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ON THE MINIMUM ORDER OF k-COP-WIN GRAPHS

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Abstract. We consider the minimum order graphs with a given cop number k, and we focus especially on the cases $k = 2, 3$. We prove that the minimum order of a connected graph with cop number 3 is 10, and show that the Petersen graph is the unique isomorphism type of graph with this property. We provide the results of a computational search on the cop number of all graphs up to and including order 10. A relationship is presented between the minimum order of graph with cop number k and Meyniel's conjecture on the asymptotic maximum value of the cop number of a connected graph.

1. Introduction

Cops and Robbers is a vertex-pursuit game played on graphs that has been the focus of much recent attention. Throughout, we only consider finite, connected, and simple undirected graphs. There are two players consisting of a set of *cops* and a single *robber*. The game is played over a sequence of discrete time-steps or *rounds*, with the cops going first in the first round and then playing alternate time-steps. The cops and robber occupy vertices, and more than one cop may occupy a vertex. When a player is ready to move in a round they must move to a neighboring vertex. Players can *pass* by remaining on their own vertex. Observe that any subset of cops may move in a given round. The cops win if after some finite number of rounds, one of them can occupy the same vertex as the robber. This is called a *capture*. The robber wins if he can avoid capture indefinitely. A *winning strategy for the cops* is a set of rules that, if followed, result in a win for the cops, and a *winning strategy for the robber* is defined analogously.

If we place a cop at each vertex, then the cops are guaranteed to win. Therefore, the minimum number of cops required to win in a graph G is a well defined positive integer, named the *cop number* of the graph G. We

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write $c(G)$ for the cop number of a graph G, and say that a graph satisfying $c(G) = k$ is *k-cop-win*. For example, the Petersen graph is 3-cop-win. If $k = 1$, then we say that G is *cop-win*. Nowakowski and Winkler [13], and independently Quilliot [15], considered the game with one cop only. The introduction of the cop number came in [1]. Many papers have been written on cop number since these three early works; see the book [5] for additional references and background on the cop number.

Meyniel's conjecture is one of the deepest unsolved problems on the cop number. It states that for a connected graph G of order n, $c(G) \in O(\sqrt{n})$. Hence, the largest cop number of a graph is asymptotically bounded above by $d\sqrt{n}$ for a constant d. The conjecture has so far resisted all attempts to resolve it, and the best known bounds (see, for example, [11]) do not even prove that $c(G) \in O(n^{\varepsilon})$, for $\varepsilon < 1$. For a fixed positive integer k, define m_k to be the minimum order of a connected graph G satisfying $c(G) \geq k$. Define M_k to be the minimum order of a connected k-cop-win graph. It is evident that the m_k are monotonically increasing, and $m_k \leq M_k$.

Up until this study, only the first two values of these parameters were known: $m_1 = M_1 = 1$ and $m_2 = M_2 = 4$ (witnessed by the 4-cycle). We derive that $m_3 = M_3 = 10$. Interestingly, the Petersen graph is the unique isomorphism type of 3-cop-win graphs with order 10. In addition to a proof of this fact, we use a computer search to calculate the cop number of every connected graph on 10 or fewer vertices (there are nearly 12 million such unlabelled graphs). We perform this categorization by checking for cop-win orderings [13] and using an algorithm provided in [3]. We present these computational results in the next section.

In addition, we prove the following theorems.

Theorem 1.1. *If G is a graph on at most 9 vertices, then $c(G) \leq 2$.*

Theorem 1.2. *The Petersen graph is the unique isomorphism type of graphs on 10 vertices that are 3-cop-win.*

The proofs of Theorems 1.1 and 1.2—which are deferred to Section 4—exploit new ideas which are of interest in their own right. In particular, we prove a series of structural lemmas concerning the cop number of graphs containing a vertex whose co-degree is a small constant, namely with maximum degree at least $n - 7$, where n is the order of the graph.

Furthermore, we prove that Meyniel's conjecture is equivalent to bounds on the values m_k (see Theorem 3.1). We give lower bounds on the growth rates of the number of non-isomorphic k-cop-win graphs of a given order in Theorem 3.2.

For background on graph theory see [17]. We use the notation $v(G) = |V(G)|$. We write V and E, respectively, for the vertex set and the edge set of a graph G when G is clear from context. For $u, v \in V$, we write $u \sim v$ when $uv \in E$. For $S \subseteq V$, we write $u \sim S$ when $u \notin S$ and there exists $v \in S$ such that $u \sim v$. Given a vertex v, its *neighborhood* is $N(v) = \{u \in V \mid (v, u) \in E\}$, and its *closed neighborhood* is $N[v] = \{v\} \cup N(v)$. We define

$N(S) = \bigcup_{v \in S} N(v) \setminus S$ and $N[S] = \bigcup_{v \in S} N[v]$. For convenience, we use the notation $N(u, v) = N(\{u, v\})$. A vertex v is *dominated* by the vertex w if $N[v] \subseteq N[w]$. For $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. We use the notation $X \setminus Y$ for the difference of sets. We write $G - S$ for $G[V \setminus S]$, and $G - H$ for $G - V(H)$ when H is an induced subgraph of G. For disjoint sets of vertices $S, T \subseteq V$, we denote the set of edges between the two sets by $(S : T) = \{st \in E \mid s \in S, t \in T\}$, and we use $|(S : T)|$ to denote the cardinality of this set. We denote the minimum degree of a vertex in G by $\delta(G)$ and the maximum degree by $\Delta(G)$. We generalize the latter symbol to subsets of vertices: for $S \subseteq V$, $\Delta(S) = \max_{s \in S} \deg(s)$.

