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THE ERDŐS–KO–RADO BASIS FOR A LEONARD SYSTEM

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Abstract. We introduce and discuss an *Erdős–Ko–Rado basis* of the vector space underlying a Leonard system $\Phi = \big(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\big)$ that satisfies a mild condition on the eigenvalues of A and A^*. We describe the transition matrices to/from other known bases, as well as the matrices representing A and A^* with respect to the new basis. We also discuss how these results can be viewed as a generalization of the linear programming method used previously in the proofs of the "Erdős–Ko–Rado theorems" for several classical families of Q-polynomial distance-regular graphs, including the original 1961 theorem of Erdős, Ko, and Rado.

1. Introduction

Leonard systems [23] naturally arise in representation theory, combinatorics, and the theory of orthogonal polynomials (see e.g. [25, 28]). Hence they are receiving considerable attention. Indeed, the use of the name "Leonard system" is motivated by a connection to a theorem of Leonard [12], [2, pp. 263–274], which involves the q-Racah polynomials [1] and some related polynomials of the Askey scheme [10]. Leonard systems also play a role in coding theory; see [11].

Let $\Phi = \big(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\big)$ be a Leonard system over a field $\mathbb{K}$, and V the vector space underlying Φ (see Section 2 for formal definitions). Then $V = \bigoplus_{i=0}^d E_i^* V$ and $\dim E_i^* V = 1$ $(0 \leqslant i \leqslant d)$. We have a "canonical" (ordered) basis of V associated with this direct sum decomposition, called a *standard basis*. There are 8 variations for the standard basis. Next, let $U_\ell = \big(\sum_{i=0}^{\ell} E_i^* V\big) \cap \big(\sum_{j=\ell}^{d} E_j V\big)$ $(0 \leqslant \ell \leqslant d)$. Then, again it follows that $V = \bigoplus_{\ell=0}^d U_\ell$ and $\dim U_\ell = 1$ $(0 \leqslant \ell \leqslant d)$. We have a "canonical" basis of V associated with this *split decomposition*, called a *split basis*. The split

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decomposition is crucial in the theory of Leonard systems,[1] and there are 16 variations for the split basis. Altogether, Terwilliger [24] defined 24 bases of V and studied in detail the transition matrices between these bases as well as the matrices representing A and A^* with respect to them.

In the present paper, we introduce another basis of V, which we call an *Erdős–Ko–Rado* (or *EKR*) *basis* of V, under a mild condition on the eigenvalues of A and A^* (see below). As its name suggests, this basis arises in connection with the famous *Erdős–Ko–Rado theorem* [6] in extremal set theory. Indeed, Delsarte's *linear programming method* [4], which is closely related to Lovász's ϑ-function bound [13, 16] on the Shannon capacity of graphs, has been successfully used in the proofs of the "Erdős–Ko–Rado theorems" for certain families of *Q-polynomial distance-regular graphs*[2] [29, 7, 17, 20] (including the original 1961 theorem of Erdős et al.), and constructing appropriate feasible solutions to the dual programs amounts to describing the EKR bases for the Leonard systems associated with these graphs; see Section 4. It seems that the previous constructions of the feasible solutions depend on the geometric/algebraic structures which are more or less specific to the family of graphs in question. Our results give a uniform description of such feasible solutions in terms of the *parameter arrays* of Leonard systems.

The contents of the paper are as follows. Section 2 reviews basic terminology, notation and facts concerning Leonard systems. In Section 3, we first study the subspaces $W_t = \big(E_0^*V + \sum_{i=d-t+1}^{d} E_i^*V\big) \cap \big(E_0V + \sum_{j=t+1}^{d} E_jV\big)$ $(0 \leqslant t \leqslant d)$. We show that $\dim W_t = 1$ $(0 \leqslant t \leqslant d)$, and that $V = \bigoplus_{t=0}^{d} W_t$ if and only if $q \neq -1$, or $q = -1$ and d is even, where q denotes a *base* of Φ (which is determined by the recurrence satisfied by the eigenvalues of A and A^*). Assuming that this is the case, we then define an EKR basis associated with this direct sum decomposition. We describe the transition matrices to/from 3 bases out of the 24 bases mentioned above (2 standard, 1 split), as well as the matrices representing A and A^* with respect to the EKR basis. Our main results are Theorems 3.9, 3.12, and 3.13. Section 4 is devoted to discussions of the connections and applications of these results to the Erdős–Ko–Rado theorems.

