Contributions to Discrete Mathematics

Volume 10, Number 1, Pages 99–112
ISSN 1715-0868

A CURIOUS POLYNOMIAL INTERPOLATION OF CARLITZ–RIORDAN'S q-BALLOT NUMBERS

FRÉDÉRIC CHAPOTON AND JIANG ZENG

Abstract. We study a polynomial sequence $C_n(x|q)$ defined as a solution of a q-difference equation. This sequence, evaluated at q-integers, interpolates Carlitz–Riordan's q-ballot numbers. In the basis given by some kind of q-binomial coefficients, the coefficients are again some q-ballot numbers. We obtain another curious recurrence relation for these polynomials in a combinatorial way.

1. Introduction

This paper was motivated by a previous work of the first author on flows on rooted trees [8], where the well-known Catalan numbers and the closely related ballot numbers played an important role. In fact, one can easily introduce one more parameter q in this work, and then Catalan numbers and ballots numbers get replaced by their q-analogues introduced a long time ago by Carlitz–Riordan [6], see also [5, 13].

These q-Catalan numbers have been recently considered by many people, see for example [9, 4, 15, 3], including some work by Reineke [16] on moduli space of quiver representations.

Inspired by an analogy with another work of the first author on rooted trees [7], it is natural to try to interpolate the q-ballot numbers. In the present article, we prove that this is possible and study the interpolating polynomials.

Our main object of study is a sequence of polynomials in x with coefficients in $\mathbb{Q}(q)$, defined by the q-difference equation:

$$\Delta_q C_{n+1}(x|q) = qC_n(q^2x + q + 1|q), \tag{1.1}$$

where $\Delta_q f(x) = (f(1 + qx) - f(x))/(1 + (q - 1)x)$ is the Hahn operator.

After reading a previous version of this paper, Johann Cigler has kindly brought the two related references [10, 11] to our attention, where a sequence of more general polynomials $G_n(x, r)$ was introduced through a q-difference

Received by the editors November 27, 2013, and in revised form October 10, 2014.
2000 *Mathematics Subject Classification.* 05A10, 05A19, 05A30.
Key words and phrases. q-Catalan numbers, lattice paths, q-ballot numbers.

operator for q-integer x and positive integer r. Comparing these two sequences one has

$$G_n(qx+1,2) = C_{n+1}(x|q).$$

In the next section, we recall classical material on Carlitz–Riordan's q-analogue for Catalan and ballot numbers and define our polynomials. In the third section, we evaluate our polynomials at q-integers in terms of q-ballot numbers and prove a product formula when $q = 1$. In the fourth section, we find their expansion in a basis made of a kind of q-binomial coefficients and obtain another recurrence for these polynomials. This recurrence is not usual even in the special $x = q = 1$ case and we have only a combinatorial proof in the general case. We conclude the paper with some open problems.

Nota Bene: Figures are best viewed in color.

2. Carlitz–Riordan's q-ballot numbers

Recall that the Catalan numbers $C_n = \binom{2n}{n}/(n+1)$ may be defined as solutions to

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad (n \geq 0), \quad C_0 = 1. \tag{2.1}$$

The first values are

n	0	1	2	3	4	5	6	7	8
C_n	1	1	2	5	14	42	132	429	1430

It is well known that C_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$, which do not pass above the line $y = x$. As a natural generalization, one considers the set $\mathcal{P}(n,k)$ of lattice paths from $(0,0)$ to $(n+1,k)$ with steps $(1,0)$ and $(0,1)$, such that the last step is $(1,0)$ and they never rise above the line $y = x$. Let $f(n,k)$ be the cardinality of $\mathcal{P}(n,k)$. The first values of $f(n,k)$ are given in Table 1. These numbers are called ballot numbers and have a long history in the literature of combinatorial theory. Moreover, one (see [12]) has the explicit formula

$$f(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{k} \qquad (n \geq k \geq 0). \tag{2.2}$$

Carlitz and Riordan [6] introduced the following q-analogue of these numbers

$$f(n,k|q) = \sum_{\gamma \in \mathcal{P}(n,k)} q^{A(\gamma)}, \tag{2.3}$$

where $A(\gamma)$ is the area under the path (and above the x-axis). The first values of $f(n,k|q)$ are given in Table 2. Furthermore, Carlitz [5] uses a variety of elegant techniques to derive several basic properties of the $f(n,k|q)$, among which the following is the basic recurrence relation

$$f(n,k|q) = qf(n,k-1|q) + q^k f(n-1,k|q) \qquad (n,k \geq 0), \tag{2.4}$$

where $f(n,k|q) = 0$ if $n < k$ and $f(0,0|q) = 1$.

n\k	0	1	2	3	4	5	6
0	**1**						
1	1	**1**					
2	**1**	2	**2**				
3	1	**3**	5	**5**			
4	**1**	4	**9**	14	**14**		
5	1	**5**	14	**28**	42	**42**	
6	**1**	6	**20**	48	**90**	132	**132**

Table 1. The first values of ballot numbers $f(n,k)$.

n\k	0	1	2	3	4
0	1				
1	1	q			
2	1	$q[2]_q$	q^2+q^3		
3	1	$q[3]_q$	$q^2+q^3+2q^4+q^5$	$q^3+q^4+2q^5+q^6$	
4	1	$q[4]_q$	$q^2+q^3+2q^4+2q^5+2q^6+q^7$	q^3Y	q^4Y

Table 2. The first values of q-ballot numbers $f(n,k|q)$ with $Y = q^6+3q^5+3q^4+3q^3+2q^2+q+1$.

