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ON THE CHROMATIC INDEX OF LATIN SQUARES

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Abstract. A proper coloring of a Latin square of order n is an assignment of colors to its elements triples such that each row, column and symbol is assigned n distinct colors. Equivalently, a proper coloring of a Latin square is a partition into partial transversals. The chromatic index of a Latin square is the least number of colors needed for a proper coloring. We study the chromatic index of the cyclic Latin square which arises from the addition table for the integers modulo n. We obtain the best possible bounds except for the case when $n/2$ is odd and divisible by 3. We make some conjectures about the chromatic index, suggesting a generalization of Ryser's conjecture (that every Latin square of odd order contains a transversal).

1. Introduction

A *Latin square* L of order n is an $n \times n$ array of symbols chosen from a set S of size n such that each symbol occurs exactly once in each row and exactly once in each column. If we replace "exactly" by "at most" in the definition, such an object is a *partial Latin square*, where some cells are possibly empty. For convenience we assume $S = \{0, 1, \ldots, n-1\}$ (also labelling rows and columns with the set S) and we represent any partial Latin square as a set of ordered triples (i, j, k) where symbol k occurs in row i and column j.

An *m-coloring* is any surjective map $C : L \to M$ (where $|M| = m$); C is said to be *proper* if $C((i,j,k)) = C((i',j',k'))$ and $(i,j,k) \neq (i',j',k')$ implies that $i \neq i'$, $j \neq j'$ and $k \neq k'$. (That is, the mapping C is injective on the rows, columns and symbols.) The *chromatic index* $\chi'(L)$ of a Latin square L is the least integer m such that a proper m-coloring exists. (We use the notation χ' as we are properly coloring triples as if they are edges in a hypergraph; this is consistent with work done on Steiner triple systems; see below).

A *partial transversal* P of order n is a partial Latin square of order n such that each row, column or symbol is used at most once. Observe that a proper m-coloring of a Latin square L of order n is equivalent to a partition of L into m partial transversals. A partial transversal of order n and size n is a

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transversal. A recent survey on transversals (and generalizations thereof) is given in [7]. The following observation is immediate.

Lemma 1.1. *For any Latin square L of order n, $\chi'(L) \geqslant n$ with equality if and only if L has an orthogonal mate.*

We make the following conjectures.

Conjecture 1.2. *If L is a Latin square of odd order n, $\chi'(L) \leqslant n+1$.*

Conjecture 1.3. *If L is a Latin square of even order n, $\chi'(L) \leqslant n+2$.*

Ryser [6], (see also the survey of [2, Section 1.4]) conjectured that every Latin square of odd order has a transversal. In a conjecture attributed to Brualdi [3, p. 103], every Latin square of order n possesses a partial transversal of size $n-1$.

Conjectures 1.2 and 1.3 (if true) may, in general, be difficult to prove because of the following observations.

Lemma 1.4. *Conjecture* 1.2 *implies Ryser's conjecture.*

Proof. If Ryser's conjecture is false, there exists a Latin square L of odd order such that any partial transversal has size at most $n-1$. If $\chi'(L) \leqslant n+1$, we may color at most $(n+1)(n-1)$ triples of L, a contradiction. □

The next lemma follows by a similar argument.

Lemma 1.5. *Conjectures* 1.2 *and* 1.3 *together imply Brualdi's conjecture.*

Indeed, if $\chi'(L) \leqslant n+2$ for any Latin square L of order n, then Brualdi's conjecture is true.

We show next that Conjectures 1.2 and 1.3 cannot be improved. To this end, for each $n \geqslant 1$, we define B_n to be the Latin square given by the operation table for the integers modulo n. (That is, any cell (i,j) contains the symbol $i+j$ evaluated modulo n.) It is well known that B_n contains a transversal if and only if n is odd [2]. The following lemma is immediate.

Lemma 1.6. *If n is even, there exists a Latin square L of order n such that $\chi'(L) \geqslant n+2$.*

Moreover, in [8], it is shown that for each order $n > 3$, there exists a Latin square without a disjoint mate.

Lemma 1.7. *For each $n > 3$, there exists a Latin square L such that $\chi'(L) \geqslant n+1$.*

An *equitable* m-coloring of a Latin square L is a proper m-coloring of L in which the cardinality of any two color classes differ by at most 1. The least integer m such that an equitable m-coloring of L exists is called the equitable chromatic index of L. The notation we use is $\chi'_{eq}(L)$. We conclude our conjecturing with the following.