2. Leonard systems

Let $\mathbb{K}$ be a field, d a positive integer, $\mathscr{A}$ a $\mathbb{K}$-algebra isomorphic to the full matrix algebra $\mathrm{Mat}_{d+1}(\mathbb{K})$, and V an irreducible left $\mathscr{A}$-module. We remark that V is unique up to isomorphism, and that V has dimension $d+1$. An element A of $\mathscr{A}$ is said to be *multiplicity-free* if it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Let A be a multiplicity-free element of $\mathscr{A}$ and

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[1] In some cases, V has the structure of an evaluation module of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, and the split decomposition corresponds to its weight space decomposition; see e.g. [9].

[2] Q-polynomial distance-regular graphs are thought of as finite/combinatorial analogues of compact symmetric spaces of rank one; see [2, pp. 311–312].

$\{\theta_i\}_{i=0}^d$ an ordering of the eigenvalues of A. Let $E_i : V \to V(\theta_i)$ $(0 \leqslant i \leqslant d)$ be the projection map onto $V(\theta_i)$ with respect to $V = \bigoplus_{i=0}^d V(\theta_i)$, where $V(\theta_i) = \{\boldsymbol{u} \in V : A\boldsymbol{u} = \theta_i \boldsymbol{u}\}$. We call E_i the *primitive idempotent* of A associated with θ_i. Notice that the E_i are polynomials in A.

A *Leonard system* in $\mathscr{A}$ ([23, Definition 1.4]) is a sequence

$$\Phi = \Big(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\Big) \tag{1}$$

satisfying the following axioms (LS1)–(LS5):

(LS1) Each of A, A^* is a multiplicity-free element in $\mathscr{A}$.[3]

(LS2) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.

(LS3) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.

(LS4) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

(LS5) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

We say that Φ is *over* $\mathbb{K}$. We refer the reader to [23, 26, 28] for background on Leonard systems.

Throughout the paper, $\Phi = \big(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d\big)$ shall always denote the Leonard system (1). Notice that the following are Leonard systems:

$$\begin{aligned}
\Phi^* &= \Big(A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d\Big), \\
\Phi^{\downarrow} &= \Big(A; A^*; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d\Big), \\
\Phi^{\Downarrow} &= \Big(A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d\Big).
\end{aligned}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \quad \Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow\Downarrow = \Downarrow\downarrow.$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the above relations is the dihedral group D_4 with 8 elements. We shall use the following notational convention:

Notation 2.1. For any $g \in D_4$ and for any object f associated with Φ, we let f^g denote the corresponding object for $\Phi^{g^{-1}}$; an example is $E_i^*(\Phi) = E_i(\Phi^*)$.

It is known ([26, Theorem 6.1]) that there is a unique antiautomorphism $\dagger$ of $\mathscr{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. From now on, let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ be a nondegenerate bilinear form on V such that ([26, Section 15])

$$\langle X\boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{u}_1, X^\dagger \boldsymbol{u}_2 \rangle \quad (\boldsymbol{u}_1, \boldsymbol{u}_2 \in V,\ X \in \mathscr{A}).$$

We shall write

$$||\boldsymbol{u}||^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle \quad (\boldsymbol{u} \in V).$$

[3]It is customary that A^* denotes the conjugate transpose of A. It should be stressed that we are *not* using this convention.

