reasoning that every face in the 1-skeleton of H'/C in P^* also occurs in the 1-skeleton of H/C in $(P')^*$. Since the 1-skeletons of H'/C and H/C in P^* share faces (since C is incident to both H and H' in P, it contains faces in $\mathcal{F}$, and hence contains $(k-1)$-faces and $(k-2)$-faces incident to faces in $\mathcal{F}$ which are incident to both H and H'), and since digonal contraction does not affect the incidences between $(k-1)$-faces and $(k-2)$-faces for $k > i+2$, we find that the 1-skeleton of H/C in $(P')^*$ is connected. Thus C/H is connected in P' (and is therefore flag-connected).

We will now proceed by showing that P' is strongly flag-connected (and therefore strongly connected). In order to do this, we choose an arbitrary section $S' = B/A$ of P' and demonstrate that it is flag-connected. Since every face of P' corresponds to at least one face of P, we can examine the corresponding section $S = B/A$ of P. Consider two flags Φ and Ψ of S': they correspond to two flags Φ and Ψ of S, so we have a sequence $\Phi = \Phi_0, ..., \Phi_r = \Psi$ of successively adjacent flags of S taking Φ to Ψ in S. If this sequence does not contain a flag that passes through any of the faces in $\mathcal{F}$, we can produce a corresponding sequence of successively adjacent flags in S', as desired, since all the relevant flags will still exist in P', up to the renaming of the face H' as H, if it occurs in any of the flags.

Suppose now that the sequence contains a flag that passes through a face in $\mathcal{F}$. If the maximum face of P is an element of $\mathcal{F}$, and hence $\mathcal{F} = \{F\}$, then we are done, since we then have $P' = H/F_{-1}$, which we already showed is strongly flag-connected. Suppose that this is not the case and let Φ_j be the first flag containing any member of $\mathcal{F}$ in the sequence; call this member F'. Then a number of subsequent flags will all contain F' until we reach some flag Φ_k again not containing F'. Note that Φ_k cannot contain any of the faces in $\mathcal{F}$ because no two of these faces are contained in a common $(i+1)$-face, by Condition (1).

We now find a sequence of successively adjacent flags taking Φ_{j-1} to Φ_k in S'. Any flag of S containing F' must also contain H or H'; Condition (3) with Lemma 3.2 tells us that F' has only H and H' as corank-1 faces, and of course we cannot have $A = F'$ since A is a face of P', whereas F' does not occur in P'. Therefore, Φ_{j-1} and Φ_k must each also contain H or H', since they are i-adjacent to flags containing F'. Hence, in S', the flags Φ_{j-1} and

Φ_k both contain H (the combination of H and H' from P). Since H/F_{-1} is strongly flag-connected in P', as established earlier, we know that H/A is flag-connected in P'. Hence, we can move Φ_{j-1} to a flag Φ' identical to Φ_{j-1} for all faces containing H, and identical to Φ_k for all faces contained in H. Since we know that B/H is flag-connected in P', as established earlier, we can move Φ' to Φ_k, since these two flags coincide in all faces contained in H. Hence, we can move Φ_{j-1} to Φ_k in S'. This means that we can take $\Phi = \Phi_0$ to Φ_k in S'. Now, we can continue taking the original sequence of successively adjacent flags in S from Φ_k onwards to get from Φ to Ψ; if we again pass through a face in $\mathcal{F}$, we can find another alternative sequence of subsequently adjacent flags, as we just did from Φ_{j-1} to Φ_k. Hence, we can modify the sequence of subsequently adjacent flags from Φ to Ψ in S to get a sequence of subsequently adjacent flags from Φ to Ψ in S', and so S' is flag-connected. Since S' is an arbitrary section of P', we see that every section of P' is flag-connected, and so P' is strongly flag-connected, and therefore strongly connected. Therefore, since we have demonstrated all four conditions (P1) through (P4) on P', we see that P' is a polytope. □

In Figure 2, we see four polytopes with digons that cannot be contracted. In particular, the first polytope violates Condition (1) of Theorem 4.1 while satisfying (2) and (3), the second polytope violates Condition (2) of Theorem 4.1 while satisfying (1) and (3), and the third polytope violates Condition (3) of Theorem 4.1 while satisfying (1) and (2). The combinatorial structure of these examples is too tight to allow the contraction of the specified digons. The fourth polytope violates Condition (4) of Theorem 4.1 while satisfying the other three conditions. We do not examine Condition (4) on the first three polytopes: we do not know whether (4) actually implies some of the other conditions, but even if it were the case that (4) implied another condition, we would want to include all four conditions in the theorem, since (4) is a condition on P' with a complicated corresponding condition on P which we would like to keep as simple as possible.