It is also easy to see that the polynomial $f(n,k|q)$ is of degree $kn-\binom{k}{2}$ and satisfies the equation $f(n,n|q) = qf(n,n-1|q)$. If one defines the q-Catalan numbers by

$$C_{n+1}(q) = \sum_{k=0}^{n} f(n,k|q) = q^{-n-1} f(n+1,n+1|q) \qquad (n \geq 0), \tag{2.5}$$

then one obtains the following analogue of (2.1) for the Catalan numbers

$$C_{n+1}(q) = \sum_{i=0}^{n} C_i(q)C_{n-i}(q)q^{(i+1)(n-i)}, \tag{2.6}$$

where $C_0(q) = 1$. Setting $\widetilde{C}_n(q) = q^{\binom{n}{2}}C_n(q^{-1})$, one has a simpler q-analog of (2.1),

$$\widetilde{C}_{n+1}(q) = \sum_{i=0}^{n} q^i \widetilde{C}_i(q)\widetilde{C}_{n-i}(q). \tag{2.7}$$

The first values are $C_1(q) = 1$, $C_2(q) = 1+q$, $C_3(q) = 1+q+2q^2+q^3$ and

$$C_4(q) = 1+q+2q^2+3q^3+3q^4+3q^5+q^6.$$

Except for a determinant formula [1], no explicit formula is known for Carlitz–Riordan's q-Catalan numbers. However, Andrews [1] proved the following recurrence formula,

$$C_n(q) = \frac{q^n}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q + q\sum_{j=0}^{n-1}(1-q^{n-j})q^{(n+1-j)j}\begin{bmatrix}2j+1\\ j\end{bmatrix}_q C_{n-1-j}(q), \tag{2.8}$$

where $[x]_q = (q^x-1)/(q-1)$.

Recall that the q-shifted factorial $(x;q)_n$ is defined by

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad (n\geq 1) \quad \text{and} \quad (x;q)_0 = 1.$$

The two kinds of q-binomial coefficients are defined by

$$\begin{bmatrix}n\\ k\end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad \binom{x}{k}_q := \frac{x(x-1)\dots(x-[k-1]_q)}{[k]_q!},$$

with $\binom{x}{0}_q = 1$. Note that

$$\binom{[n]_q}{k}_q = q^{\binom{k}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_q, \qquad \binom{[-n]_q}{k}_q = (-1)^k q^{-kn}\begin{bmatrix}k+n-1\\ k\end{bmatrix}_q.$$

The q-derivative operator $\mathcal{D}_q$ and Hahn operator Δ_q are defined by

$$\mathcal{D}_q f(x) = \frac{f(qx)-f(x)}{(q-1)x} \quad \text{and} \quad \Delta_q f(x) = \frac{f(1+qx)-f(x)}{1+(q-1)x}. \tag{2.9}$$

Definition. *The sequence of polynomials $\{C_n(x|q)\}_{n\geq 1}$ is defined by the q-difference equation (1.1) or equivalently*

$$\frac{C_{n+1}(x|q) - C_{n+1}(q^{-1}x - q^{-1}|q)}{1+(q-1)x} = C_n(qx+1|q) \qquad (n\geq 1), \tag{2.10}$$

with the initial condition $C_1(x|q) = 1$ and $C_n(-\frac{1}{q}|q) = 0$ for $n\geq 2$.

For example, we have,

$$C_2(x|q) = 1 + q\binom{x}{1}_q,$$

$$C_3(x|q) = (1+q) + (q+q^2+q^3)\binom{x}{1}_q + q^4\binom{x}{2}_q,$$

$$\begin{aligned}C_4(x|q) = (q^3+q^2+2q+1) &+ (q^6+q^5+2q^4+2q^3+2q^2+q)\binom{x}{1}_q \\ &+ (q^9+q^8+q^7+q^6+q^5)q^{-1}\binom{x}{2}_q + q^9\binom{x}{3}_q.\end{aligned}$$

It is clear that $C_n(x|q)$ is a polynomial in $\mathbb{Q}(q)[x]$ of degree $n-1$ for $n\geq 1$.

3. Some preliminary results

We first show that the evaluation of the polynomials $C_n(x|q)$ at q-integers is always a polynomial in $\mathbb{N}[q]$. Note that formulae (3.1) and (3.4) were implicitly given in [10].

