Contributions to Discrete Mathematics

Volume 12, Number 1, Pages 14–19
ISSN 1715-0868

BOUNDS ON SEVERAL VERSIONS OF RESTRAINED DOMINATION NUMBERS

BABAK SAMADI, HAMID R. GOLMOHAMMADI, AND ABDOLLAH KHODKAR

Abstract. We investigate several versions of restrained domination numbers and present new bounds on these parameters. We generalize the concept of restrained domination and improve some well-known bounds in the literature. In particular, for a graph G of order n and minimum degree $\delta \geq 3$, we prove that the restrained double domination number of G is at most $n - \delta + 1$. In addition, for a connected cubic graph G of order n, we show that the total restrained domination number of G is at least $n/3$ and the restrained double domination number of G is at least $n/2$.

1. Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$, edge set $E = E(G)$, minimum degree $\delta = \delta(G)$, and maximum degree $\Delta = \Delta(G)$. We use [13] for terminology and notation which are not defined here. For any vertex $v \in V$, $N(v) = \{u \in G \mid uv \in E(G)\}$ denotes the *open neighbourhood* of v in G, and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighbourhood*. A set $S \subseteq V(G)$ is a *dominating set* in G if each vertex in $V \setminus S$ is adjacent to at least one vertex in S. A dominating set S is a *restrained dominating set (total restrained dominating set)* if the subgraph(s) induced by the vertices of $V \setminus S$ ($V \setminus S$ and S) has (have) no isolated vertices. The restrained domination number $\gamma_r(G)$ (total restrained domination number $\gamma_t^r(G)$) is the minimum cardinality of a restrained dominating set (total restrained dominating set). The concept of restrained domination was introduced by Telle and Proskurowski [12], while its total version was introduced by Ma, Chen and Sun [11].

A generalization of the concept of total restrained domination was exhibited in [10] as k-tuple total restrained domination. The *k-tuple total restrained domination number*, $\gamma^r_{\times k,t}(G)$, of a graph G is the minimum cardinality of a k-tuple total restrained dominating set as a subset $S \subseteq V$ with this property that every vertex in V has at least k neighbors in S and every

Received by the editors November 30, 2014, and in revised form April 12, 2015.
2000 *Mathematics Subject Classification.* 05C69.
Key words and phrases. restrained domination, restrained double domination, total restrained domination.

vertex in $V \setminus S$ has at least k neighbors in $V \setminus S$. This concept was introduced by Kazemi in [10].

Kala and Nirmala Vasantha [9] defined the concept of restrained double domination. In fact, a set $S \subseteq V(G)$ is a *restrained double dominating set* in G if every vertex in V is dominated by at least two vertices in S and the subgraph induced by $V \setminus S$ has no isolated vertices. The minimum cardinality of a restrained double dominating set is called the *restrained double domination number* of G and is denoted by $\gamma_{2r}(G)$.

In this paper, we first obtain some bounds on $\gamma_t^r(G)$ and $\gamma_{2r}(G)$ of a graph G that improve their corresponding results in [1, 2, 8, 9, 11]. Then for a graph G we introduce the concept of the (k, k', k'')-domination number, denoted $\gamma_{(k,k',k'')}(G)$, as a generalization of the parameters $\gamma_r(G)$, $\gamma_t^r(G)$, $\gamma_{2r}(G)$ and $\gamma^r_{\times k,t}(G)$. Finally, we find a lower bound for $\gamma_{(k,k',k'')}(G)$. This bound improves some of the known results presented in [2, 3, 4, 6, 7, 10]. In particular, for a cubic graph G we conclude $\gamma_r(G) \geq n/4$ (see [5]), $\gamma_t^r(G) \geq n/3$ and $\gamma_{2r}(G) \geq n/2$.

2. Bounds on $\gamma_t^r(G)$ and $\gamma_{2r}(G)$

We make use of the following lemma to show that the bounds found in this section are sharp.

Lemma 2.1. *Let m and n be positive integers. Then*

(i) $\gamma_t^r(K_{m,n}) = \begin{cases} m+n, & \textit{if } \min\{m,n\} = 1, \\ 2, & \text{otherwise}, \end{cases}$

(ii) $\gamma_t^r(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$,

(iii) $\gamma_{2r}(K_n) = \begin{cases} 2, & \textit{if } n \neq 3, \\ 3, & \textit{if } n = 3, \end{cases}$

(iv) $\gamma_{2r}(K_{1,n}) = n + 1$.