The first polytope is a "dihedron over a digon." This is a polyhedron consisting of two digonal faces sharing both edges and both vertices of the polyhedron. Both digonal faces are in $\mathcal{F}$ (so $i = 2$), and they are both contained in the $(i+1)$-face P, violating Condition (1) of Theorem 4.1.

The diagram showing the second polytope does not actually show the polytope itself, but a related polytope. To get the polytope that we are actually interested in, we increase the rank of every face of the displayed polytope (which is a polyhedron made up of two triangles and a square) by 1, and then have two vertices contained in every higher-rank face. The resulting polytope is then rank-4 and consists of two vertices, four edges (corresponding to the four vertices of the displayed polytope), two of which are H and H', five digons (corresponding to the five edges of the displayed polytope), one of which is F, and three polyhedra (corresponding to the

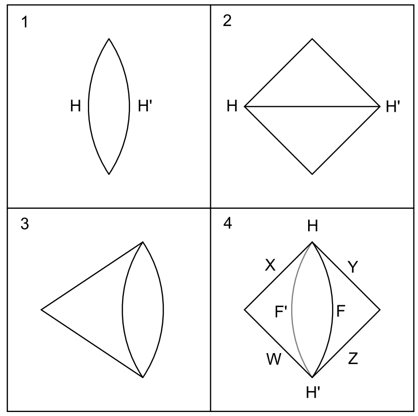

FIGURE 2. Four non-contractible digons.

triangles and square of the displayed polytope). We see that the polyhedron corresponding to the square of the displayed polytope contains both H and H' but does not contain F, violating Condition (2) of Theorem 4.1.

The third polytope is a “pyramid over a digon”. The apex of the pyramid, shown on the left, is contained in two triangles, which are H and H'. Then F is the polytope itself. We see that Condition (3) of Theorem 4.1 is violated since, taking R as either of the vertices of the polyhedron other than the apex, the vertex figure of R is a triangle rather than a digon.

For the fourth polytope, as with the second, the diagram does not actually show the polytope itself. Again, to get the polytope P that we are actually interested in, we increase the rank of every face of the displayed polytope (which is a polyhedron made up of four triangles, two in the front containing F and two in the back containing F') by 1, and then have two vertices contained in every higher-rank face. The resulting polytope is then rank-4. Figure 3 demonstrates that Condition (4) of Theorem 4.1 is in fact violated by this polytope, with $I = P$.

We denote the four facets of P corresponding to the four triangles in the diagram as follows: I corresponds to the triangle XWF, II to YZF, III to XWF', and IV to YZF'. The first two diagrams in Figure 3 show the sections P/H and P/H'. The result of digonal contraction is shown in the third diagram in Figure 3: F and F' have been removed, and the remaining faces and their incidences have been adjusted accordingly. As we can see, P'/H is not connected, hence violating Condition (4). If the original polytope was ‘twisted’ in such a way that the order of the faces in P/H was the same but the faces I and II were swapped in P/H' (so that I contained Z and II contained W), then P'/H would be a square, and hence connected. Hence, we see the complexity of Condition (4) in P: in order to see whether

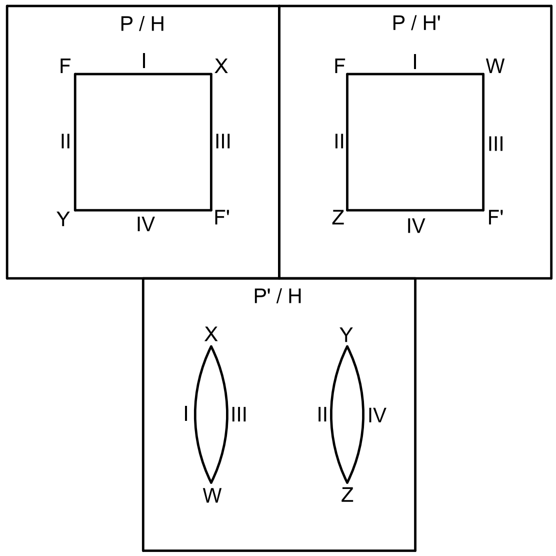

FIGURE 3. An example of Condition (4) breaking for digonal contraction.