Proposition 3.1. *When $x = [k]_q$ we have*

$$C_{n+1}([k]_q|q) = q^{kn+\frac{n(n+1)}{2}} f(k+n, n|q^{-1}) \qquad (n,\, k \geq 0). \tag{3.1}$$

Proof. When $x = [k]_q$, (2.10) becomes

$$C_{n+1}([k]_q|q) = q^k C_n([k+1]_q|q) + C_{n+1}([k-1]_q|q). \tag{3.2}$$

It is easy to see that (3.1) is equivalent to (2.4). □

Corollary 3.2. *We have*

$$C_{n+1}(0|q) = C_n(1|q) \quad \textit{and} \quad C_{n+1}(1|q) = \widetilde{C}_{n+1}(q). \tag{3.3}$$

Proof. Letting $x = 0$ in (2.10) we get $C_{n+1}(0|q) = C_n(1|q)$. Letting $k = 1$ in (3.1) we have

$$\begin{aligned} C_{n+1}(1|q) &= q^{n+\frac{n(n+1)}{2}} f(1+n, n|q^{-1}) \\ &= q^{n+1+\frac{n(n+1)}{2}} f(n+1, n+1|q^{-1}) \\ &= q^{\frac{n(n+1)}{2}} C_{n+1}(q^{-1}), \end{aligned}$$

which is equal to $\widetilde{C}_{n+1}(q)$ by definition. □

The shifted factorial is defined by

$$(x)_0 = 1, \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{and} \quad (x)_{-n} = \frac{1}{(x-n)_n},$$

where $n = 1, 2, 3, \ldots$

Proposition 3.3. *When $q = 1$, we have the explicit formula*

$$C_{n+1}(x|1) = \frac{(x+1)(x+n+2)_{n-1}}{n!} = \frac{x+1}{x+1+n}\binom{x+2n}{n} \quad (n \geq 0). \tag{3.4}$$

Proof. When $q = 1$, equation (2.10) reduces to

$$C_{n+1}(x|1) = C_{n+1}(x-1|1) + C_n(x+1|1). \tag{3.5}$$

Since $C_{n+1}(x|1)$ is a polynomial in x of degree n, it suffices to prove that the right-hand side of (3.4) satisfy (3.5) for x being positive integers k. By Proposition 3.1 and (2.2) it suffices to check the following identity

$$\frac{k+1}{k+1+n}\binom{k+2n}{n} = \frac{k}{k+n}\binom{k-1+2n}{n} + \frac{k+2}{k+1+n}\binom{k+2n-1}{n-1}. \tag{3.6}$$

This is straightforward. □

To motivate our result in the next section we first prove two q-versions of a folklore result on the polynomials which take integral values on integers (see [17, p. 38]). Introduce the polynomials $p_k(x)$ by

$$p_0(x) = 1 \quad \text{and} \quad p_k(x) = (-1)^k q^{-\binom{k}{2}} \frac{(x-1)(x-q)\cdots(x-q^{k-1})}{(q;q)_k}, \quad k \geq 1.$$

So $p_k(q^n) = {n \brack k}_q$ for $n \in \mathbb{N}$.

Proposition 3.4. *The following statements hold true.*

(i) *The polynomial $f(x)$ of degree k assumes values in $\mathbb{Z}[q]$ at $x = 1, q, \ldots, q^k$ if and only if*

$$f(x) = c_0 + c_1 p_1(x) + \cdots + c_k p_k(x), \tag{3.7}$$

where $c_j = q^{\binom{j}{2}}(1-q)^j \mathcal{D}_q^j f(1)$ are polynomials in $\mathbb{Z}[q]$ for $0 \leq j \leq k$.

(ii) *The polynomial $\tilde{f}(x)$ of degree k assumes values in $\mathbb{Z}[q]$ at $x = 0, [1]_q, \ldots, [k]_q$ if and only if*

$$\tilde{f}(x) = \sum_{j=0}^{k} \tilde{c}_j q^{-\binom{j}{2}} \binom{x}{j}_q, \tag{3.8}$$

where $\tilde{c}_j = q^{\binom{j}{2}} \Delta_q^j \tilde{f}(0)$ are polynomials in $\mathbb{Z}[q]$ for $0 \leq j \leq k$.

Proof. Clearly we can expand any polynomial $f(x)$ of degree k in the basis $\{p_j(x)\}_{0 \leq j \leq k}$ as in (3.7). It is easy to see that

$$\mathcal{D}_q p_k(x) = \frac{q^{1-k}}{1-q} p_{k-1}(x) \Longrightarrow \mathcal{D}_q^j p_k(x) = \frac{q^{\binom{j+1}{2}-jk}}{(1-q)^j} p_{k-j}(x). \tag{3.9}$$

Hence, applying $\mathcal{D}_q^j$ to the two sides of (3.7) we obtain

$$\mathcal{D}_q^j f(1) = c_j \frac{q^{-\binom{j}{2}}}{(1-q)j} \Longrightarrow c_j = q^{\binom{j}{2}}(1-q)^j \mathcal{D}_q^j f(1).$$