2. Computational results

In this section, we present the results of a computer search on the cop number of small order graphs. For a positive integer n, define $f_k(n)$ to be the number of non-isomorphic connected k-cop-win graphs of order n (that is, the *unlabelled* graphs G of order n with $c(G) = k$). Define $g(n)$ to be the number of non-isomorphic (not necessarily connected) graphs of order n, and $g_c(n)$ the number of non-isomorphic connected graphs of order n. Trivially, for all k, $f_k(n) \leq g(n)$. The following table presents the values of g, g_c, f_1, f_2 and f_3 for small orders.

order n	$g(n)$	$g_c(n)$	$f_1(n)$	$f_2(n)$	$f_3(n)$
1	1	1	1	0	0
2	2	1	1	0	0
3	4	2	2	0	0
4	11	6	5	1	0
5	34	21	16	5	0
6	156	112	68	44	0
7	1,044	853	403	450	0
8	12,346	11,117	3,791	7,326	0
9	274,668	261,080	65,561	195,519	0
10	12,005,168	11,716,571	2,258,313	9,458,257	1

The values of g and g_c come from [16], f_1 was computed by checking for cop-win orderings [13], while f_2 and f_3 were computed using Algorithm 1 in [3]. Among these graphs there is only one graph G of order 10 that requires 3 cops to win, and a graph with fewer vertices has cop number strictly less than 3. The graph G must be the Petersen graph, since it is 3-cop win. All of these facts together independently verify Theorems 1.1 and 1.2.

3. Meyniel's conjecture and growth rates

The following theorem, while straightforward to prove, sets up an unexpected connection between Meyniel's conjecture and the order of m_k.

Theorem 3.1.
(1) For any positive integer k, $m_k \in O(k^2)$.
(2) Meyniel's conjecture is equivalent to the property that $m_k \in \Omega(k^2)$, for all $k \in \mathbb{N}$.

Proof. The incidence graphs of projective planes have order $2(q^2 + q + 1)$, where q is a prime power, and have cop number $q+1$ (see [2] or [14]). Hence, this family of graphs shows that for q a prime power,

$$m_{q+1} \in O(q^2).$$

Now fix k a positive integer. Bertrand's postulate (which states that for all integers $x > 1$, there is a prime q between x and $2x$ [6, 8]) provides a prime q with $k < q < 2k$. Hence,

$$m_k \leq m_q \leq m_{q+1} \in O(q^2) = O((2k)^2) = O(k^2).$$

For (2), if $m_k \in o(k^2)$, then there is some connected graph G with order $o(k^2)$ and cop number k. But Meyniel's conjecture implies that $c(G) \in o(k)$, a contradiction. Hence, Meyniel's conjecture implies that $m_k \in \Omega(k^2)$.

For the reverse direction, suppose that $m_k \in \Omega(k^2)$. For a contradiction, suppose that Meyniel's conjecture is false. Then there is a connected graph G of order n with $c(G) = k \in \Omega(\sqrt{n})$. Thus $\sqrt{n} \in o(k)$, and so $n \in o(k^2)$. But then $m_k \leq n \in o(k^2)$, a contradiction. □

Hence, if Meyniel's conjecture holds, then Theorem 3.1 implies that

$$m_k = \Theta(k^2).$$

While the parameters m_k are non-decreasing, an open problem is to determine whether the M_k are in fact non-decreasing. A possibly more difficult problem is to settle whether $m_k = M_k$ for all $k \geq 1$. The gap in our knowledge of the parameters m_k and M_k points to the question of growth rates for the classes of connected k-cop-win graphs. It is well known (see [16] for example) that

$$\begin{aligned} g(n) &\in (1+o(1))\frac{2^{\binom{n}{2}}}{n!} \\ &= 2^{\frac{1}{2}n^2 - \frac{1}{2}n - n\log_2 n + n\log_2 e - \Theta(\log n)}, \end{aligned}$$

where the second equality follows by Stirling's formula. The following theorem supplies a super-exponential lower bound for the parameters f_k. A vertex is *universal* if it is adjacent to all others. If $f : X \to Y$ is a function and $S \subseteq X$, then we denote the restriction of f to S by $f \restriction S$.

Theorem 3.2.
(1) For all $n > 1$, $g(n-1) \leq f_1(n)$.
(2) For $k > 1$, and all $n > 2m_k$, $g(n - m_k - 1) \leq f_k(n)$.

Proof. For (1), fix a graph G of order $n-1$. Form G' by adding a universal vertex to G. If $G \not\cong H$, then it is an exercise to show that $G' \not\cong H'$. The result now follows since G' is cop-win.

For (2), given G of order $n - m_k - 1$, form a graph G^{+k} as follows. First form G' with the new universal vertex labelled x_G. Fix a k-cop-win graph H of order m_k (which we label as H_G), and specify a fixed vertex y_G of H_G. Add the bridge $x_G y_G$ connecting H_G to G'.

We first claim that G^{+k} is k-cop-win. We have $c(G^{+k}) \geq k$, since a winning strategy for the robber, if there are fewer than k cops, is to remain in H_G. To show that $c(G^{+k}) \leq k$, a set of k cops plays as follows. At the beginning of the game, one cop is on x_G, while the remaining cops stay in G. Then the robber, R, cannot move to G' without being caught, so the robber moves in H_G. All the cops then move to H_G and play their winning strategy there, with the following caveat. If R moves outside H_G, then the cops play as if R is on y_G. Eventually, the robber is caught in H_G, or the robber is in G' and at least one cop occupies y_G. But then that cop moves to x_G to win.