Condition (4) is satisfied in P, we need to examine the incidences between the faces in $\mathcal{F}$ and the faces in I/H and I/H'.

However, there exist simple conditions on P which are sufficient for Condition (4) to be satisfied.

Remark 4.2: Suppose that for a polytope P as in Theorem 4.1, we have $\mathcal{F} = \{F\}$ and for every $(i+2)$-face I in P, one of the polygonal sections I/H and I/H' is finite. Then Condition (4) is satisfied.

We see why this is true in Figure 4. Here, we note that the $(i+1)$-faces A and B coincide in I/H and in I/H', since both faces A and B contain F, which contains H and H'. It is possible that x and y coincide, or that z and w coincide, but this does not meaningfully affect the result. If both polygonal sections I/H and I/H' in P are infinite, then we see that the resulting section I/H in P' cannot be connected, since there is no way to get from A to B. Similarly, we see that the following condition is also sufficient for Condition (4) to be satisfied:

Remark 4.3: Suppose that for a polytope P as in Theroem 4.1, every $(i+2)$-face I in P incident to both H and H' has the property that I contains exactly one face from $\mathcal{F}$, and one of the polygonal sections I/H and I/H' is finite. Then Condition (4) is satisfied.

Another important note about Theorem 4.1 is that it extends naturally to the question of contracting an edge. Suppose P is an n-polytope with two

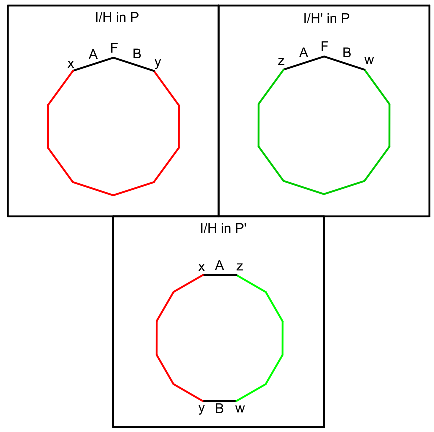

Figure 4. Merging two polygons with digonal contraction.

vertices H and H' joined by an edge F, and denote by $\mathcal{F}$ the set of all edges containing both H and H'. Suppose that we have produced a poset $\hat{P}$ by combining H and H' and removing the faces in $\mathcal{F}$, as in digonal contraction, although of course F is not a digon but an edge. Then $\hat{P}$ is a polytope if and only if the conditions of Theorem 4.1 hold for P (note that Condition (3) holds trivially since H and H' have no corank-2 faces); this follows from the proof of Theorem 4.1. In particular, if P is a polytope lattice, then the edge F is contractible, i.e., $\hat{P}$ is a polytope, if and only if for every facet I in P containing both vertices H and H', one of the polygonal sections I/H and I/H' is finite. Note that this is always the case if P is in fact a convex polytope.

5. Reversing Digonal Contraction

A natural question is if, and how, we may perform the opposite operation to digonal contraction, which we might call digonal insertion. Whereas digonal contraction replaces a digon with an edge, digonal insertion would replace an edge with a digon. To formalize this, let P' be an n-polytope with an $(i-1)$-face H (for $i \leq n$). We wish to produce a poset P by replacing H by two copies of itself, H and H', which are both contained in a set of new i-faces $\mathcal{F}$. Faces of P' other than H and H' are copied over to P, and their incidences between each other are preserved, but we must determine the number of faces in $\mathcal{F}$, their incidences, and the incidences of the faces H and H', with the goal of P being a polytope so that performing digonal contraction on P produces again P'.

It is not clear if there is any simple way of deciding how to set the number of faces in $\mathcal{F}$ and the incidences of the faces in $\mathcal{F}$, H, and H' so that P is a polytope and performing digonal contraction on P produces P', or moreover if there is any simple condition on P' which tells us whether it is even possible to produce such a polytope P. The following considerations highlight some of the difficulties.

First, we see that the new i-faces in $\mathcal{F}$ each should only have two corank-1 faces, i.e., H and H'. Since our goal is to reverse digonal contraction, the poset P should have the properties required for digonal contraction, and in particular each face in $\mathcal{F}$ can only have two corank-1 faces, by Lemma 3.2 with Condition (3) for digonal contraction. Moreover, the faces H and H' must have identical lower-rank incidences: given any corank-1 face I of H, (P4) for F/I (for any $F \in \mathcal{F}$) tells us that I must be incident to another $(i-1)$-face of P, but we know that only one other $(i-1)$-face is incident to F, i.e., H', so I is also a corank-1 face of H'; we can make a similar argument assuming first that I is instead a corank-1 face of H'. Hence, for the new faces we have produced, i.e., for H, H', and the faces in $\mathcal{F}$, it is clear how to set their lower-rank incidences. The problem is in setting the higher-rank incidences so that P is a polytope.