Since $\mathcal{D}_q^j f(1)$ involves only the values of $f(x)$ at $x = 0, [1], \ldots, [k]_q$ for $0 \leq j \leq k$, the result follows. In the same manner, since

$$\Delta_q \binom{x}{k}_q = \binom{x}{k-1}_q,$$

we obtain the expansion (3.8). $\square$

Remark. (1) *We can also derive* (3.8) *from* (3.7) *as follows. Let $y = (x-1)/(q-1)$. For any polynomial $f(x)$ define $\tilde{f}(y) = f(1+(q-1)y)$. Since $q^n = 1 + (q-1)[n]_q$, it is clear that $f(q^n) \in \mathbb{Z}[q]$ if and only if $\tilde{f}([n]_q) \in \mathbb{Z}[q]$. Writing $1 + qx - [j]_q = q(x - [j-1]_q)$ we see that*

$$\tilde{f}_j(x) = q^{-\binom{j}{2}} \binom{x}{j}_q.$$

The expansion (3.8) *follows from* (3.7) *immediately.*

(2) *When* $\tilde{f}(x)=x^n$, *it is known (see, for example,* [18]*) that*

$$\tilde{c}_k=\Delta_q 0^n=[k]_q!S_q(n,k),$$

where $[k]_q!=[1]_q\cdots[k]_q$ *and* $S_q(n,k)$ *are classical* q*-Stirling numbers of the second kind defined by*

$$S_q(n,k)=S_q(n-1,k-1)+[k]_qS_q(n-1,k) \qquad \textit{for} \quad n\geq k\geq 1,$$

with $S_q(n,0)=S_q(0,k)=0$ *except* $S_q(0,0)=1$.

(3) *The two formulas* (3.7) *and* (3.8) *are special cases of the Newton interpolation formula, namely, for any polynomial* f *of degree less than or equal to* n *one has*

$$f(x)=\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\frac{f(b_j)}{\prod_{r=0,r\neq j}^{k}(b_j-b_r)}\right)(x-b_0)\cdots(x-b_{k-1}), \tag{3.10}$$

where $b_0,b_1,\ldots,b_{n-1}$ *are distinct complex numbers. Some recent applications of* (3.10) *in the computation of moments of Askey–Wilson polynomials are given in* [14].

4. Main results

In the light of Propositions 3.1 and 3.4, it is natural to consider the expansion of $C_{n+1}(x|q)$ and $C_{n+1}(qx+1|q)$ in the basis $\binom{x}{j}_q$ $(j\geq 0)$. It turns out that the coefficients in such expansions are Carlitz–Riordan's q-ballot numbers. Note that formula (4.2) was implicitly given in [10].

Theorem 4.1. *For* $n\geq 0$ *we have*

(4.1)
$$C_{n+1}(x|q)=\sum_{j=0}^{n}f(n+j,n-j|q^{-1})q^{jn+\frac{1}{2}(n-j)(n+j+1)}\binom{x}{j}_q,$$

(4.2)
$$C_n(qx+1|q)=\sum_{j=0}^{n-1}f(n+j,n-1-j|q^{-1})q^{jn+\frac{1}{2}n(n+1)-\frac{1}{2}(j+1)(j+2)}\binom{x}{j}_q.$$

Proof. It is sufficient to prove the theorem for $x=[k]_q$ with $k=0,1,\ldots,n$. By Proposition 3.1, the two equations (4.1) and (4.2) are equivalent to

$$f(k+n,n|q^{-1})=\sum_{j=0}^{k}f(n+j,n-j|q^{-1})q^{jn-kn-j}\begin{bmatrix}k\\j\end{bmatrix}_q, \tag{4.3}$$

$$f(k+n,n-1|q^{-1})=\sum_{j=0}^{k}f(n+j,n-j-1|q^{-1})q^{jn-kn-2j+k}\begin{bmatrix}k\\j\end{bmatrix}_q, \tag{4.4}$$

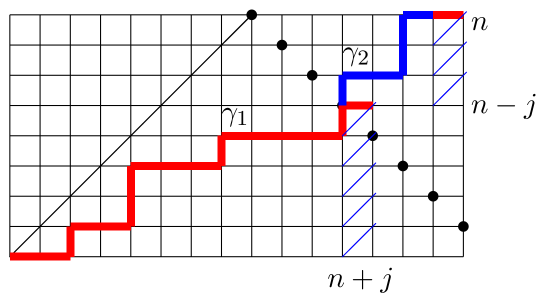

FIGURE 1. The decomposition $\gamma \mapsto (\gamma_1, \gamma_2)$.

for $n \geq k \geq j$. Replacing q by $1/q$ and using $\begin{bmatrix} k \\ j \end{bmatrix}_{q^{-1}} = q^{-j(k-j)} \begin{bmatrix} k \\ j \end{bmatrix}_q$ we get

$$f(n+k, n|q) = \sum_{j=0}^{k} f(n+j, n-j|q) q^{(n-j)(k-j)+j} \begin{bmatrix} k \\ j \end{bmatrix}_q, \tag{4.5}$$

$$f(k+n, n-1|q) = \sum_{j=0}^{k} f(n+j, n-j-1|q) q^{(n-j-1)(k-j)+j} \begin{bmatrix} k \\ j \end{bmatrix}_q. \tag{4.6}$$