Notation 2.2. Henceforth we fix a nonzero vector $\boldsymbol{v}^g$ in $E_0^g V$ for each $g \in D_4$. We abbreviate $\boldsymbol{v} = \boldsymbol{v}^1$ where 1 is the identity of D_4. For convenience, we also assume $\boldsymbol{v}^{g_1} = \boldsymbol{v}^{g_2}$ whenever $E_0^{g_1} V = E_0^{g_2} V$ ($g_1, g_2 \in D_4$). We remark that $||\boldsymbol{v}^g||^2$, $\langle \boldsymbol{v}^g, \boldsymbol{v}^{*g} \rangle$ are nonzero for any $g \in D_4$; cf. [26, Lemma 15.5].

We now recall a few direct sum decompositions of V, as well as (ordered) bases of V associated with them. First, $\dim E_i^* V = 1$ ($0 \leqslant i \leqslant d$) and $V = \bigoplus_{i=0}^d E_i^* V$. By [26, Lemma 10.2], $E_i^* \boldsymbol{v} \neq 0$ ($0 \leqslant i \leqslant d$), so that $\{E_i^* \boldsymbol{v}\}_{i=0}^d$ is a basis of V, called a Φ-*standard basis* of V. Next, let $U_\ell = \left(\sum_{i=0}^\ell E_i^* V\right) \cap \left(\sum_{j=\ell}^d E_j V\right)$ ($0 \leqslant \ell \leqslant d$). Then, again $\dim U_\ell = 1$ ($0 \leqslant \ell \leqslant d$) and $V = \bigoplus_{\ell=0}^d U_\ell$, which is referred to as the Φ-*split decomposition* of V [28]. We observe $U_0 = E_0^* V$ and $U_d = E_d V$. For $0 \leqslant i \leqslant d$, let θ_i be the eigenvalue of A associated with E_i. Then it follows that $(A - \theta_\ell I) U_\ell = U_{\ell+1}$ and $(A^* - \theta_\ell^* I) U_\ell = U_{\ell-1}$ for $0 \leqslant \ell \leqslant d$, where $U_{-1} = U_{d+1} = 0$ [23, Lemma 3.9]. For $0 \leqslant i \leqslant d$, let τ_i, η_i be the following polynomials in $\mathbb{K}[z]$:

$$\tau_i(z) = \prod_{h=0}^{i-1} (z - \theta_h), \quad \eta_i(z) = \tau_i^{\Downarrow}(z) = \prod_{h=0}^{i-1} (z - \theta_{d-h}).$$

From the above comments it follows that $\tau_\ell(A) \boldsymbol{v}^* \in U_\ell$ ($0 \leqslant \ell \leqslant d$) and $\{\tau_\ell(A) \boldsymbol{v}^*\}_{\ell=0}^d$ is a basis of V, called a Φ-*split basis* of V. Moreover, there are nonzero scalars φ_i ($1 \leqslant i \leqslant d$) in $\mathbb{K}$ such that $A^* \tau_\ell(A) \boldsymbol{v}^* = \theta_\ell^* \tau_\ell(A) \boldsymbol{v}^* + \varphi_\ell \tau_{\ell-1}(A) \boldsymbol{v}^*$ ($1 \leqslant \ell \leqslant d$).

Let $\phi_i = \varphi_i^{\Downarrow}$ ($1 \leqslant i \leqslant d$). The *parameter array* of Φ is

$$p(\Phi) = \Big(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d \Big).$$

Terwilliger [23, Theorem 1.9] showed that the isomorphism class[4] of Φ is determined by $p(\Phi)$ and gave a classification of the parameter arrays of Leonard systems; cf. [27, Section 5]. In particular, the sequences $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ are recurrent in the sense that there is a scalar $\beta \in \mathbb{K}$ such that

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} = \beta + 1 \quad (2 \leqslant i \leqslant d-1). \tag{2}$$

It also follows that

$$\phi_i = \varphi_1 \vartheta_i + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leqslant i \leqslant d), \tag{3}$$

where

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leqslant i \leqslant d).$$

[4] A Leonard system Ψ in a $\mathbb{K}$-algebra $\mathscr{B}$ is *isomorphic* to Φ if there is a $\mathbb{K}$-algebra isomorphism $\gamma : \mathscr{A} \to \mathscr{B}$ such that $\Psi = \Phi^\gamma := \left(A^\gamma; A^{*\gamma}; \{E_i^\gamma\}_{i=0}^d; \{E_i^{*\gamma}\}_{i=0}^d\right)$.