In P', we have a set of i-faces containing H, and in P, for each such i-face we need to decide whether it is incident to H or to H' (it cannot be incident to both, as any i-face containing both H and H' in P is a member of the set $\mathcal{F}$ and is removed in P'). Clearly, we cannot make this decision completely arbitrarily: if we view the 1-skeleton of P'/H as a graph, we need to assign each vertex to H or to H' in P, i.e., to partition the vertices of the graph into two sets, so that each corresponding subgraph is nonempty and connected.

Then the edges of the graph, corresponding to $(i+1)$-faces in P' containing H, have obvious lower-rank incidences in P: if such an edge contains two vertices both assigned to H, then the corresponding $(i+1)$-face in P must be incident to H; if such an edge contains two vertices both assigned to H', then the corresponding $(i+1)$-face in P must be incident to H'; and if such an edge contains one vertex assigned to H and one vertex assigned to H', then the corresponding $(i+1)$-face in P must be incident to both H and H', and therefore must be incident to exactly one of the faces in $\mathcal{F}$.

However, this still leaves us, in general, many different ways to assign the vertices of the graph to H or H' in P, and also many ways to decide on how many faces are in $\mathcal{F}$ and which edges of the graph they are incident to. It is not clear if there is any simple condition telling us which choices give us an actual polytope for P. Hence, it seems as though reversing digonal contraction is more complicated than performing digonal contraction.

6. The Helfand Construction

Helfand gives us a rank-preserving construction [2, Section 2.2] that we can apply to any (abstract) polytope of rank $n \geq 3$ to get another n-polytope. This construction has useful connections to digonal contraction.

Let P be an n-polytope. For Helfand's construction, we choose a rank k, with $0 \leq k \leq n-1$, and replace (combinatorially) every k-face F of P

with a number of new k-faces, one for each $(k+1)$-face containing F, and add an i-face for $i > k$ for every incident pair of a k-face F of P and an $(i+1)$-face G of P. All faces of P with rank other than k are retained, as well as all incidences between faces with rank other than k. The new k-faces each contain the same proper faces as their corresponding original faces did, and each is contained in the original $(k+1)$-face it corresponds to. Every new i-face for $i > k$ is contained in its corresponding original $(i+1)$-face G, and contains the new faces corresponding to the original faces above F contained in G. We call this process the *global Helfand construction*, and the resulting polytope, $[P]_k$, is called the *k-bubble* of P.

In Figure 5, we have an example of what happens when the global Helfand construction is applied to the cube. The first figure is the cube itself.

If $k = 0$, we get the second figure, known as the truncated cube (in general, applying the Helfand construction with $k = 0$ to any convex polyhedron produces a convex polyhedron obtained by truncating the vertices of the original polyhedron). Here, we see that each vertex (i.e., each k-face) is replaced with three new vertices, corresponding to the three edges of the original cube containing the vertex. Additionally, we have three new edges for each vertex, corresponding to the three squares of the original cube containing the vertex. Finally, we have a new triangle for each vertex, corresponding to the incidence between the vertex and the cube itself.

If $k = 1$, we get the third figure, in which every edge of the cube is replaced by a digon. This occurs because every edge is contained in exactly two squares in the cube, and so each edge of the cube is replaced with two new edges. Additionally, we have a new 2-face for each edge (the digon), corresponding to the incidence between the edge and the cube itself.

If $k = 2$, the construction can be applied, but we just get the same cube back.

We produce a local version of this construction, again combinatorially: instead of removing every k-face of P, we only remove one k-face F of P, and replace it with a collection of new k-faces, one for each $(k+1)$-face of P containing F. We also add an i-face for $i > k$ for every incident pair (F, G) of P with G an $(i+1)$-face of P. All faces other than F are retained, as are all incidences between faces other than F. The new k-faces each contain the same proper faces as F, and each is contained in the $(k+1)$-face containing F from which it was derived. Every new i-face for $i > k$ is contained in its corresponding original $(i+1)$-face G, and contains the new faces corresponding to the original faces above F contained in G. We call this operation the *local Helfand construction* (at level k), or for short, the *Helfand construction.*