To finish the proof of (2), we must show that if $G \not\cong J$, then $G^{+k} \not\cong J^{+k}$. For a contradiction, let $h : G^{+k} \to J^{+k}$ be an isomorphism. Then we must have $h(x_G) = x_J$ by noting that x_G and x_J are the only vertices with the maximum degree $n - m_k$ (note that y_G has degree at most $m_k < n - m_k$ by hypothesis). The vertex y_G is unique with the property that it is adjacent to x_G and has neighbors not adjacent to x_G (the same holds by replacing the subscript G by J). But then $h(H_G) = H_J$, which implies the contradiction that the restricted mapping $h \restriction G : G \to J$ is an isomorphism. □

We do not know the asymptotic order for f_k (even if $k = 1$). A recent result [4] proves that the number of distinct *labelled* cop-win graphs is $2^{\frac{1}{2}n^2 - \frac{1}{2}n + o(n)}$.

4. Proofs of Theorems 1.1 and 1.2

We now proceed to the proofs of Theorems 1.1 and 1.2, but first introduce notation for the state of the game. We fix a connected graph G on which the game is played. In this section, we play with k cops labelled $C_1, C_2, \ldots, C_k$. The *state* of the game is a pair $(C\,;\,r)$, where C is a k-tuple of vertices $C = (c_1, c_2, \ldots, c_k)$, $c_i \in V(G)$ is the current position of cop C_i, and $r \in V(G)$ is the current position of the robber R. For notational convenience, we write $(c_1, \ldots, c_k\,;\,r)$ for $((c_1, \ldots, c_k)\,;\,r)$. When we need to specify whose turn it is to act, we underline the position of the player whose turn it is, that is, $(\underline{C}\,;\,r)$ denotes that it is the cops' turn to move, and $(C\,;\,\underline{r})$ the robber's. The reader should not be confused by the notation for the state of the game and the notation (u, v) commonly used to denote a directed edge.

We use a shorthand notation to describe moves. The notation

$$(\underline{c_1, \ldots, c_k}\,;\,r) \to (c_1', \ldots, c_k'\,;\,\underline{r})$$

denotes the cop move where each C_i moves from c_i to c_i'. Similarly

$$(c_1, \dots, c_k\,;\, \underline{r}) \rightarrow (\underline{c_1, \dots, c_k}\,;\, r')$$

denotes the robber's move from r to r'. We will concatenate moves and we use the shorthand $\twoheadrightarrow$, meaning a cop move followed by a robber move. That is,

$$(\underline{c_1, \dots, c_k}\,;\, r) \twoheadrightarrow (\underline{c_1', \dots, c_k'}\,;\, r')$$

is equivalent to

$$(\underline{c_1, \dots, c_k}\,;\, r) \rightarrow (c_1', \dots, c_k'\,;\, \underline{r}) \rightarrow (\underline{c_1', \dots, c_k'}\,;\, r').$$

There will be cases where the strategy allows for either the robber or the cops to be in one of several positions. In general, for $T_i \subseteq V$, $S \subseteq V$, the state of the game has the form $(T_1, \dots, T_k\,;\, S)$, meaning that $c_i \in T_i$, and $r \in S$.

The robber's *safe neighborhood*, denoted $S(R)$, is the connected component of $G - N[C]$ containing the robber. We say that the robber is *trapped* when $S(R) = \emptyset$. This condition is equivalent to having both $r \in N(C)$ and $N(r) \subseteq N[C]$. Once the robber is trapped, he will be caught on the subsequent cop move, regardless of the robber's next action. When the robber is trapped, we are in a *cop-winning position*, denoted by appending the symbol $\mathcal{C}$ to the configuration: $(c_1, c_2, \dots, c_k; r)\,\mathcal{C}$.

4.1. **The end game.** We frequently use the following facts to identify cop-win strategies for two cops in the end game. We state a more general version of these results for k cops.

We say that a vertex v is *no-backtrack-winning* if there is a winning strategy for the cop starting at v such that the cop never repeats a vertex during the pursuit. For example, when G is a tree, every vertex is no-backtrack-winning.

Next we fix some notation. For a set $U = \{u_1, u_2, \dots, u_t\} \subseteq V$, let $N_U'(u_j) = N(u_j) \setminus N[U \setminus u_j]$ be the neighbors of u_j that are not adjacent to any other vertex in U.

Lemma 4.1. *Let $(\underline{C}\,;\, r)$ be the state of the game. Suppose that there exists a $c_j \in C$ such that either (a) $(S(R) : N_C'(c_j)) = \emptyset$ and $G[S(R)]$ is cop-win; or (b) $N(S(R)) \cap N_C'(c_j) = \{v\}$ such that $H = G[S(R) \cup \{v\}]$ is cop-win and v is no-backtrack-winning in H. Then the cops can win from this configuration.*

Proof. Let $S = S(R)$ be the initial safe neighborhood of R. In both cases, only cop C_j is active, while the others remain stationary. In case (a), the cop C_j moves (possibly over several rounds of the game) to $S(R)$, and then follows the cop-win strategy on $G[S(R)]$. The robber cannot escape $S(R)$ during these moves of the cop C_j, because the condition in (a) implies that $S(R)$ would be the same if we removed C_j from the game.

In case (b), cop C_j moves to v and then follows a no-backtrack strategy on $G[S \cup \{v\}]$. This prevents the robber from ever reaching v. In both cases, the only way for the robber to avoid capture by C_j is to move into the neighborhood of the remaining cops. $\square$

We highlight two useful consequences that are used heavily for $k = 2$ in subsequent proofs.