We only prove (4.5). By definition, the left-hand side $f(n+k, n|q)$ is the enumerative polynomial of lattice paths from $(0,0)$ to $(n+k+1, n)$ with $(1,0)$ as the last step. Each such path γ must cross the line $y = -x + 2n$. Suppose it crosses this line at the point $(n+j, n-j)$, $0 \leq j \leq k$. Then the path corresponds to a unique pair (γ_1, γ_2), where γ_1 is a path from $(0,0)$ to $(n+j, n-j)$ and γ_2 is a path from $(n+j, n-j)$ to $(n+k, n)$. It is clear that the area under the path γ is equal to $S_1 + S_2 + S_3 + j$, where

- S_1 is the area under the path γ_1', which is obtained from γ_1 plus the last step $(n+j, n-j) \to (n+j+1, n-j)$;
- S_2 is the area under the path γ_2 and above the line $y = n-j$;
- S_3 is the area of the rectangle delimited by the four lines $y = 0$, $y = n-j$, $x = n+j+1$ and $x = n+k+1$, i.e., $(n-j)(k-j)$.

This decomposition is depicted in Figure 1. Clearly, summing over all such lattice paths gives the summand on the right-hand side of (4.5). This completes the proof.

□

Remark. *When $q = 1$, by (2.2), the above theorem implies that*

$$\frac{x+1}{x+n+1}\binom{x+2n}{n} = \sum_{j=0}^{n} \frac{2j+1}{n+j+1}\binom{2n}{n-j}\binom{x}{j}, \tag{4.7}$$

$$\frac{x+2}{x+n+1}\binom{x+2n-1}{n-1} = \sum_{j=0}^{n-1} \frac{2j+2}{n+j+1}\binom{2n-1}{n-j-1}\binom{x}{j}. \tag{4.8}$$

Note that the two sequences

$$\{f(n+j,n-j|1)\} \quad \text{and} \quad \{f(n+j,n-j-1|1)\} \qquad (0\le j\le n)$$

correspond, respectively, to the $(2n-1)$*-th and* $2n$*-th anti-diagonal coefficients of the triangle* $\{f(n,k)\}_{0\le k\le n}$*, see Table 1.*

Theorem 4.2. *The polynomials* $C_n(x|q)$ *satisfy* $C_1(x|q)=1$ *and*

$$[n]_qC_{n+1}(x|q) = ([2n-1]_q + xq^{2n-1})C_n(x|q) + \sum_{j=0}^{n-2}[n-j-1]_q\widetilde{C}_j(q)C_{n-j}(x|q)q^{2j+1}. \tag{4.9}$$

Proof. Since $C_{n+1}(x|q)$ is a polynomial in x of degree n, it suffices to prove (4.9) for $x=[k]_q$, where k is any positive integer, namely,

$$[n]_qC_{n+1}([k]_q|q) = [2n+k-1]_qC_n([k]_q|q) + \sum_{j=0}^{n-2}[n-j-1]_q\widetilde{C}_j(q)C_{n-j}([k]_q|q)q^{2j+1}. \tag{4.10}$$

Let $m\ge n$ and

$$\tilde f(m,n|q) = q^{(m-n)n+\binom{n+1}{2}}f(m,n|q^{-1}). \tag{4.11}$$

In view of the definition (2.3) it is clear that

$$\tilde f(m,n|q) = \sum_{\gamma\in\mathcal{P}(m,n)} q^{A'(\gamma)},$$

where $A'(\gamma)$ denotes the area above the path γ and under the line $y=x$ and $y=n$. Since

$$\widetilde{C}_j(q) = q^{\binom{j}{2}}C_j(q^{-1}) = q^{\binom{j+1}{2}}f(j,j|q^{-1}) = \tilde f(j,j|q),$$

using Proposition 3.1 and (4.11) with $m=k+n$, we can rewrite (4.10) as

$$[n]_q\tilde f(m,n|q) = [n+m-1]_q\tilde f(m-1,n-1|q) + \sum_{j=0}^{n-2} q^j[n-j-1]_q\tilde f(j,j|q)q^{j+1}[n-j-1]_q\tilde f(m-j-1,n-j-1|q). \tag{4.12}$$

A pointed lattice path is a pair (α,γ) such that $\alpha\in\{(1,1),\dots,(n,n)\}$ and $\gamma\in\mathcal{P}(m,n)$. If $\alpha=(i,i)$ we call i the height of α and write $h(\alpha)=i$. Let $\mathcal{P}^*(m,n)$ be the set of all such pointed lattice paths. It is clear that the left-hand side of (4.12) has the following interpretation

$$[n]_q\tilde f(m,n|q) = \sum_{(\alpha,\gamma)\in\mathcal{P}^*(m,n)} q^{h(\alpha)-1+A'(\gamma)}. \tag{4.13}$$

Now, we compute the above enumerative polynomial of $\mathcal{P}^*(m,n)$ in another way in order to obtain the right-hand side of (4.12). We distinguish two cases.