It is easy to see the effect of the local Helfand construction on the cube: looking at Figure 5, the case $k = 0$ would have us truncate a single vertex of the cube instead of every vertex, and the case $k = 1$ would have us replacing a single edge by a digon instead of every edge. Note that the global Helfand construction is the same as applying the local Helfand construction in turn to

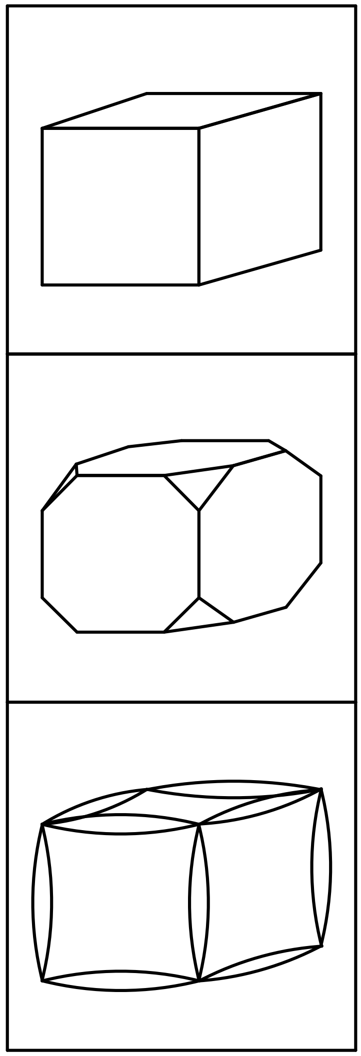

Figure 5. Two different applications of the global Helfand construction to the 3-cube.

every (original) k-face of a polytope. Helfand proves that the global Helfand construction always produces a polytope; it is straightforward to show that the local Helfand construction always produces a polytope as well.

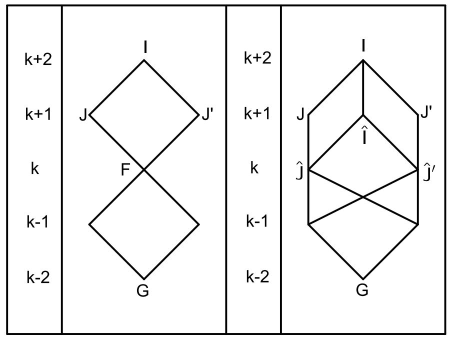

FIGURE 6. Producing a digon via the Helfand construction.

7. Connections Between the Helfand Construction and Digonal Contraction

We now note that the (local) Helfand construction, when applied to a k-face F of an n-polytope P for $1 \leq k \leq n-2$, produces digons (as rank-2 sections) with greatest faces of rank $k+1$. We see this illustrated in the Hasse diagrams in Figure 6. A digon $\hat{I}/G$ is produced for every incident pair of faces (G, I) of P such that G is a corank-2 face of F and F is a corank-2 face of I. This follows from the fact that there are two $(k+1)$-faces J and J' between F and I. After the construction, we get a new k-face $\hat{J}$ corresponding to J and a new k-face $\hat{J}'$ corresponding to J'. Both of these faces contain the $(k-2)$-face G and the two $(k-1)$-faces between F and G in the original polytope, and both are contained in the new $(k+1)$-face $\hat{I}$ corresponding to I. Hence, we get a digon $\hat{I}/G$ in $\hat{P}$. We can now ask when a single such digon can be removed via digon contraction.

Theorem 7.1. *Let P be an n-polytope with a k-face F, let $1 \leq k \leq n-2$, and let $\hat{P}$ be the n-polytope obtained from P by applying the Helfand construction at F. Furthermore, let G be a $(k-2)$-face of P and I be a $(k+2)$-face of P with $G < F < I$, and let J and J' be the two proper $(k+1)$-faces of the section I/F in P. Finally, let $\mathcal{I}$ be the set of $(k+2)$-faces in P containing both J and J' (so that $I \in \mathcal{I}$). Then the digon $\hat{I}/G$ in $\hat{P}$ is contractible in the sense that the resulting structure is an n-polytope if and only if the following conditions hold in P:*

(A) *No two faces in $\mathcal{I}$ are contained in a common $(k+3)$-face;*
(B) *Every face containing both J and J' also contains a face in $\mathcal{I}$;*

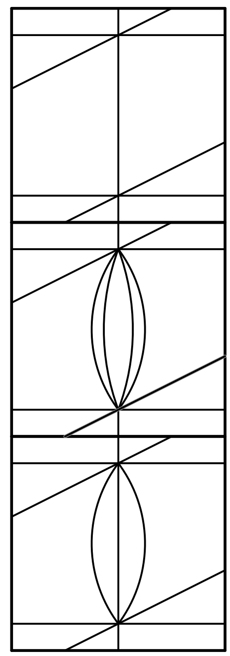

FIGURE 7. Producing and contracting digons with the Helfand construction.