See [2] *for* (i), (ii) *and see* [9] *for* (iii), (iv).

The following theorem gives lower bounds on the total restrained and restrained double domination numbers of a graph G with ℓ leaves and s support vertices.

Theorem 2.2. *Let G be a graph of order n without isolated vertices. Supposed that G has ℓ leaves and s support vertices. Then*

(i)
$$\gamma_t^r(G) \geq \ell + \max\left\{\frac{n-s}{\Delta}, \frac{n}{\Delta+1}\right\},$$

(ii)
$$\gamma_{2r}(G) \geq \max\left\{\frac{2n + (\Delta+1)\ell}{\Delta+2}, \frac{2n + \ell\Delta + (\Delta-2)n_2 - s}{\Delta+1}\right\},$$

where n_2 is the number of vertices of degree two that are not support vertices.

In addition, these bounds are sharp.

Proof. To prove (i), we first prove the following claims.

Claim 1. $\gamma_t^r(G) \geq \ell + (n-s)/\Delta$.

Let S be a total restrained dominating set in G. Let L be the set of leaves and let $\{w_i\}_{i=1}^s$ be the set of support vertices. Suppose l_i is the number of leaves adjacent to w_i for every $1 \leq i \leq s$. Since all leaves and support vertices belong to S, it follows that no leaf has neighbors in $V \setminus S$ and each w_i has at most $\Delta - \ell_i$ neighbors in $V \setminus S$, for every $1 \leq i \leq s$. Also every vertex in $S \setminus (L \cup \{w_i\}_{i=1}^s)$ has at most $\Delta - 1$ neighbors in $V \setminus S$. Now we consider the set $[S, V \setminus S]$ of edges with a vertex in S and a vertex in $V \setminus S$. We have

$$\sum_{i=1}^{s}(\Delta - \ell_i) + (|S| - \ell - s)(\Delta - 1) \geq |[S, V \setminus S]|. \tag{2.1}$$

On the other hand, every vertex in $V \setminus S$ has at least one neighbor in S. Hence

$$|[S, V \setminus S]| \geq n - |S|. \tag{2.2}$$

Since $\ell_1 + ... + \ell_s = \ell$, the inequalities (2.1) and (2.2) imply

$$s\Delta - \ell + (|S| - \ell - s)(\Delta - 1) \geq n - |S|.$$

This implies the desired lower bound.

Claim 2. $\gamma_t^r(G) \geq \ell + n/(\Delta + 1)$.

Let S and L be as in the proof of Claim 1. It is easy to check that $S \setminus L$ is a dominating set in G. Suppose α is the number of edges with at least one vertex in $S \setminus L$. Then

$$(n - |S|) + \ell \leq \alpha \leq \Delta(|S| - \ell).$$

This completes the proof of Claim 2.

The result now follows from Claim 1 and Claim 2. The bound is sharp for P_{4n+2} and the star $K_{1,n}$ by Lemma 2.1. This completes the proof of (i).

The proof of (ii) is similar to the proof of (i). Note that all leaves, support vertices and vertices of degree two belong to every restrained double dominating set. The bound is sharp for the star $K_{1,n}$. □

In [1, 8, 11], it was proved that $\gamma_t^r(T) \geq \Delta(T) + 1$, where T is a tree. Note that $\ell + (n-s)/\Delta \geq \Delta(T) + 1$ and $\ell + n/(\Delta + 1) \geq \Delta(T) + 1$. Thus the lower bound in (i) of Theorem 2.2 is tighter than its corresponding bound given in [1, 8, 11].

It is easy to check that the lower bound given in (ii) of Theorem 2.2 is an improvement of the lower bound $\gamma_{2r}(G) \geq 2n/(\Delta + 1)$, which was first given in [9] for a graph G of order n without isolated vertices.

We conclude this section by establishing an upper bound on the restrained double domination number of a graph in terms of its order and its minimum degree.