Conjecture 1.8. *For each Latin square L, $\chi'_{eq}(L) = \chi'(L)$.*

We exhibit equitable colorings of B_4 and B_6 found by hand in Figure 1. The subscripts denote the colors. These examples prove that $\chi'(B_4) = \chi'_{eq}(B_4) = 6$ and $\chi'(B_6) = \chi'_{eq}(B_6) = 8$.

0_A	1_C	2_D	3_B
1_F	2_A	3_C	0_D
2_B	3_E	0_F	1_A
3_D	0_B	1_E	2_C

0_A	1_B	2_H	3_G	4_F	5_E
1_H	2_A	3_D	4_E	5_G	0_B
2_E	3_F	4_A	5_B	0_D	1_C
3_B	4_G	5_C	0_H	1_A	2_F
4_C	5_D	0_E	1_F	2_B	3_A
5_F	0_C	1_G	2_D	3_E	4_H

FIGURE 1. Equitable colorings of B_4 and B_6.

A trivial upper bound for $\chi'(L)$ is given by applying Brook's theorem; since each triple of L shares a row, column or symbol with exactly $3(n-1)$ other triples of L, $\chi'(L) \leqslant 3(n-1)$. Asymptotic upper bounds for $\chi'(L)$ are implied by [5, Theorem 1.1] which applies to a more general class of hypergraphs. Specifically,

Theorem 1.9 ([5])**.** *For each $\delta > 0$, there exists $n_0 > 0$ such that if L is a Latin square of order $n \geqslant n_0$, $\chi'(L) \leqslant (1+\delta)n$.*

In fact, [5, Theorem 1.1] implies a similar result on coloring sufficiently dense partial Latin squares.

It would be amiss not to mention some of the analogous work on the chromatic index of Steiner triple systems, the inspiration of this paper. A *Steiner triple system* S is a set V of *points* and a set B of *blocks* (each subsets of V of size 3) such that each pair of points lies in exactly one block. A *block-color class* is a set of pairwise disjoint triples; while an *m-block coloring* is a partition of B into m block-color classes. The *chromatic index* $\chi'(S)$ of a Steiner triple system S is the least value for m such that an m-block coloring exists.

It is not hard to show the lower bound $\chi'(S) \geqslant (|V|-1)/2$ and applying Brooks' theorem to the block intersection graph yields the upper bound $\chi'(S) \leqslant 3(|V|-3)/2$. Pippenger and Spencer showed in [5] that $\chi'(S)/|V|$ is asymptotically equal to $1/2$. Meszka, Nedela and Rosa showed in [4] that if S may be defined cyclically, then $\chi'(S) \leqslant 2|V|/3 - c$ (where $c \in \{2/3, 1\}$), improving a result by Colbourn and Colbourn [1].

We now focus on whether the above conjectures are true for the Latin square B_n, $n \geqslant 1$. From the discussion above, $\chi'(B_n) = n$ if n is odd. Thus we focus on the case when n is even. Our main results are as follows.

Theorem 1.10. *Let $n \equiv 2 \pmod 4$ and $n \not\equiv 0 \pmod 3$. Then $\chi(B_n) = \chi'_{eq}(B_n) = n+2$.*

Theorem 1.11. *Let $n \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 3$. Then $\chi(B_n) \leqslant n+3$.*

Theorem 1.12. *Let $n \equiv 0 \pmod 4$. Then $\chi(B_n) = n + 2$.*

We prove these theorems in Sections 2, 3 and 4, respectively.

2. The case when $n/2$ is odd and n is not divisible by 3.

Let $n/2$ be odd and not divisible by 3. We construct an equitable coloring for B_n as follows. Since any cell (i, j) contains $i + j$ (evaluated mod n), it suffices to specify only the cells of each partial transversal.

All calculations below are taken modulo n, unless otherwise stated. We define an $\oplus$ operation so that for a set S of cells,

$$S \oplus (i, j) = \{(i' + i, j' + j) \mid (i', j') \in S\}.$$

Let $N = n/2$. Consider the sets of cells

$$L_0 := \{(i, i), (i, i + 1) \mid 0 \leqslant i \leqslant n - 1\}$$

and

$$T_0 := \{(i, i) \mid 0 \leqslant i \leqslant N - 1\} \cup \{(N + i, N + 1 + i) \mid 0 \leqslant i \leqslant N - 2\}.$$

Define $T'_0 = T_0 \oplus (N, N)$. Observe that T_0 (and thus T'_0) are each partial transversals of size $n - 1$, $T_0, T'_0 \subset L_0$, $T_0 \cap T'_0 = \emptyset$ and that

$$L_0 \setminus (T_0 \cup T'_0) \quad = \quad \{(n - 1, 0), (n - 1, 0) \oplus (N, N)\}. \tag{2.1}$$

Next, for each j such that $0 \leqslant j \leqslant N - 1$, we define

$$L_j = L_0 \oplus (2j, 4j), \quad T_j = T_0 \oplus (2j, 4j) \quad \text{and} \quad T'_j = T_j \oplus (N, N).$$

Clearly the sets L_j, $0 \leqslant j \leqslant N - 1$ partition the sets of cells of B_n. It follows that the sets T_j, T'_j, $(0 \leqslant j \leqslant N - 1)$ are pairwise disjoint partial transversals, each of size $n - 1$.