(C) *Denote by P' the poset obtained by combining the faces J and J' in P as in digonal contraction, removing every face in $\mathcal{I}$, and leaving all other faces and incidences of P unchanged. Then for every $(k+4)$-face L containing both J and J' in P, the rank-2 section L/J is connected in P'.*

In Figure 7, we see an example of such a process: we start with the cubic tessellation of 3-space (this is a 4-polytope; in the diagram, only the neighborhood of one edge is shown - the edge is the intersection of four cubes), apply the Helfand construction to an edge (replacing the edge with a 3-polytope consisting of four digons sharing the two vertices and joined edge-to-edge to form the tessellation $\{2, 4\}$ of the 2-sphere), then contract one of the digons in the result (the new 3-polytope now has three digons sharing the two vertices joined edge-to-edge to form the tessellation $\{2, 3\}$ of the sphere).

Proof. We first examine the incidence structure of $\hat{P}$. An *old* face C in P/F (a face that exists in P as well) contains exactly (i) the old faces in P that C contains, and (ii) the *new* faces (i.e. those newly-introduced in $\hat{P}$, each corresponding to an old face in P/F) corresponding to either C itself or a face in P/F contained in C. A new face $C = \hat{D}$ contains exactly the new faces corresponding to the faces in P/F contained in D, as well as all faces properly contained in F in P.

We note that in $\hat{P}$, the digon we are considering contracting is $\hat{I}/G$, which means that the faces $\hat{J}$ and $\hat{J}'$ correspond to the faces H and H' in Theorem 4.1, (since they are the edges of the digon), and the set $\hat{\mathcal{I}} := \{\hat{I}'|I' \in \mathcal{I}\}$ corresponds to the set $\mathcal{F}$ in the digonal contraction theorem (since they are exactly the faces having $\hat{J}$ and $\hat{J}'$ as common corank-1 faces).

We first show that if any of the conditions for the present theorem do not hold, then the digon $\hat{I}/G$ is not contractible in $\hat{P}$.

First, suppose that Condition (A) does not hold, i.e., some $(k+2)$-faces $I_1, I_2 \in \mathcal{I}$ are contained in a common $(k+3)$-face C in P. Then in $\hat{P}$ we have $\hat{I}_1, \hat{I}_2 \leq \hat{C}$, so we violate Condition (1) of Theorem 4.1 (with $i = k+1$). Hence, Condition (A) is necessary for contractibility.

Next, suppose that Condition (B) does not hold, i.e., suppose that we have some face K in P containing J and J' that does not contain any face in $\mathcal{I}$. Then in $\hat{P}$, we have $\hat{J}, \hat{J}' \leq \hat{K}$, but none of the faces in $\hat{\mathcal{I}}$ are contained in $\hat{K}$. Hence, we violate Condition (2) of Theorem 4.1, so Condition (B) is necessary for contractibility.

Finally, suppose that Condition (C) does not hold, i.e., we have some $(k+4)$-face L in P containing both J and J', with the section L/J disconnected in P'. Denote by $\hat{P}'$ the poset obtained by contracting the digon $\hat{I}/G$ in $\hat{P}$. Condition (4) in Theorem 4.1 tells us that the 2-section $\hat{L}/\hat{J}$ of $\hat{P}'$ must be connected in order for the digon $\hat{I}/G$ in $\hat{P}$ to be contractible in $\hat{P}$. However, the 2-section $\hat{L}/\hat{J}$ of $\hat{P}'$ corresponds to the 2-section L/J of P' by definition of the Helfand construction: every face of $\hat{L}/\hat{J}$ corresponds to a face of L/J, and the incidence structure is the same. Therefore, $\hat{L}/\hat{J}$ is disconnected in $\hat{P}'$, and so the digon $\hat{I}/G$ in $\hat{P}$ is not contractible in $\hat{P}$. Hence, Condition (C) is necessary for contractibility.

Now suppose that Conditions (A) through (C) hold. We show that then Conditions (1) through (4) of Theorem 4.1 must hold in $\hat{P}$, so that the sufficiency of Conditions (A) through (C) follows from Theorem 4.1.