Theorem 2.3. *If G is a graph of order n and minimum degree $\delta \geq 3$, then $\gamma_{2r}(G) \leq n - \delta + 1$ and the bound is sharp.*

Proof. Let u be a vertex in G with $deg(u) = \delta$ and $v, w \in N(u)$. Since $\delta \geq 3$, it follows that $|N[u] \setminus \{v, w\}| \geq 2$ and $N[u] \setminus \{v, w\}$ is a nonempty set. Also, it is easy to see that the subgraph induced by $N[u] \setminus \{v, w\}$ has no isolated vertices. Now let $S = V(G) \setminus (N[u] \setminus \{v, w\})$. Obviously, u is adjacent to both v and w, and $v, w \in S$. Let $x \in N(u) \setminus \{v, w\}$. Then x can be joined to at most $\delta - 2$ vertices in $N[u] \setminus \{v, w\}$. Thus x has at least two neighbors in S. Finally, no vertex in S has all its neighbors in $N[u] \setminus \{v, w\}$ because $|N[u] \setminus \{v, w\}| = \delta - 1$. Thus, there are no isolated vertices in the subgraph induced by S. Therefore S is a restrained double dominating set in G. Hence,

$$\gamma_{2r}(G) \leq |S| = |V(G) - (N[u] \setminus \{v, w\})| = n - \delta + 1.$$

The upper bound is sharp for the complete graph K_n when $n \geq 4$, by Lemma 2.1. □

The authors in [9] showed that $\gamma_{2r}(G) \leq n - 2$ for every graph of order n and minimum degree $\delta(G) \geq 3$. In fact, Theorem 2.3 gives an improvement for this bound. In addition, if $\delta \geq 4$, then the upper bound $n - 2$ for $\gamma_{2r}(G)$ is not sharp by Theorem 2.3.

3. (k, k', k'')-DOMINATING SETS

Let k, k' and k'' be nonnegative integers. A set $S \subseteq V$ is a (k, k', k'')-*dominating set* in G if every vertex in S has at least k neighbors in S, every vertex in $V \setminus S$ has at least k' neighbors in S, and at least k'' neighbors in $V \setminus S$. The (k, k', k'')-*domination number* $\gamma_{(k,k',k'')}(G)$ is the minimum cardinality of a (k, k', k'')-dominating set. We note that every graph with the minimum degree at least k has a (k, k', k'')-dominating set, since $S = V(G)$ is such a set. Note that

$$\gamma_{(0,1,1)}(G) = \gamma_r(G), \quad \gamma_{(1,1,1)}(G) = \gamma_t^r(G),$$
$$\gamma_{(1,2,1)}(G) = \gamma_{2r}(G), \quad \gamma_{(k,k,k)}(G) = \gamma^r_{\times k,t}(G).$$

In this section, we calculate a lower bound on $\gamma_{(k,k',k")}(G)$, which improves the existing lower bounds on these four parameters.

Theorem 3.1 ([2, 4]). *If G is a graph without isolated vertices of order n and size m, then*

$$\gamma_t^r(G) \geq 3n/2 - m. \tag{3.1}$$

In addition this bound is sharp.

Also Hattingh et al. [6] found that

$$\gamma_r(G) \geq n - 2m/3. \tag{3.2}$$

The following known result is an immediate consequence of Theorem 3.1.

Theorem 3.2 ([3]). *If T is a tree of order $n \geq 2$, then*

$$\gamma_t^r(T) \geq \left\lceil \frac{n+2}{2} \right\rceil. \tag{3.3}$$

The inequality

$$\gamma_r(T) \geq \left\lceil \frac{n+2}{3} \right\rceil \tag{3.4}$$

on restrained domination number of a tree of order $n \geq 1$ was obtained by Hattingh and Rautenbach [7].

The author in [10] generalized Theorem 3.1 and proved that if $\delta(G) \geq k$, then

$$\gamma^r_{\times k,t}(G) \geq 3n/2 - m/k. \tag{3.5}$$

Moreover the authors in [9] proved that if G is a graph without isolated vertices, then

$$\gamma_{2r}(G) \geq \frac{5n-2m}{4}. \tag{3.6}$$

We now improve the lower bounds given in inequalities (3.1)–(3.6). For this purpose we introduce a notation. Let G be a graph with $\delta(G) \geq k$ and S be a (k, k', k'')-dominating set in G. We define

$$\delta^* = \min\{\deg(v) \mid v \in V(G) \text{ and } \deg(v) \geq k' + k''\}.$$

It is easy to see that if $v \in V \setminus S$, then $\deg(v)$ is at least $k' + k''$ and therefore is at least δ^*.