Corollary 4.2. *Let* $(\underline{C}\,; r)$ *be the state of the game, played with* $k \geq 2$ *cops. If* $|S(R)| \leq 2$ *and* $|N(S(R))| \leq 2k-1$*, then the cops can win.*

Proof. Let $S = S(R)$. We have $|N(S) \cap N(C)| \leq 2k-1$, so the pigeonhole principle ensures that there exists a cop C_j such that $|N(S) \cap N'_C(c_j)| \leq 1$. We are done by Lemma 4.1, since every vertex of a connected 2-vertex graph is no-backtrack-winning. $\square$

Corollary 4.3. *Let* $(\underline{C}\,; r)$ *be the state of the game, played with* $k \geq 2$ *cops. If*

$$\max_{v \in S(R)} \deg_G(v) \leq 3$$

and $S(R)$ *contains at most one vertex of degree* 3*, then the cops can win.*

Proof. Let $S = S(R)$. Since $G[S]$ is connected, we have $(S : N(C)) \leq 3$. Therefore, some cop C_j has $|(S : N'_C(c_j))| \leq 1$. If $G[S]$ is a tree, then we are done by Lemma 4.1. If $G[S]$ is not a tree, then $G[S]$ must be unicyclic with one degree 3 vertex, say u. Therefore, $|(S : N(C))| = 1$, and except for u, every vertex in the cycle has degree 2 in G. A winning strategy for the cops is as follows: two cops move until they both reach u. Now $S(R)$ is a path, so Lemma 4.1 completes the proof. $\square$

4.2. **Graphs with $\Delta(G) \geq n-6$.** In this section, we prove Theorem 1.1. We also make progress on the proof of Theorem 1.2 by showing that if $v(G) = 10$ and $\Delta(G) = 4$, then $c(G) \leq 2$.

Lemma 4.4. *Let* G *be a graph on* n *vertices. If there is a vertex* $u \in V(G)$ *of degree at least* $n-6$*, then either* $c(G) \leq 2$ *or the induced subgraph* $G[V \setminus N[u]]$ *is a 5-cycle.*

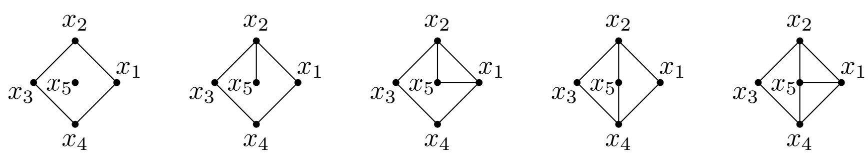
Corollary 4.5. *If* $\Delta(G) \geq n-5$*, then* $c(G) \leq 2$*.*

Lemma 4.4 and its immediate corollary are crucial tools in proving the main results. In particular, Theorem 1.1 is a quick consequence of Corollary 4.5. This reduces the search to 10 vertex graphs with $2 \leq \delta(G) \leq \Delta(G) \leq 4$.

Proof of Lemma 4.4. Let $H = G[V \setminus N[u]]$. By Lemma 4.1(a), if H is cop-win, then $c(G) \leq 2$. In particular, this holds if H does not contain an induced cycle of length at least 4. So we only need to consider the case where $v(H) = 4$ or 5, and H contains an induced 4-cycle. Let x_1, x_2, x_3, x_4 form the 4-cycle in H (in that order). Let x_5 be the additional vertex (if present).

We now distinguish some cases based on $N(x_5) \cap H$. If $x_5 \sim x_i$ for every $i \in \{1, 2, 3, 4\}$, then H is cop-win, and hence, $c(G) \leq 2$. We therefore have

5 cases to consider, depicted in Figure 1. Case (a) includes the situation when $\deg(u) = n-5$, and there is no vertex x_5.



(a) (b) (c) (d) (e)

FIGURE 1. The five cases for $G[V \setminus N[u]]$ in Lemma 4.4.

First we make some technical claims. We start by noting that moving to x_5 is in most situations a bad idea for the robber in Cases (a), (b) and (c).

Claim 1. In Cases (a), (b), and (c), if the state of the game is of the form $(\underline{N[u], V(H)}\,; x_5)$, the cops have a winning strategy.

For the proof of the claim, C_1 moves to u. In Case (a), $S(R) = \{x_5\}$ and we are already done by Corollary 4.2. In Case (b), if possible, C_2 moves directly to x_2. Otherwise, C_2 moves first to x_1 and then to x_2. In either case the robber is trapped at x_5. In Case (c), C_2 moves to x_2 or x_1 (whichever c_2 is adjacent to), again trapping the robber in x_5 The proof of the claim follows.

Next we consider the structure of $N(y) \cap V(H)$ for vertices $y \in N(u)$.

Claim 2. Suppose the state of the game has the form $(\underline{N[u], \{x_1, x_3\}}\,; y)$, where $y \in N(u)$ is such that either (a) $N(y) \cap V(H) = \{x_2, x_4\}$, or (b) y is adjacent to at most one of x_2 or x_4. Then the cops have a winning strategy.