To prove that Condition (1) of Theorem 4.1 holds for $\hat{P}$ with $i = k+1$, observe that each $\hat{I}' \in \hat{\mathcal{I}}$ is a corank-1 face of only one old face, i.e. I', and I' cannot contain any of the other faces in $\hat{\mathcal{I}}$ since each old face contains only one new corank-1 face. Hence, no two of the $(k+1)$-faces in $\hat{\mathcal{I}}$ can be contained in a common old $(k+2)$-face. Furthermore, no two faces $\hat{I}_1, \hat{I}_2$ in $\hat{\mathcal{I}}$ can be contained in a common new $(k+2)$-face $C = \hat{D}$ of $\hat{P}$; if this were the case, then we would have $D > I_1, I_2$ in P, violating Condition (A). Hence, Condition (1) of Theorem 4.1 must hold for $\hat{P}$.

To prove Condition (2) of Theorem 4.1 holds for $\hat{P}$, suppose some face C in $\hat{P}$ contains $\hat{J}$ and $\hat{J}'$. Firstly, suppose C is an old face. Then it must contain J and J', as every face contained in C is either an old face or a new face corresponding to an old face above F contained in C. Then by

Condition (B), C contains some I' in $\mathcal{I}$, and therefore it contains $\hat{I}'$ in $\hat{\mathcal{I}}$, so (2) of Theorem 4.1 holds in this case. Secondly, suppose $C = \hat{D}$ is a new face. Then D must contain J and J', so by the same reasoning as before, D contains a face from $\hat{\mathcal{I}}$, and therefore $\hat{D} = C$ does as well. Hence, in both cases, Condition (2) of Theorem 4.1 holds for $\hat{P}$.

To prove Condition (3) of Theorem 4.1 holds for $\hat{P}$, let R be a common corank-2 face of $\hat{J}$ and $\hat{J}'$ in $\hat{P}$. We note that $\hat{J}$ and $\hat{J}'$ have the same set of incident faces of lower rank, i.e., the set of faces of F in P, by definition of the Helfand construction. Then the two faces between $\hat{J}$ and R are identical to the two faces between $\hat{J}'$ and R, so $\hat{I}'/R$ is a digon for every $\hat{I}'$ in $\hat{\mathcal{I}}$. Thus Condition (3) of Theorem 4.1 holds for $\hat{P}$.

To prove Condition (4) of Theorem 4.1 holds for $\hat{P}$, let E be a $(k+4)$-face of $\hat{P}$ incident to both $\hat{J}$ and $\hat{J}'$. We now consider the rank-2 section $E/\hat{J}$ in $\hat{P}'$.

First, suppose that E is a new face in $\hat{P}$, so $E = \hat{L}$ for a $(k+5)$-face L in P. Then by Condition (C), the 2-section L/J is connected in P', and hence the corresponding rank-2 section $\hat{L}/\hat{J} = E/\hat{J}$ is connected in $\hat{P}'$.

Now suppose that E is an old face in $\hat{P}$ and consider the 2-section $E/\hat{J}$ of $\hat{P}$. Clearly, this rank-2 section contains $\hat{E}$ as an edge. Moreover, since E is an old face containing $\hat{J}$, the 2-section $E/\hat{J}$ must contain J as a vertex. Furthermore, since E is incident to both $\hat{J}$ and $\hat{J}'$, it is incident to both J and J', and therefore to some I' in $\mathcal{I}$; hence, $E/\hat{J}$ must contain I' as an edge and $\hat{I}'$ as a vertex. Finally, by (P4) on the section E/J in $\hat{P}$, we see that $E/\hat{J}$ must contain an additional edge K, which must be an old face since it lies in E/J, and a corresponding vertex $\hat{K}$. Examining the incidence structure of these three vertices and three edges, we see that $E/\hat{J}$ is in fact a triangle in $\hat{P}$, as shown in Figure 8; similarly, as shown in the same figure, $E/\hat{J}'$ is a triange in $\hat{P}$ as well (here, we have an analogous face K' to K, from (P4) on the section E/J'). After examining the effect of digonal contraction on these triangles, we see, again in Figure 8, that $E/\hat{J}$ is a square in $\hat{P}'$, and is hence connected.

Therefore, in both cases, the rank-2 section $E/\hat{J}$ in $\hat{P}'$ is connected in $\hat{P}'$, satisfying Condition (4) of Theorem 4.1 for $\hat{P}$.