Theorem 3.3. *Let G be a graph with $\delta(G) \geq k$. Then*

$$\gamma_{(k,k',k'')}(G) \geq \frac{(k'+\delta^*)n - 2m}{\delta^* + k' - k}.$$

Proof. Let S be a minimum (k, k', k'')-dominating set in G. Then, every vertex $v \in S$ is adjacent to at least k vertices in S. Therefore $|E(G[S])| \geq k|S|/2$. Let $E(v)$ be the set of edges at vertex v. Now let $v \in V \setminus S$. Since S is a (k, k', k'')-dominating set, it follows that v is incident to at least k' edges $e_1, ..., e_{k'}$ in $[S, V \setminus S]$ and at least k'' edges $e_{k'+1}, ..., e_{k'+k''}$ in $E(G[V \setminus S])$. Since $deg(v) \geq \delta^* \geq k' + k''$, v is incident to at least $\delta^* - k' - k''$ edges in $E(v) \setminus \{e_i\}_{i=1}^{k'+k''}$. The value of $|[S, V \setminus S]| + |E(G[V \setminus S])|$ is minimized if the edges in $E(v) \setminus \{e_i\}_{i=1}^{k'+k''}$ belong to $E(G[V \setminus S])$. Therefore

$$\begin{aligned} 2m &= 2|E(G[S])| + 2|[S, V \setminus S]| + 2|E(G[V \setminus S])| \\ &\geq k|S| + 2k'(n - |S|) + k''(n - |S|) + (\delta^* - k' - k'')(n - |S|). \end{aligned}$$

This leads to $\gamma_{(k,k',k'')}(G) = |S| \geq ((k' + \delta^*)n - 2m)/(\delta^* + k' - k)$. □

We note that when $(k, k', k'') = (1, 1, 1)$, then Theorem 3.3 gives improvements for inequalities (3.1) and (3.3). When $(k, k', k'') = (0, 1, 1)$, then it will be improvements of its corresponding inequalities given by (3.2) and (3.4). Also, if $(k, k', k'') = (k, k, k)$, Theorem 3.3 improves inequality (3.5) and if $(k, k', k'') = (1, 2, 1)$, it improves inequality (3.6).

The following result of Hattingh and Joubert is an immediate consequence of Theorem 3.3.

Corollary 3.4 ([5]). *If G is a cubic graph of order n, then $\gamma_r(G) \geq n/4$.*

Also, for total restrained and restrained double domination numbers of a cubic graph G, we obtain $\gamma_t^r(G) \geq n/3$ and $\gamma_{2r}(G) \geq n/2$ by Theorem 3.3.

References

1. X. Chen, J. Liu, and J. Meng, *Total restrained domination in graphs*, Comput. Math. Appl. **62** (2011), 2892–2898.
2. J. Cyman and J. Raczek, *On the total restrained domination number of a graph*, Australas. J. Combin. **36** (2006), 91–100.
3. J.H. Hattingh, E. Jonck, E.J. Joubert, and A.R. Plummer, *Total restrained domination in trees*, Discrete Math. **307** (2007), 1643–1650.
4. ———, *Total restrained domination in unicyclic graphs*, Util. Math. **82** (2010), 81–96.
5. J.H. Hattingh and E.J. Joubert, *Restrained domination in cubic graphs*, J. Combin. Optim. **22** (2011), 166–179.
6. J.H. Hattingh, E.J. Joubert, M. Loizeaux, A.R. Plummer, and L. Van Der Merwe, *Restrained domination in unicyclic graphs*, Discuss. Math. **29** (2009), 71–86.
7. J.H. Hattingh and D. Rautenbach, *Further results on weak domination in graphs*, Util. Math. **61** (2002), 193–207.
8. N. Jafari Rad, *Results on total restrained domination in graphs*, Int. J. Contemp. Math. Sci. **3** (2008), 383–387.
9. R. Kala and T.R. Nirmala Vasantha, *Restrained double domination number of a graph*, AKCE Int. J. Graphs Combin. **5** (2008), 73–82.
10. A.P. Kazemi, *k-tuple total restrained domination/domatic in graphs*, Bull. Iranian Math. Soc. **40** (2014), 751–763.
11. S. Ma, X. Chen, and L. Sun, *On total restrained domination in graphs*, Czechoslovak Math. J. **55** (2005), 165–173.
12. J.A. Telle and A. Proskurowski, *Algorithms for vertex partitioning problems on partial k-trees*, SIAM J. Discrete Math. **10** (1997), 529–550.
13. D.B. West, *Introduction to graph theory*, 2nd ed., Prentice Hall, USA, 2001.

Department of Mathematics
University of Mazandaran, Babolsar, IRI
E-mail address: samadibabak62@gmail.com

Department of Mathematics
University of Tafresh, Tafresh, IRI
E-mail address: h.golmohamadi@tafreshu.ac.ir

Department of Mathematics
University of West Georgia Carrollton, GA 30118, USA
E-mail address: akhodkar@westga.edu