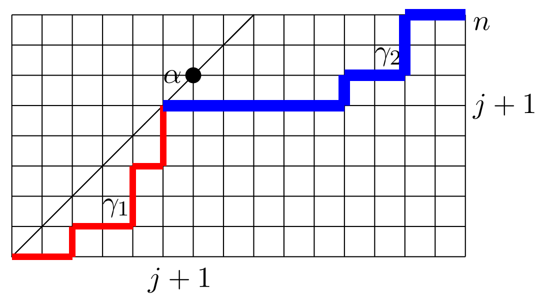

FIGURE 2. $(\alpha,\gamma) \to (\gamma_1, (\alpha', \gamma_2))$

- Let $\mathcal{P}_1^*(m,n,j)$ be the set of all pointed lattice paths (α,γ) in $\mathcal{P}^*(m,n)$ such that $h(\alpha) \in \{j+2,\ldots,n\}$, where j is the smallest integer such that $(j+1,j) \to (j+1,j+1)$ is a step of γ, i.e., the first step of γ touching the line $y=x$. If $(\alpha,\gamma) \in \mathcal{P}_1^*(m,n,j)$, then we have the correspondence $(\alpha,\gamma) \to (\gamma_1,(\alpha',\gamma_2))$, where γ_1 is a lattice path from $(0,0)$ to $(j+1,j+1)$ which touches the line $y=x$ only at the two extremities, and (α',γ_2) is a pointed lattice path from $(0,0)$ to $(m-j,n-j-1)$ with $h(\alpha') = h(\alpha)-j-1$. This decomposition is depicted in Figure 2.

  Thus the corresponding enumerative polynomial of such paths for the fixed j is

$$\sum_{(\alpha,\gamma)\in\mathcal{P}_1^*(m,n,j)} q^{h(\alpha)-1+A(\gamma)}$$
$$= q^j \tilde{f}(j,j|q)\cdot q^{j+1}[n-j-1]_q \tilde{f}(m-j-1,n-j-1|q).$$

  Summing over all j $(0 \le j \le n-2)$ we obtain the second term on the right-hand side of (4.12).

- Let $\mathcal{P}_2^*(m,n)$ be the set of all pointed lattice paths (α,γ) in $\mathcal{P}^*(m,n)$ such that $h(\alpha) \in \{1,\ldots,n\}$ and $h(\alpha) \le j+1$ where j (if any) is the smallest integer such that $(j+1,j) \to (j+1,j+1)$ is a step of γ, i.e., the first step of γ touching the line $y=x$. If $(\alpha,\gamma)\in\mathcal{P}_2^*(m,n)$, where $\gamma = (p_0,\ldots,p_{m+n+1})$ with $p_0=(0,0)$ and $p_{m+n+1} = (m+n+1,n)$, we can associate a pair (i,γ') where $\gamma' \in \mathcal{P}(m-1,n-1)$ is obtained from γ by deleting the vertical step $(x,h(\alpha)-1) \to (x,h(\alpha))$ and the first horizontal step $(0,0)\to(1,0)$, i.e.,

$$\gamma' = (p_1',\ldots,p_i',p_{i+2}',\ldots,p_{n+m+1}')$$

  where $i = x+h(\alpha)-1$, $p_k' = p_k-(1,0)$ if $k=1,\ldots,i$ and $p_k' = p_k-(0,1)$ if $k=i+2,\ldots,m+n+1$. It is easy to see that the mapping $(\alpha,\gamma)\mapsto(i,\gamma')$ is a bijection, which is depicted in Figure 3.

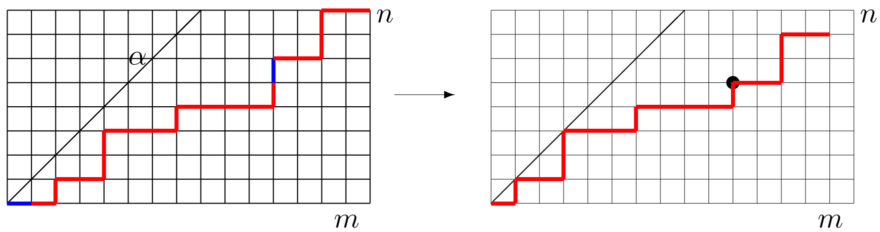

FIGURE 3. $(\alpha,\gamma)\mapsto(i,\gamma')$ with $m=15$, $n=8$, $\alpha=(6,6)$ and $i=15$.