From (2.1), the set of cells not so far used in a partial transversal is

$$\{(2j - 1, 4j), (N + 2j - 1, N + 4j) \mid 0 \leqslant j \leqslant N - 1\}.$$

Since N is odd, this set is in turn equal to

$$\{(i, (i + 1)(N + 2)) \mid 0 \leqslant i \leqslant n - 1\}.$$

Let $S := \{(i, (i + 1)(N + 2)) \mid 0 \leqslant i \leqslant N - 1\}$ and $S' := S \oplus (N, N)$. Since N is not divisible by 3, S and S' are each partial transversals of size N. Together with the partial transversals already constructed, we now have a set of $n + 2$ transversals which partition the cells of B_n.

Thus $\chi'(B_n) = n + 2$. We next adjust the above transversals to make an equitable coloring. To do so we need to extend the lengths of the partial transversals S and S'. With this aim in mind, let

$$U = \{(2N - 1 - i', (i' - 1)(N - 2)) \mid 1 \leqslant i' \leqslant N - 1\}$$

and let $U' = U \oplus (N, N)$. We claim that $U \cup S$ (and hence $U' \cup S'$) is a partial transversal of size $n - 2$. It is clear that S and U share no common row. Suppose that S and U share a common column. Then

$$(i' - 1)(N - 2) \equiv (i + 1)(N + 2) \pmod{2N}$$

for some i and i' in the ranges above. Thus

$$2i + 2i' \equiv (i' - i)N \pmod{2N}.$$

This implies that $i' - i$ is even so that $2i + 2i' \equiv 0 \pmod{2N}$, $i + i' \equiv 0 \pmod{N}$ and $i' \equiv N - i \pmod{2N}$ or $i' \equiv 2N - i \pmod{2N}$. The former contradicts $i' - i$ being even so we have $i' \equiv 2N - i \pmod{2N}$. Since $0 \leqslant i \leqslant N - 1$, we have that $2N - 1 \geqslant i' \geqslant N + 1$ or $i' = 0$, a contradiction.

Next, we claim that U and S share no common entries. If not, then

$$2N - 1 - i' + (i' - 1)(N - 2) \equiv i + (i + 1)(N + 2) \pmod{2N},$$

for i and i' in the above ranges. We then obtain a contradiction by simplifying modulo 2. Thus $U \cup S$ (and hence $U' \cup S'$) are each transversals of size $n - 1$.

It remains to show that U intersects $T_j \cup T'_j$ at most once for each j such that $0 \leqslant j \leqslant N - 1$. (If so, we can truncate partial transversals which intersect U by 1 to obtain an equitable partition into partial transversals.) We first check that $U \cap S' = \emptyset$. If not, working modulo N, we obtain $i \equiv -i' - 1 \pmod{N}$ and $2(i + 1) \equiv -2(i' - 1) \pmod{N}$, which together give a contradiction.

It is sufficient now to show that U intersects L_j at most once for each j such that $0 \leqslant j \leqslant N - 1$. First observe that if cell $(I(i'), J(i'))(= (I, J)) = (2N - 1 - i', (i' - 1)(N - 2)) \in U$, then $J - I \equiv 3 - i' + N(i' - 1) \pmod{2N}$. It follows that for any such i', $J - I$ is odd. However, cell $(I, J) \in L_x$ if and only if $\lfloor (J - I)/2 \rfloor \equiv x \pmod{N}$, so cells $(I(i'), J(i'))$ and $(I(i''), J(i''))$ belong to the same set L_x if and only if

$$(2 - i')/2 + N(i' - 1)/2 \equiv (2 - i'')/2 + N(i'' - 1)/2 \pmod{N},$$

implying that $i' \equiv i'' \pmod{N}$, which is what we needed to show.