Notice that $\vartheta_1 = \vartheta_d = 1$. Moreover,

$$\vartheta_{d-i+1} = \vartheta_i, \quad \vartheta_i^* = \vartheta_i \quad (1 \leqslant i \leqslant d). \tag{4}$$

The parameter array behaves nicely with respect to the D_4 action:

Lemma 2.3 ([23, Theorem 1.11])**.** *The following hold.*

(i) $p(\Phi^*) = \Big(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d\Big)$.

(ii) $p(\Phi^{\downarrow}) = \Big(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d\Big)$.

(iii) $p(\Phi^{\Downarrow}) = \Big(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d\Big)$.

The following can be easily read off [24, 26].

Lemma 2.4 ([24, 26])**.** *The following hold.*

(i) $$E_i^* \boldsymbol{v} = \frac{||E_i^* \boldsymbol{v}||^2}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \sum_{\ell=0}^{i} \frac{\tau_\ell^*(\theta_i^*)}{\varphi_1 \cdots \varphi_\ell} \tau_\ell(A) \boldsymbol{v}^* \quad (0 \leqslant i \leqslant d).$$

(ii) $$\tau_\ell(A)\boldsymbol{v}^* = \langle \boldsymbol{v}, \boldsymbol{v}^* \rangle \cdot \varphi_1 \cdots \varphi_\ell \times \sum_{i=0}^{\ell} \frac{\eta_{d-\ell}^*(\theta_i^*)}{\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)} \cdot \frac{1}{||E_i^* \boldsymbol{v}||^2} E_i^* \boldsymbol{v} \quad (0 \leqslant \ell \leqslant d).$$

(iii) $$E_j \boldsymbol{v}^* = \sum_{\ell=j}^{d} \frac{\eta_{d-\ell}(\theta_j)}{\tau_j(\theta_j)\eta_{d-j}(\theta_j)} \tau_\ell(A)\boldsymbol{v}^* \quad (0 \leqslant j \leqslant d).$$

(iv) $$\tau_\ell(A)\boldsymbol{v}^* = \sum_{j=\ell}^{d} \tau_\ell(\theta_j) E_j \boldsymbol{v}^* \quad (0 \leqslant \ell \leqslant d).$$

(v) $$E_j \boldsymbol{v}^{*\downarrow} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \frac{\phi_{d-j+1} \cdots \phi_d}{\varphi_1 \cdots \varphi_j} E_j \boldsymbol{v}^* \quad (0 \leqslant j \leqslant d).$$

Finally, it follows that ([26, Lemma 9.2, Theorem 17.12])

$$E_0^* E_i E_0^* = \frac{\varphi_1 \cdots \varphi_i \phi_1 \cdots \phi_{d-i}}{\eta_d^*(\theta_0^*)\tau_i(\theta_i)\eta_{d-i}(\theta_i)} E_0^* \quad (0 \leqslant i \leqslant d),$$

from which it follows that

$$||E_i^* \boldsymbol{v}||^2 = \frac{\varphi_1 \cdots \varphi_i \phi_{i+1} \cdots \phi_d}{\eta_d(\theta_0)\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)} ||\boldsymbol{v}||^2 \quad (0 \leqslant i \leqslant d), \tag{5}$$

by virtue of Lemma 2.3 (i).

3. The Erdős–Ko–Rado basis

Let $F_\ell : V \to U_\ell$ $(0 \leqslant \ell \leqslant d)$ be the projection map onto U_ℓ with respect to the Φ-split decomposition $V = \bigoplus_{\ell=0}^d U_\ell$.