Since $1\le x\le m$ and $0\le h(\alpha)-1\le n-1$ we have $i\in\{1,\dots,m+n-1\}$. As $A'(\gamma)=x-1+A'(\gamma')$ we have

$$h(\alpha)-1+A'(\gamma)=i-1+A'(\gamma').$$

It follows that

$$\sum_{(\alpha,\gamma)\in\mathcal{P}_2^*(m,n)} q^{h(\alpha)+A'(\gamma)}=\sum_{i=1}^{m+n-1} q^{i-1}\sum_{\gamma'} q^{A'(\gamma')}=[n+m-1]_q\tilde f(m-1,n-1|q).$$

Summing up the two cases we obtain the right-hand side of (4.12). $\square$

When $q=1$ we have an alternative proof of Theorem 4.2.

Another proof of the $q=1$ case. When $q=1$ (4.9) reduces to

$$nC_{n+1}(x|1)=(2n-1+x)C_n(x|1)+\sum_{j=0}^{n-2}(n-j-1)C_jC_{n-j}(x|1)\quad (n\ge 2). \tag{4.14}$$

This yields immediately $C_1(x|1)=1$, $C_2(x|1)=x+1$, $C_3(x|1)=(x+1)(x+4)/2$, in accordance with the formula (3.4). For $n\ge 3$, letting $k=n-j-3$, $N=n-3$ and $z=x+3$, by (3.4), the recurrence (4.14) is equivalent to the following identity

$$\frac{(z+N+2)_N}{N!}=\sum_{k=0}^{N}4^{N-k}\frac{(3/2)_{N-k}(z+k)_k}{(3)_{N-k}k!}\qquad (N\ge 0).$$

Notice that we can rewrite the right-hand side as

$$\frac{(3/2)_N}{(3)_N}4^N\sum_{k=0}^{N}\frac{(-2-N)_k((z+1)/2)_k(z/2)_k}{k!(-1/2-N)_k(z)_k}$$

$$= \frac{(3/2)_N}{(3)_N}4^N\Bigg({}_3F_2\left(\begin{matrix}-2-N, & (z+1)/2, & z/2 \\ -1/2-N, & z, & \end{matrix};1\right) - \frac{(-2-N)_{N+1}((z+1)/2)_{N+1}(z/2)_{N+1}}{(-1/2-N)_{N+1}(z)_{N+1}(N+1)!} - \frac{(-2-N)_{N+2}((z+1)/2)_{N+2}(z/2)_{N+2}}{(-1/2-N)_{N+2}(z)_{N+2}(N+2)!}\Bigg).$$

Invoking the Pfaff-Saalschütz formula [2, Theorem 2.2.6] we obtain

$${}_3F_2\left(\begin{matrix}-2-N, & (z+1)/2, & z/2 \\ -1/2-N, & z, & \end{matrix};1\right) = \frac{(z/2)_{N+2}((z-1)/2)_{N+2}}{(z)_{N+2}(-1/2)_{N+2}}.$$

Substituting this in the previous expression yields $(z+N+2)_N/N!$ after simplification. □

When $x = 1$ Eq. (4.14) reduces to the following identity for Catalan numbers:

$$nC_{n+1} = 2nC_n + \sum_{j=0}^{n-2}(n-j-1)C_jC_{n-j}. \tag{4.15}$$

5. Concluding remarks

We conclude this paper with a few open problems. By Theorem 4.2 it is clear that $C_{n+1}(x|q)$ is a polynomial in x of degree n with leading coefficient $q^{n^2}/[1]_q[2]_q\cdots[n]_q$. Let

$$P_n(x|q) := [1]_q[2]_q\cdots[n-1]_qC_n(x|q).$$

It follows from Theorem 4.1 that

$$P_{n+1}(x|q) = ([2n-1]_q + xq^{2n-1})P_n(x|q) + \sum_{j=0}^{n-2}\widetilde{C}_j(q)[n-j-1]_q\cdots[n-1]_qP_{n-j}(x|q)q^{2j+1}. \tag{5.1}$$

This immediately implies that $P_n(x|q)$ is a polynomial in x and q with positive integral coefficients. On the other hand, it is clear from Theorem 4.2 that $P_n(x|q)$ is divisible by $qx+1$ for $n \geq 2$. The following observation has been checked up to $n = 27$.

Conjecture (Irreducible polynomial). *For $n \geq 6$ the quotient*

$$P_n(x|q)/(qx+1)$$

is an irreducible polynomial in x and q with positive integral coefficients.

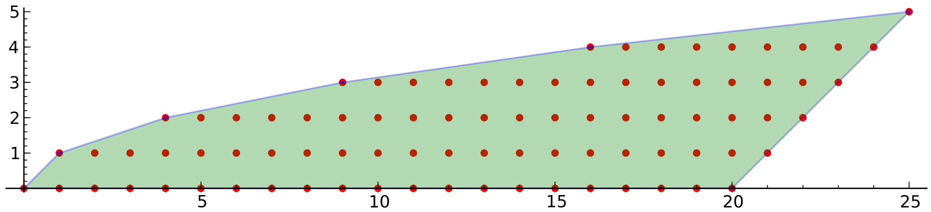

FIGURE 4. Newton polygon of the numerator of $C_n(x|q)$ for $n = 6$.