We show how this construction works for $n = 10$ and $n = 14$ in Figures 2 and 3 respectively. For the sake of clarity, we only give the undashed partial transversals. The entry in a cell denotes the subscript of a transversal rather than the entry within the Latin square (rows and columns are ordered $0, 1, \ldots, n - 1$). The cells with ∞ denote elements of S and the cells with an ∞ subscript belong to U.

3. The case when $n/2$ is odd and n is divisible by 3.

In this section we prove Theorem 1.11. Let $N = n/2$ be odd and divisible by 3. We construct a proper $n + 3$-coloring for B_n as follows.

As in the previous section we define

$L_0 = \{(i, i), (i, i + 1) | 0 \leqslant i \leqslant n - 1\}$,
$T_0 = \{(i, i) |\ 0 \leqslant i \leqslant N - 1\} \cup \{(N + i, N + 1 + i) |\ 0 \leqslant i \leqslant N - 2\}$,
$T'_0 = T_0 \oplus (N, N)$.

Next, for each j such that $0 \leqslant j \leqslant N - 1$, we define

$$L_j = L_0 \oplus (j, 3j), \quad T_j = T_0 \oplus (j, 3j) \quad \text{and} \quad T'_j = T_j \oplus (N, N).$$

0			1		2	3	∞	4	
	0			∞		2		3	4
4	∞	0		1			2		3
3		4	0		1			∞	
	3		4	0	∞	1		2	
				4		0	1		2_∞
2		3			4	∞	0	1	
1	2		3_∞					0	
∞	1	2		3		4			0
		1		2	3		4		

FIGURE 2. An equitable coloring of B_{10} with $\chi'_{eq}(B_{10})$ colors.

0			1		2		3	4	∞	5		6	
	0			∞		2		3		4	5		6
6		0		1			2		3		4	5	∞
5	6		0		1			∞		3		4	
	5	6	∞	0		1		2			3		4
4		5		6	0		1		2			∞	
	4		5		6	0	∞	1		2		3	
				5		6		0	1		2_∞		3
3		4			5	∞	6		0	1		2	
	3_∞		4					6		0		1	2
2		3		4		5			6	∞	0		1
1		2	3		4_∞		5					0	
∞	1		2	3		4		5		6			0
		1		2		3	4		5		6		

FIGURE 3. An equitable coloring of B_{14} with $\chi'_{eq}(B_{14})$ colors.

Observe that the sets L_j, $0 \leqslant j \leqslant N-1$ partition the cells of B_n. It follows that the sets T_j, T'_j are pairwise disjoint partial transversals, each of size $n-1$.

From (1), the set of cells not so far used in a partial transversal is

$$\{(j-1, 3j)|\ 0 \leqslant j \leqslant n-1\}.$$

Let $Q = n/3$. These remaining cells can be partitioned into the following partial transversals:

$$\begin{aligned} T_{\infty_1} &= \{(j-1, 3j)|\ 0 \leqslant j \leqslant Q-1\}, \\ T_{\infty_2} &= \{(j-1, 3j)|\ Q \leqslant j \leqslant 2Q-1\}, \\ T_{\infty_3} &= \{(j-1, 3j)|\ 2Q-1 \leqslant j \leqslant 2n-1\}. \end{aligned}$$

Together with the partial transversals already constructed, we now have a set of $n+3$ transversals which partition the cells of B_n. Thus $\chi'(B_n) \leqslant n+3$ and $\chi'(B_n) \in \{n+2, n+3\}$.

Figure 4 contains an example of the construction for $N = 9$. As in the previous examples, we only give the undashed partial transversals. The entry in a cell of a partial transversal denotes the subscript of a transversal rather than the entry within the Latin square.

0			∞_1				3		4		5		6		7		8
8	0		1			∞_1		3		4		5		6		7	
	8	0		1		2			∞_1		4		5		6		7
7		8	0		1		2		3			∞_1		5		6	
	7		8	0		1		2		3		4			∞_1		6
∞_2		7		8	0		1		2		3		4		5		
6			∞_2		8	0		1		2		3		4		5	
	6		7			∞_2	0		1		2		3		4		5
5		6		7		8		0	∞_2	1		2		3		4	
	5		6		7		8			0	1	∞_2	2		3		4
4		5		6		7		8			0		1	2	∞_2	3	
∞_3	4		5		6		7		8			0		1		2	3
	3	4	∞_3	5		6		7		8			0		1		2
2		3		4	5	∞_3	6		7		8			0		1	
	2		3		4		5	6	∞_3	7		8			0		1
1		2		3		4		5		6	7	∞_3	8			0	
	1		2		3		4		5		6		7	8	∞_3		0
∞_1		1		2		3		4		5		6		7		8	

FIGURE 4. A coloring of B_{18} with $\chi'(B_{18})$ colors.