Lemma 3.1 (cf. [8, Lemma 5.4])**.** *The following hold.*

(i) $F_\ell E_i^* = 0$ *if* $\ell > i$ $\quad (0 \leqslant i, \ell \leqslant d)$.

(ii) $F_\ell E_j = 0$ *if* $\ell < j$ $\quad (0 \leqslant j, \ell \leqslant d)$.

Proof. Immediate from $E_i^*V \subseteq \sum_{\ell=0}^{i} U_\ell$ and $E_jV \subseteq \sum_{\ell=j}^{d} U_\ell$. $\square$

We shall mainly work with the $\Phi^\downarrow$-split decomposition $V = \bigoplus_{\ell=0}^{d} U_\ell^\downarrow$, where

$$U_\ell^\downarrow = \left(\sum_{i=d-\ell}^{d} E_i^*V\right) \cap \left(\sum_{j=\ell}^{d} E_jV\right) \quad (0 \leqslant \ell \leqslant d).$$

We now "modify" the $U_\ell^\downarrow$ and introduce the subspaces W_t $(0 \leqslant t \leqslant d)$ of V defined by[5]

$$W_t = \left(E_0^*V + \sum_{i=d-t+1}^{d} E_i^*V\right) \cap \left(E_0V + \sum_{j=t+1}^{d} E_jV\right) \quad (0 \leqslant t \leqslant d).$$

Observe $W_t \neq 0$ $(0 \leqslant t \leqslant d)$, $W_0 = E_0^*V$, and $W_d = E_0V$. Notice also that

$$W_t^* = W_{d-t} \quad (0 \leqslant t \leqslant d). \tag{6}$$

Our aim is to show $\dim W_t = 1$ $(0 \leqslant t \leqslant d)$, and then to determine precisely when $V = \bigoplus_{t=0}^{d} W_t$. Pick $\boldsymbol{w} \in W_t$. Then from Lemma 3.1 (applied to $\Phi^\downarrow$) it follows that

$$F_\ell^\downarrow \boldsymbol{w} = \sum_{i=0}^{d-\ell} F_\ell^\downarrow E_i^* \boldsymbol{w} = \sum_{j=0}^{\ell} F_\ell^\downarrow E_j \boldsymbol{w} \quad (0 \leqslant \ell \leqslant d).$$

Hence

$$F_\ell^\downarrow \boldsymbol{w} = \begin{cases} F_\ell^\downarrow E_0 \boldsymbol{w} & \text{if } 0 \leqslant \ell \leqslant t, \\ F_\ell^\downarrow E_0^* \boldsymbol{w} & \text{if } t \leqslant \ell \leqslant d, \end{cases} \tag{7}$$

from which it follows that

$$\boldsymbol{w} = \sum_{\ell=0}^{t} F_\ell^\downarrow E_0 \boldsymbol{w} + \sum_{\ell=t+1}^{d} F_\ell^\downarrow E_0^* \boldsymbol{w} = E_0 \boldsymbol{w} + \sum_{\ell=t+1}^{d} F_\ell^\downarrow (E_0^* - E_0) \boldsymbol{w}. \tag{8}$$

By Lemma 2.4 (i) and Lemma 2.3 (ii), we have

$$\begin{aligned} F_\ell^\downarrow E_0^* \boldsymbol{w} &= F_\ell^\downarrow E_d^{*\downarrow} \boldsymbol{w} \\ &= \frac{\langle \boldsymbol{w}, E_d^{*\downarrow} \boldsymbol{v}^\downarrow \rangle}{\|E_d^{*\downarrow} \boldsymbol{v}^\downarrow\|^2} F_\ell^\downarrow E_d^{*\downarrow} \boldsymbol{v}^\downarrow \\ &= \frac{\langle \boldsymbol{w}, E_d^{*\downarrow} \boldsymbol{v}^\downarrow \rangle}{\langle \boldsymbol{v}^\downarrow, \boldsymbol{v}^{*\downarrow} \rangle} \cdot \frac{\tau_\ell^{*\downarrow}(\theta_d^{*\downarrow})}{\varphi_1^\downarrow \cdots \varphi_\ell^\downarrow} \tau_\ell^\downarrow(A^\downarrow) \boldsymbol{v}^{*\downarrow} \\ &= \frac{\langle \boldsymbol{w}, E_0^* \boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle} \cdot \frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_d} \tau_\ell(A) \boldsymbol{v}^{*\downarrow} \end{aligned} \tag{9}$$