Corollary 5.1. *For $n \geq 1$ there are polynomials $g_n(q) \in \mathbb{N}[q]$ such that*

$$\lim_{x \to -1/q} \frac{C_{n+1}(x|q)}{1+qx} = \frac{g_n(q)}{[n]_q}, \tag{5.2}$$

where $g_0(q) = 1$ and

$$g_n(q) = (1+q^{n-1})g_{n-1}(q) + \sum_{j=0}^{n-2} \widetilde{C}_j(q) g_{n-j-1}(q) q^{2j+1}. \tag{5.3}$$

Proof. It is clear that (5.2) is true for $n = 1$. Suppose that it is true until $n-1$ with $n \geq 2$. We derive (5.3) from (4.9). □

The first values of $g_n(q)$ are $1,\ 1,\ 2q+1,\ 3q^3+3q^2+3q+1$. Let $F(x) := \sum_{n\geq 0} \widetilde{C}_n(q)x^n$ and $G(x) := \sum_{n\geq 1} g_n(q)x^n$. We conjecture that the following functional equation holds true:

$$G(x) = xF(x)F(qx)\left(1+G(qx)\right). \tag{5.4}$$

It is not difficult to see that the above equation would provide a combinatorial interpretation for the polynomials $g_n(q)$ in the model of lattice paths.

Let us mention that the Newton polygon of the polynomial P_n has a simple shape. This is illustrated in Figure 4, where the horizontal axis is associated with powers of q and the vertical axis with powers of x. The slopes of the upper part are given in general by the odd integers $1, 3, \ldots, 2n-3$. This shape follows in a direct way by induction from the formula (5.1) for the polynomials P_n. In fact, the Newton polygon of the first term of the right hand side of (5.1) contains the Newton polygons of the other terms.

Finally, there should be analogs of our results for Cigler's polynomials $G_n(x, r)$ for any r, as introduced in [10, 11].

REFERENCES

1. G. Andrews, *Identities in combinatorics. II. A q-analog of the Lagrange inversion theorem*, Proc. Amer. Math. Soc. **53** (1975), no. 1, 240–245.
2. G. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
3. S. A. Blanco and T. Kyle Petersen, *Counting Dyck paths by area and rank*, arXiv:1206.0803v1.

4. L. M. Butler and W. P. Flanigan, *A note on log-convexity of q-Catalan numbers*, Ann. Comb. **11** (2007), no. 3–4, 369–373.
5. L. Carlitz, *Sequences, paths, ballot numbers*, Fibonacci Quart. **10** (1972), no. 5, 531–549.
6. L. Carlitz and J. Riordan, *Two element lattice permutation numbers and their q-generalization*, Duke Math. J. **31** (1964), no. 3, 371–388.
7. F. Chapoton, *Sur une série en arbres à deux paramètres*, Séminaire Lotharingien de Combinatoire **70** (2013), 1–20.
8. ———, *Flows on rooted trees and the Menous-Novelli-Thibon idempotents*, Mathematica Scandinavica **115** (2014), no. 1.
9. J. Cigler, *q-Catalan numbers and q-Narayana polynomials*, arXiv:math/0507225.
10. ———, *Operatormethoden für q-Identitäten. IV. Eine Klasse von q-Gould-Polynomen*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **205** (1996), 169–174.
11. ———, *Operatormethoden für q-Identitäten. VI. Geordnete Wurzelbäume und q-Catalan-Zahlen*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **206** (1997), 253–266.
12. L. Comtet, *Advanced combinatorics*, D. Reidel, Dordrecht, Holland, 1974.
13. J. Fürlinger and J. Hofbauer, *q-Catalan numbers*, J. Combin. Theory Ser. A **40** (1985), 248–264.
14. V. J. W. Guo, M. Ishikawa, H. Tagawa, and J. Zeng, *A quadratic formula for basic hypergeometric series related to Askey–Wilson polynomials*, Proc. Amer. Math. Soc. **143** (2015), no. 5, 2003–2015.
15. J. Haglund, *The q,t-Catalan numbers and the space of diagonal harmonics with an appendix on the combinatorics of Macdonald polynomials*, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008.
16. M. Reineke, *Cohomology of noncommutative Hilbert schemes*, Algebr. Represent. Theory **8** (2005), no. 4, 541–561.
17. R. Stanley, *Enumerative combinatorics. Vol. I. with a forword by Gian-Carlos Rota*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Moneterey, CA, 1986.
18. J. Zeng, *The Akiyama–Tanigawa algoritheorem for Carlitz's q-Bernoulli numbers*, Integers **6** (2006), no. A05.

Université de Lyon, Université Lyon 1, Institute Camille Jordan, UMR 5028 du CNRS, 69622 Villeurbanne, France
E-mail address: `chapoton@math.univ-lyon1.fr`

Université de Lyon, Université Lyon 1, Institute Camille Jordan, UMR 5028 du CNRS, 69622 Villeurbanne, France
E-mail address: `zeng@math.univ-lyon1.fr`