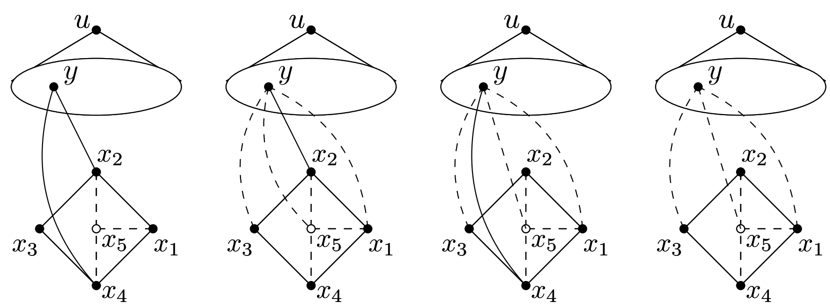
For the proof of the claim, Figure 2 shows the four classes of possible graph structures. Let us first consider the structure (B1). Let $z = x_2$. The cop C_1 moves to u, and C_2 moves to z. Now the robber is trapped in all cases of Figure 1 except Case (a). In Case (a), the robber's only move is to x_5. After this move, the robber can be caught by Claim 1. The same cop strategy works for structures (B2) and (B3), taking $z = x_4$. A simplified version of this proof shows that the same cop strategy works for structure (A), taking $z = x_2$. The proof of the claim follows.

We remark that in Cases (d) and (e) of Figure 1, x_5 and x_1 are symmetric, so the statement also holds for the configuration $(\underline{N[u], x_5}\,; y)$.

The next claim concerns the situation where there are two vertices in $N(u)$ that do not satisfy the condition of the previous claim.

Claim 3. If there are two vertices $y, z \in N(u)$ such that $\{x_2, x_3, x_4\} \subseteq N(y)$, and $\{x_1, x_2, x_4\} \subseteq N(z)$, then $c(G) \leq 2$.

For the proof of Claim 3, first we deal with all cases but Case (d). The cops start at u and z. If the robber starts at x_3, the cops' winning strategy is: $(\underline{u, z}\,; x_3) \to (u, y\,; \underline{x_3})\,\mathcal{C}$. (Recall that appending the symbol $\mathcal{C}$ denotes



(A) (B1) (B2) (B3)

FIGURE 2. The four classes of possible structures of G for Claim 2. Vertex x_5 might not be present, and dashed edges might not be present.

that the robber is trapped, and will be caught on the subsequent cop turn, regardless of robber's next move.) If the robber starts at x_5, the strategy will depend on the structure of H. In Cases (a), (b), and (c), we are done by Claim 1. In Case (e), $(\underline{u, z}\,; x_5) \rightarrow (u, x_1\,; \underline{x_5})\,\mathcal{C}$ is a winning strategy.

The remainder of the proof deals with Case (d), which requires a more involved argument.

First suppose that there exists $w \in N(u)$ such that $\{x_2, x_4, x_5\} \subseteq N(w)$. Then the cops start at u and z. The robber can start at x_3 or x_5 in either case the cops have a winning strategy: $(\underline{u, z}\,; x_3) \rightarrow (y, u\,; \underline{x_3})\,\mathcal{C}$, or $(\underline{u, z}\,; x_5) \rightarrow (w, u\,; \underline{x_5})\,\mathcal{C}$.

Now assume that no such w exists. Start the cops at u and y. The robber starts in $\{x_1, x_5\}$. If the robber starts at x_1, then $(\underline{u, y}\,; x_1) \rightarrow (y, u\,; \underline{x_1})\,\mathcal{C}$. So we may assume the robber starts at x_5. If $|N(x_5) \cap N(u)| \leq 1$, we are done by Corollary 4.2. Otherwise, the cops move by $(\underline{u, x_3}\,; x_5) \rightarrow (v, z\,; \underline{x_5})$, for some $v \in N(x_5) \cap N(u)$. The robber is forced to move to some $w \in N(x_5) \cap N(u)$ (if no such w exists, then R is trapped). By our initial argument, w cannot be adjacent to both x_2 and x_4, so the state satisfies the conditions of Claim 2(b), and the proof of the claim follows.

Claim 4. Either $c(G) \leq 2$, or we can relabel the vertices of H via an automorphism of H so that x_1 is adjacent to $N(u)$.

To prove the claim, suppose that no such relabelling exists. We will show a winning strategy for the cops, starting at u and x_3. In Cases (a) and (b), the claim follows from Corollary 4.2 (either $S(R) = \{x_1\}$ or $S(R) = \{x_5\}$). In Cases (d) and (e), $S(R) \subseteq \{x_1, x_5\}$, and we are assuming that both x_1 and x_5 have no edges to $N(u)$. Hence, $|N(S(R))| \leq 2$, and we are again done by Corollary 4.2. In Case (c), $S(R) = \{x_1, x_5\}$, and we are assuming that both x_1 and x_2 do not have neighbors in $N(u)$. By Claim 1, we may assume R does not start at x_5, and so R starts at x_1. Let $v \in N(u) \cap N(x_5)$ (if $x_5 \nsim N(u)$, then $N(S(R))$ is dominated by $c_2 = x_3$). Now the cops can win by following the strategy: $(\underline{u, x_3}\,; x_1) \rightarrow (\underline{v, x_3}\,; x_1)\,\mathcal{C}$. The proof of the claim follows.

Having established the above claims, we now conclude the proof of Lemma 4.4. By Claim 4, we may assume $x_1 \sim w \in N(u)$. Initially place C_1 at u and C_2 at x_1. The robber could start at x_3 or, in Cases (a), (b), and (d), at x_5. If the robber starts at x_5 in Cases (a) and (b), then we are done by Claim 1. In Case (d), x_5 and x_3 are symmetric, so without loss of generality, $r = x_3$, and the initial state is $(\underline{u, x_1}\,; x_3)$.