[5] The subscript t is chosen in accordance with the concept of *t-intersecting families* in the Erdős–Ko–Rado theorem; see Section 4.

for $0 \leqslant \ell \leqslant d$. Likewise, by Lemma 2.4 (iii) and Lemma 2.3 (ii), we have

$$
\begin{aligned}
F_\ell^\downarrow E_0 \boldsymbol{w} &= F_\ell^\downarrow E_0^\downarrow \boldsymbol{w} \\
&= \frac{\langle \boldsymbol{w}, E_0^\downarrow \boldsymbol{v}^{*\downarrow}\rangle}{||E_0^\downarrow \boldsymbol{v}^{*\downarrow}||^2} F_\ell^\downarrow E_0^\downarrow \boldsymbol{v}^{*\downarrow} \\
&= \frac{\langle \boldsymbol{w}, E_0 \boldsymbol{v}^{*\downarrow}\rangle}{||E_0 \boldsymbol{v}^{*\downarrow}||^2} \cdot \frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A) \boldsymbol{v}^{*\downarrow}
\end{aligned} \tag{10}
$$

for $0 \leqslant \ell \leqslant d$. Since $F_t^\downarrow E_0^* \boldsymbol{w} = F_t^\downarrow E_0 \boldsymbol{w}$ by (7), we have in particular:

$$
\frac{\langle \boldsymbol{w}, E_0^* \boldsymbol{v}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle} \cdot \frac{\eta_t^*(\theta_0^*)}{\phi_{d-t+1} \cdots \phi_d} = \frac{\langle \boldsymbol{w}, E_0 \boldsymbol{v}^{*\downarrow}\rangle}{||E_0 \boldsymbol{v}^{*\downarrow}||^2} \cdot \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)}. \tag{11}
$$

Combining these comments, it follows from (8), Lemma 2.4 (iv) and (v) that

$$
\begin{aligned}
\boldsymbol{w} &= E_0 \boldsymbol{w} + \frac{\langle \boldsymbol{w}, E_0 \boldsymbol{v}^{*\downarrow}\rangle}{||E_0 \boldsymbol{v}^{*\downarrow}||^2} \cdot \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \\
&\quad \times \sum_{\ell=t+1}^{d} \left(\frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} - \frac{\eta_t^*(\theta_0^*)\eta_{d-\ell}(\theta_0)}{\eta_{d-t}(\theta_0)} \right) \tau_\ell(A) \boldsymbol{v}^{*\downarrow} \\
&= E_0 \boldsymbol{w} + \frac{\langle \boldsymbol{w}, E_0 \boldsymbol{v}^{*}\rangle}{||E_0 \boldsymbol{v}^{*}||^2} \cdot \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \sum_{j=t+1}^{d} \frac{\phi_{d-j+1} \cdots \phi_d}{\varphi_1 \cdots \varphi_j} \\
&\quad \times \sum_{\ell=t+1}^{j} \tau_\ell(\theta_j) \left(\frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} - \frac{\eta_t^*(\theta_0^*)\eta_{d-\ell}(\theta_0)}{\eta_{d-t}(\theta_0)} \right) E_j \boldsymbol{v}^*.
\end{aligned}
$$