Given the construction above, it is easy to show that $\chi'_{eq}(B_n) \leqslant 3n+3$. Let $1 \leqslant j \leqslant N$. Since each partial transversal T_j has size $n-1$, T_j can be partitioned into two partial transversals of size Q and one of size $Q-1$. The same can be said of T'_j and T_{∞_i} has size Q for $i = 1, 2, 3$. This implies $3n+3$ partial transversals of size either Q or $Q-1$. As indicated by Conjecture 1.8, we believe this bound can be improved considerably.

4. The case when $n/2$ is even.

In this section we prove Theorem 1.12. The case $n = 4$ is done in Figure 1. We may thus assume that $n \geqslant 8$. Let $M = n/4$ and $N = n/2$. We use L_0, T_0, and T'_0 as they are defined in the previous sections. We next define the following sets L_j, T_j, and T'_j:

$$L_j = \begin{cases} L_0 \oplus (3j, j) & 0 \leqslant j \leqslant M-1; \\ L_{j-M} \oplus (2, N+2) & M \leqslant j \leqslant N-1 \text{ and } n \equiv 4 \pmod 8; \\ L_{j-M} \oplus (1, N+1) & M \leqslant j \leqslant N-1 \text{ and } n \equiv 0 \pmod 8; \end{cases}$$

$$T_j = \begin{cases} T_0 \oplus (3j, j) & 0 \leqslant j \leqslant M-1; \\ T_{j-M} \oplus (2, N+2) & M \leqslant j \leqslant N-1 \text{ and } n \equiv 4 \pmod 8; \\ T_{j-M} \oplus (1, N+1) & M \leqslant j \leqslant N-1 \text{ and } n \equiv 0 \pmod 8; \end{cases}$$

$$T'_j = T_j \oplus (N, N).$$

The sets L_j are pairwise disjoint and so partition the cells of B_n. It follows that the sets T_j, T'_j are pairwise disjoint partial transversals, each of size of $n-1$. From (1), the sets of cells not so far used in a partial transversal are

$$\begin{aligned} S_1 &= \{(3j-1, j) | \ 0 \leqslant j \leqslant M-1\}, \\ S_2 &= \{(3j+1, j+N+2)) | \ 0 \leqslant j \leqslant M-1\}, \\ S_3 &= S_1 \oplus (N, N), \\ S_4 &= S_2 \oplus (N, N) \end{aligned}$$

if $n \equiv 4 \pmod 8$, and

$$\begin{aligned} S_1 &= \{(3j-1, j) | \ 0 \leqslant j \leqslant M-1\}, \\ S_2 &= \{(3j, j+N+1)) | \ 0 \leqslant j \leqslant M-1\}, \\ S_3 &= S_1 \oplus (N, N), \\ S_4 &= S_2 \oplus (N, N) \end{aligned}$$

if $n \equiv 0 \pmod 8$. These remaining cells can be partitioned into the following partial transversals:

$$\begin{aligned} T_\infty &= S_1 \cup S_2, \\ T'_\infty &= T_\infty \oplus (N, N) \end{aligned}$$

if $n \equiv 4 \pmod 8$, and

$$\begin{aligned} T_\infty &= S_3 \cup S_4, \\ T'_\infty &= T_\infty \oplus (N, N) \end{aligned}$$

if $n \equiv 0 \pmod 8$. Thus, $\chi'(B_n) = n + 2$.

Figure 5 contains examples of the construction for $N = 4$ and $N = 6$.

Since each partial transversal above has size either N or $n-1$, it follows that $\chi'_{eq}(B_n) \leqslant 2n + 2$. Although the constructions in Sections 3 and 4 do not lend themselves well to Conjecture 1.8, we believe there is room for considerable improvement to $\chi'_{eq}(B_n)$.

0			3		∞		1
1	0			3	2		
	∞	0			3	2	
	1		0			∞	2
2		1			0	3	
		2	1			0	3
3			2	1			0
∞	3			2		1	

0		5			4		3		2		1
1	0		5			4		∞		2	
	∞	0			5		4	3			2
2	1		0			5		4	3		
	2	1		0			5		∞	3	
		∞	1		0			5	4		3
3		2		1			0		5	4	
	3		2		1			0		∞	4
4			3	2		1			0	5	
	4			3	2			1		0	5
5		4			3	2			1		0
∞	5			4		3	2			1	

FIGURE 5. A coloring of B_8 and B_{12} with $\chi'(B_8)$ and $\chi'(B_{10})$ colors respectively.

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