If $x_3 \nsim N(u)$, then the cops win by Corollary 4.2. Otherwise, let $v \in N(x_3) \cap N(u)$. Then C_1 moves from u to v, while C_2 remains fixed at x_1, forcing R to some $y \in N(u) \cap N(x_3)$, with $y \nsim v, y \nsim x_1$. If no such y exists, then R is trapped. If y is adjacent to only one of x_2 or x_4, we are in the state $(\underline{v \in N(u), x_1}\,; y)$, which satisfies the conditions of Claim 2 (b), and hence, the cops have a winning strategy.

Otherwise, y is adjacent to x_2, x_3, and x_4. In this case, the cops move $(\underline{v, x_1}\,; y) \rightarrow (x_3, w\,; \underline{y})$, for some $w \in N(x_1) \cap N(u)$. If $y \sim x_5$, and R moves to x_5, then the cops win: in Cases (a),(b),(c) we are done by Claim 1; in Case (d), $(\underline{x_3, w}\,; x_5) \rightarrow (y, u\,; \underline{x_5})\,\mathcal{C}$; in Case (e), the cops can adopt a different strategy $(\underline{u, y}\,; x_1) \rightarrow (u, x_5\,; \underline{x_1})\,\mathcal{C}$ from the beginning. The only other option is for R to move to some $z \in N(u)$, $z \nsim x_3$. So the state is $(\underline{x_3, w}\,; z)$. Either the pair y, z satisfies the conditions of Claim 3, or the current state satisfies the conditions of Claim 2(b) or (a). In either case, we are done. This concludes the proof of Lemma 4.4. $\square$

We now state some quick but useful consequences of Lemma 4.4.

Corollary 4.6. *Let G be a graph on n vertices. If there is a vertex $u \in V$ of degree at least $n - 6$, and a vertex $v \in V \setminus N[u]$ such that $|N(v) \setminus N(u)| \geq 3$, then $c(G) \leq 2$.*

Proof. The vertex v has three neighbors in $G[V \setminus N[u]]$, and hence, $G[V \setminus N[u]]$ cannot be a 5-cycle. $\square$

Corollary 4.7. *Let G be a graph on n vertices. If there is a vertex u of degree at least $n - 6$ and a vertex $v \in V \setminus N[u]$ with $\deg(v) \leq 3$, then $c(G) \leq 2$.*

Proof. By Lemma 4.4, we only need to consider the case where $G[V \setminus N[u]]$ is a 5-cycle, x_1, x_2, x_3, x_4, x_5 (in that order). Without loss of generality, let $\deg(x_1) \leq 3$, and $\deg(x_2) \geq 3$. For each $i = 1, \dots, 5$ such that $\deg(x_i) \geq 3$, pick some $y_i \in N(x_i) \cap N(u)$ arbitrarily (we allow $y_i = y_j$ for $i \neq j$). The game starts as $(\underline{u, x_4}\,; \{x_1, x_2\}) \twoheadrightarrow (\underline{u, \{x_3, x_4\}}\,; x_1)$. First we deal with the case where $\deg(x_1) = 2$ and the case where $\deg(x_1) = 3$ and $y_1 \sim x_4$. The cops' winning strategy for these two cases is the same, namely

$$(\underline{u, \{x_3, x_4\}}\,;\, x_1) \twoheadrightarrow (\underline{y_2, x_4}\,;\, x_1) \rightarrow (x_2, x_4\,;\, \underline{x_1})\,\mathcal{C}.$$

Now we may assume that all x_i have degree 3, and hence, y_i exists for all i. We may further assume that $x_4 \neq y_1$, and, since x_3 and x_4 are symmetric, we are also done in the case $y_1 \sim x_3$. The only remaining possibility is $N(y_1) \cap (V \setminus N[u]) \subseteq N[x_1]$. Since x_3 and x_4 are symmetric, without loss of generality, the state is $(\underline{u, x_4}\,;\, x_1)$. The cops first move to y_2 and x_5, forcing the robber to y_1, then in one more move

$$(\underline{y_2, x_5}\,;\, y_1) \rightarrow (u, x_1\,;\, \underline{y_1})\,\mathcal{C},$$

the robber is trapped at y_1. □

These corollaries are enough to prove that every 9-vertex graphs is 2 cop-win, and to show that if $v(G) = 10$ and $\Delta(G) = 4$ then $c(G) \leq 2$.

Proof of Theorem 1.1. If $\Delta(G) \geq 4$, then we are done by Lemma 4.4. If $\Delta(G) = 3$, then we are done by Corollary 4.7. □

Lemma 4.8. *If $v(G) = 10$ and $\Delta(G) \geq 4$, then $c(G) \leq 2$.*

Proof. Let $u \in V(G)$ have degree at least 4. By Lemma 4.4, either $c(G) \leq 2$ or $\deg(u) = 4$, and $G[V \setminus N[u]]$ is a 5-cycle. Now, by Corollary 4.7, either $c(G) \leq 2$, or every $u \in V \setminus N[u]$ has $\deg(u) \geq 4$. In the latter case, $|(N(u) : V \setminus N[u])| \geq 10$. Thus, by the pigeonhole principle, there exists $v \in N(u)$ such that $|N(v) \cap (V \setminus N[u])| \geq 3$. We now deal with this case, namely u and v have degree 4, and $N(u) \cap N(v) = \emptyset$.

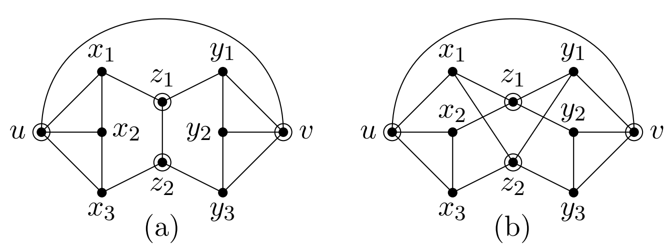


FIGURE 3. The two possible starting structures in the proof of Lemma 4.8. Circled vertices cannot have additional edges.