The coefficient of the last sum is equal to $(\theta_j - \theta_0)^{-1}$ times

$$
\begin{aligned}
&\sum_{\ell=t+1}^{j} (\theta_j - \theta_\ell + \theta_\ell - \theta_0) \cdot \tau_\ell(\theta_j) \left(\frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} - \frac{\eta_t^*(\theta_0^*)\eta_{d-\ell}(\theta_0)}{\eta_{d-t}(\theta_0)} \right) \\
&= \sum_{\ell=t+1}^{j-1} \tau_{\ell+1}(\theta_j) \left(\frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} - \frac{\eta_t^*(\theta_0^*)\eta_{d-\ell}(\theta_0)}{\eta_{d-t}(\theta_0)} \right) \\
&\qquad - \sum_{\ell=t+1}^{j} \tau_\ell(\theta_j) \left(\frac{\eta_\ell^*(\theta_0^*)(\theta_0 - \theta_\ell)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} - \frac{\eta_t^*(\theta_0^*)\eta_{d-\ell+1}(\theta_0)}{\eta_{d-t}(\theta_0)} \right) \\
&= \sum_{\ell=t+1}^{j} \tau_\ell(\theta_j) \left(\frac{\eta_{\ell-1}^*(\theta_0^*)}{\phi_{d-\ell+2} \cdots \phi_{d-t}} - \frac{\eta_\ell^*(\theta_0^*)(\theta_0 - \theta_\ell)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} \right) \\
&= \sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta_{\ell-1}^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} \big(\phi_{d-\ell+1} - (\theta_0^* - \theta_{d-\ell+1}^*)(\theta_0 - \theta_\ell) \big) \\
&= \sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta_{\ell-1}^*(\theta_0^*)}{\phi_{d-\ell+1} \cdots \phi_{d-t}} \varphi_1 \vartheta_\ell,
\end{aligned}
$$

where we have used (3) and (4). Hence

Proposition 3.2. *Let $\boldsymbol{w} \in W_t$. Then the following hold.*

(i)
$$\boldsymbol{w} = E_0\boldsymbol{w} + \frac{\langle \boldsymbol{w}, E_0\boldsymbol{v}^*\rangle}{||E_0\boldsymbol{v}^*||^2} \cdot \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)} \times \sum_{j=t+1}^{d} \frac{\phi_{d-j+1}\cdots\phi_d}{\varphi_2\cdots\varphi_j(\theta_j-\theta_0)} \left(\sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta^*_{\ell-1}(\theta^*_0)\vartheta_\ell}{\phi_{d-\ell+1}\cdots\phi_{d-t}} \right) E_j\boldsymbol{v}^*.$$

(ii)
$$\boldsymbol{w} = E^*_0\boldsymbol{w} + \frac{\langle \boldsymbol{w}, E^*_0\boldsymbol{v}\rangle}{||E^*_0\boldsymbol{v}||^2} \cdot \frac{\eta^*_t(\theta^*_0)}{\eta^*_d(\theta^*_0)\eta_{d-t}(\theta_0)} \times \sum_{i=d-t+1}^{d} \frac{\phi_1\cdots\phi_i}{\varphi_2\cdots\varphi_i(\theta^*_i-\theta^*_0)} \left(\sum_{\ell=d-t+1}^{i} \frac{\tau^*_\ell(\theta^*_i)\eta_{\ell-1}(\theta_0)\vartheta_\ell}{\phi_{d-t+1}\cdots\phi_\ell} \right) E^*_i\boldsymbol{v}.$$

In particular, $E_0W_t \neq 0$, $E^*_0W_t \neq 0$, and $\dim W_t = 1$.

Proof. (i): Clear.

(ii): By virtue of (6), the result follows from (i) above, together with Lemma 2.3 (i) and (4).

The last line follows by noting that each of $E_0\boldsymbol{w}$, $E^*_0\boldsymbol{w}$ determines $\boldsymbol{w}$. □

Notation 3.3. Henceforth we let q be a nonzero scalar in the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ such that $q + q^{-1} = \beta$, where the scalar β is from (2). We call q a *base* for Φ.[6] By convention, if $d < 3$ then q can be taken to be *any* nonzero scalar in $\overline{\mathbb{K}}$.