Thus, Conditions (1) through (4) of Theorem 4.1 must hold for $\hat{P}$, and so the digon $\hat{I}/G$ of $\hat{P}$ is contractible. □

Corollary 7.2. *Suppose P is a polytope lattice and we apply the (local) Helfand construction to a k-face F of P to obtain a polytope $\hat{P}$ as in the previous theorem. Let $\hat{I}/G$ be an arbitrary digon of $\hat{P}$ obtained as a result of this construction, with $G < F < I$ in P, $rk(I) = k+2$, and $rk(G) = k-2$. Denote by J and J' the two proper $(k+1)$-faces of the section I/F in P. Then $\hat{I}/G$ is contractible in $\hat{P}$ if and only if for every $(k+4)$-face L of P*

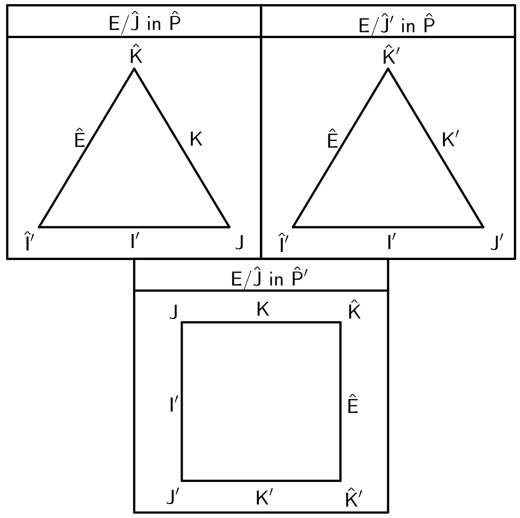

FIGURE 8. Merging two triangles into a square with digonal contraction.

containing both J and J', one of the polygonal sections L/J and L/J' in P is finite.

Proof. We will proceed with this proof with only the assumption that P is a polytope lattice. We first note that we must have $\mathcal{I} = \{I\}$, since I is the supremum of J and J' in P. Hence Condition (A) of the previous theorem must hold trivially. Similarly, Condition (B) of the previous theorem must hold, again since I is the supremum of J and J' in P.

We now consider Condition (C) of the previous theorem. Denote by P' the poset obtained by combining the faces J and J' in P, removing I, and leaving all other faces and incidences of P unchanged. Let L be a $(k+4)$-face of P containing both J and J'. We see the effect of contracting J and J' on the polygons L/J and L/J' in Figure 9. The faces A and B are the $(k+3)$-faces containing I in L/J. Note that A and B must also contain I in L/J' because I contains both J and J'. Since $\mathcal{I} = \{I\}$, the rank-2 section L/J in P' is connected if and only if at least one of the paths shown from x to y or from z to w is finite; thus, the digon $\hat{I}/G$ is contractible in $\hat{P}$ if and only if one of the polygonal sections L/J and L/J' is finite for every such choice of L. □

Remark 7.3: Suppose we apply the (local or global) Helfand construction with $1 \leq k \leq n-2$ to an n-polytope P, and then contract one of the new digons of $\hat{P}$ to obtain a polytope $\hat{P}'$. We can then contract another of the new digons of $\hat{P}$ in $\hat{P}'$, as long as it was unaffected by our previous contraction (i.e. neither of the new faces H and H' of this second digon is the merged face of the first digon), by the same reasoning. We can then

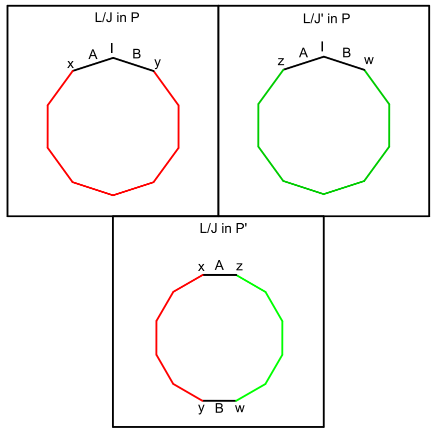

FIGURE 9. Merging two polygons with digonal contraction, in connection with the Helfand construction.

continue contracting digons produced by the Helfand construction, again under the assumption that all contracted digons are unaffected by previous contractions. In general, however, we cannot repeat this process to get the original polytope P back: for instance, in Figure 7, after contracting one digon (the digon in front), there is only one other digon that can be chosen that was not affected by the first contraction (the digon in back); after contracting both, we cannot contract further, since Condition (1) of Theorem 4.1 is violated.

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