By Lemma 4.4, both $G[V(G) \setminus N[u]]$ and $G[V(G) \setminus N[v]]$ are 5-cycles. The resulting graph structure must be one of the two shown in Figure 3. Considering the structure in Figure 3(a), we note that $\deg(z_1) = \deg(z_2) = 3$ in order to maintain the induced 5-cycle structures, and hence, we are done by Corollary 4.7.

Now suppose that G has the structure in Figure 3(b). In this case, we show $\deg(x_3) = 3$, and we are again done by Corollary 4.7. To show that

$\deg(x_3) = 3$, we look at each potential additional edge, and show that $V \setminus N[x_3]$ is not a 5-cycle, and hence, we are done by Lemma 4.4. We only need to consider edges to y_1, y_2 or y_3 as other potential edges would not maintain the induced 5-cycle structure. We have $x_3 \nsim y_1$ because $\{v, y_2, y_3\}$ forms a triangle. We have $x_3 \nsim y_3$ because z_1 is adjacent to each of x_1, y_1, y_2. Finally, $x_3 \nsim y_2$ because the existence of this edge would force $y_3 \sim x_1$, which is symmetric to the forbidden $x_3 \sim y_1$. □

4.3. **Graphs with $\Delta(G) = n - 7$.** In this section, we complete the proof of Theorem 1.2.

Lemma 4.9. *Let G be a graph with a vertex u with $\Delta(G) = \deg(u) = n - 7$ and such that $\deg(v) \leq 3$ for every $v \in V \setminus N[u]$. Then either $c(G) = 2$ or the induced subgraph $G[V \setminus N[u]]$ is a 6-cycle.*

This lemma can be generalized a bit more. In particular, if we remove the restriction on the maximum of degree of vertices in $V \setminus N[u]$, then the proofs of Lemmas 4.4 and 4.9 can be adapted to show that H must contain an induced 5-cycle or 6-cycle. However, the case analysis is cumbersome, so we have opted for this simpler formulation. The version stated above is sufficient to prove one of our main results that the Petersen graph is the only 10-vertex graph requiring 3 cops.

Proof of Lemma 4.9. Let $H = G[V \setminus N[u]]$ and suppose that $c(G) > 2$. First, we observe that H must be connected. Otherwise, we can adapt the proof of Corollary 4.7 to show that $c(G) = 2$. Indeed, H has at most one component H_1 whose cop number is 2. We use the strategy described in the proof of Corollary 4.7 to capture the robber. The only alteration of the strategy is to address the robber moving from $N(u)$ to $H - H_1$. However, $|V(H - H_1)| \leq 2$, so this component is cop-win. One cop responds by moving to u, while the other moves into $H - H_1$ for the win (by Lemma 4.1(a)).

Therefore, we may assume that H is connected and $c(H) \geq 2$. This means that H must contain an induced k-cycle for $k \in \{4, 5, 6\}$. Suppose that G contains an induced 4-cycle x_1, x_2, x_3, x_4. Without loss of generality, $x_5 \sim x_1$, and x_6 is adjacent to at most three of $\{x_2, x_3, x_4, x_5\}$ (because we already have $\deg(x_1) = 3$). Start the cops at u and x_1, so that $S(R)$ is one of $\{x_3\}$, $\{x_6\}$ or $\{x_3, x_6\}$. In the first two cases, $\Delta(S(R)) \leq 3$ so the cops win by Corollary 4.3. The last option occurs when $x_3 \sim x_6$. If x_6 has at most one neighbor in $N(u)$, then we are again done by Corollary 4.3, since $\Delta(S(R)) \leq 3$. When x_6 has two neighbors in $N(u)$, the game play depends on the initial location of the robber. If the robber starts at x_6, then C_1 holds at u while C_2 moves from x_1 to x_2 to x_3, trapping the robber. If the robber starts at x_3, then the roles are reversed: C_1 moves to x_6 in two steps while C_2 holds at x_1. At this point, the robber is trapped.

Next, suppose that G contains an induced 5-cycle x_1, x_2, x_3, x_4, x_5. Without loss of generality, $x_6 \sim x_1$. If x_6 is adjacent to two of the x_i, then we can place C_1 at u and C_2 at some x_j so that $|N(S(R)) \cap N(u)| \leq 1$, giving a cop winning position by Lemma 4.1(b). Indeed, by symmetry there are only 2 cases to consider: if $x_6 \sim x_2$, then C_2 starts at x_4 and $S(R) = \{x_1, x_2, x_6\}$; if $x_6 \sim x_3$, then C_2 starts x_3, and $S(R) = \{x_1, x_5\}$. So we may assume that x_6 has no additional neighbors in H. There are two cases to consider. If x_2 and x_4 do not share a neighbor in $N(u)$, then the game play begins with C_2 chasing R onto x_2

$$(\underline{u, x_1}\,;\, \{x_3, x_4\}) \twoheadrightarrow \cdots \twoheadrightarrow (\underline{u, \{x_4, x_5\}}\,;\, x_2).$$

If x_2 is not adjacent to $N(u)$, then the cops can ensure $S(R)$ satisfies Corollary 4.3 on their next move. Indeed, C_2 moves to x_4. If $N(x_6) \cap N(u) = \emptyset$, then the situation already satisfies Corollary 4.3; otherwise, C_1 moves to $N(x_6) \cap N(u)$, and now the situation satisfies Corollary 4.3.

The final case to consider is when x_2 and x_4 are both adjacent to $y \in N(u)$. By symmetry, x_3 and x_5 are adjacent to $z \in N(u)$. By symmetry, there is one game to consider, namely $(\underline{u, x_1}\,;\, x_3) \twoheadrightarrow (\underline{z, x_2}\,;\, x_4)$, which is cop-win by Corollary 4.3. Thus, the only option for H is an induced 6-cycle. □

The following lemma may be proved by checking the 18 possible 3-regular graphs of order 10 listed at [12], but we provide a short proof for completeness.

Lemma 4.10. *The Petersen graph is the only* 3*-regular graph* G *such that for every vertex* $u \in V(G)$, $G[V(G) \setminus N[u]]$ *is a* 6*-cycle.*

Proof. Pick any vertex u in G. The complement is a 6-cycle, where every vertex is adjacent to exactly one vertex in $N(u)$. Let $N(u) = \{y, z, w\}$. Label the vertices of the 6-cycle x_i, $0 \leq i \leq 5$, where edges are between consecutive indices. Without loss of generality, say $x_0 \sim y$. Because $V \setminus N[x_0]$ is a 6-cycle, we must have that $x_2 \sim w$ and $x_4 \sim z$ (by symmetry this is the only option). The only remaining edges to add are a matching between x_1, x_3, x_0 and y, z, w. To avoid a triangle in $V \setminus N[y]$, we cannot have $x_3 \sim z$ or $x_3 \sim w$; hence, $x_3 \sim y$. Similarly, $x_1 \sim z$, and $x_5 \sim w$. But this gives an isomorphic copy of the Petersen graph. □

We can now prove that the Petersen graph is the unique 3 cop-win graph of order 10.

Proof of Theorem 1.2. Let G be a graph of order 10 such that $c(G) = 3$. We have $\delta(G) \geq 2$; otherwise, the vertex $v \in V(G)$ of degree one is a dominated vertex, so $c(G) = c(G - v) \leq 2$ by Theorem 1.1. Lemma 4.8 ensures that $\Delta(G) \leq 3$. It is straightforward to see that $\Delta(G) = 3$, since a connected 2-regular graph is a cycle which is 2-cop-win.

Suppose a vertex $u \in V(G)$ has $\deg(u) = 3$. Then by Lemma 4.9, $G[V \setminus N[u]]$ must be a 6-cycle. If every vertex in $N(u)$ has degree 3, then G is 3-regular with $c(G) = 3$, and therefore, G is the Petersen graph by Lemma 4.10. Otherwise, there is a vertex $x_1 \in V \setminus N[u]$ with $\deg(v) = 2$. In the rest of the proof we give a winning strategy for the cops in this case.

Let the 6-cycle $G[V \setminus N[u]]$ be $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ with edges between consecutive indices. Without loss of generality, $\deg(x_1) = 2$ and $\deg(x_2) = 3$. Let $k = \max\{i \mid \deg(x_i) = 3\}$. The configuration is $(\underline{u, x_4}\,;\, \{x_1, x_2, x_6\})$ initially. If $k \leq 5$, then the cops win by Corollary 4.3. When $k = 6$, the strategy depends on the initial robber location. Let $y \in N(u) \cap N(x_2)$. We either have one of $(\underline{u, x_4}\,;\, x_2) \twoheadrightarrow (\underline{y, x_4}\,;\, x_1) \rightarrow (y, x_5\,;\, \underline{x_1})\, \mathcal{C}$, or $(\underline{u, x_4}\,;\, x_1) \twoheadrightarrow (\underline{y, x_5}\,;\, x_1)\, \mathcal{C}$, or $(\underline{u, x_4}\,;\, x_6) \twoheadrightarrow (\underline{u, x_5}\,;\, x_1) \rightarrow (y, x_6\,;\, \underline{x_1})\, \mathcal{C}$. In other words, the robber is trapped for every initial placement. $\square$

5. Further directions

We conclude with some reflections on our results and some open problems. The Petersen graph is the unique 3-regular graph of girth 5 of minimal order, so that Theorem 1.2 provides a tight lower bound for n when $c(G) = 3$. Recall that a (k, g)-*cage* is a k-regular graph with girth g of minimal order. See [9] for a survey of cages. The Petersen graph is the unique $(3, 5)$-cage, and in general, cages exist for any pair $k \geq 2$ and $g \geq 3$. Aigner and Fromme [1] proved that graphs with girth 5, and degree k have cop number at least k; in particular, if G is a $(k, 5)$-cage, then $c(G) \geq k$. Let $n(k, g)$ denote the order of a (k, g)-cage. Is it true that a $(k, 5)$-cage is k-cop-win? Next, since we have $m_k \geq n(k, 5)$, it is natural to speculate whether $m_k = n(k, 5)$ for $k \geq 4$. It seems reasonable to expect that this is true at least for small values of k. It is known that $n(4, 5) = 19$, $n(5, 5) = 30$, $n(6, 5) = 40$ and $n(7, 5) = 50$. Do any of these cages attain the analogous m_k? More generally, we can ask the same question for large k: is m_k achieved by a $(k, 5)$-cage? It is known that $n(k, 5) = \Theta(k^2)$, so an affirmative resolution would be consistent with Theorem 3.1.

The techniques to prove Theorems 1.1 and 1.2 may prove useful in classifying the cop number of graphs with order 11. We will consider this problem, and the value of m_4 in future work.

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