Lemma 3.4 (cf. [18, (6.4)]). *For $1 \leqslant i \leqslant d$, we have $\vartheta_i = 0$ precisely when $q = -1$, d is odd, and i is even.*

From Proposition 3.2 and Lemma 3.4, it follows that

Lemma 3.5. *Let q be as above. Then for $1 \leqslant t \leqslant d-1$, the following hold.*

(i) *Suppose $q \neq -1$, or $q = -1$ and d is even. Then $E^*_{d-t+1}W_t \neq 0$ and $E_{t+1}W_t \neq 0$.*

(ii) *Suppose $q = -1$ and d is odd. Then $E^*_{d-t+1}W_t \neq 0$ (resp. $E_{t+1}W_t \neq 0$) if and only if t is odd (resp. even).*

Corollary 3.6. *Let q be as above. Then the following hold.*

(i) *Suppose $q \neq -1$, or $q = -1$ and d is even. Then $V = \bigoplus_{t=0}^{d} W_t$. Moreover,*

$$\sum_{t=0}^{h} W_t = E^*_0V + \sum_{i=d-h+1}^{d} E^*_iV$$

[6] We may remark that if $d \geqslant 3$ then Φ has at most two bases, i.e., q and q^{-1}.

and

$$\sum_{t=h}^{d} W_t = E_0V \; + \sum_{j=h+1}^{d} E_jV$$

for $0 \leqslant h \leqslant d$.

(ii) *Suppose* $q = -1$ *and* d *is odd. Then* $W_{2s-1} = W_{2s}$ *for* $1 \leqslant s \leqslant \lfloor d/2 \rfloor$.

Proof. (i): Immediate from Lemma 3.5 (i).

(ii): It follows from Lemma 3.5 (ii) that

$$W_{2s-1} = \left(E_0^*V + \sum_{i=d-2s+2}^{d} E_i^*V\right) \cap \left(E_0V + \sum_{j=2s+1}^{d} E_jV\right) = W_{2s}$$

for $1 \leqslant s \leqslant \lfloor d/2 \rfloor$. □

By virtue of Corollary 3.6, we make the following assumption.

Assumption 3.7. With reference to Notation 3.3, for the rest of the paper we shall assume $q \neq -1$, or $q = -1$ and d is even.[7]

We are now ready to introduce an Erdős–Ko–Rado basis of V.

Definition 3.8. With reference to Assumption 3.7, for $0 \leqslant t \leqslant d$ let $\boldsymbol{w}_t$ be the (unique) vector in W_t such that $E_0\boldsymbol{w}_t = E_0\boldsymbol{v}^*$. We call $\{\boldsymbol{w}_t\}_{t=0}^d$ a (Φ-)*Erdős–Ko–Rado* (or (Φ-)*EKR*) *basis* of V.

Notice that the basis $\{\boldsymbol{w}_t\}_{t=0}^d$ linearly depends on the choice of $\boldsymbol{v}^* \in E_0^*V$. In particular, we have $\boldsymbol{w}_0 = \boldsymbol{v}^*$ and $\boldsymbol{w}_d = E_0\boldsymbol{v}^*$. Our preference for the normalization $E_0\boldsymbol{w}_t = E_0\boldsymbol{v}^*$ comes from the applications to the Erdős–Ko–Rado theorem; see Section 4. The following theorem gives the transition matrix from each of the $\Phi^{\downarrow}$-split basis $\{\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\}_{\ell=0}^d$, the Φ^*-standard basis $\{E_j\boldsymbol{v}^*\}_{j=0}^d$, and the Φ-standard basis $\{E_i^*\boldsymbol{v}\}_{i=0}^d$, to the EKR basis $\{\boldsymbol{w}_t\}_{t=0}^d$.

Theorem 3.9. *The following hold for* $0 \leqslant t \leqslant d$.

(i) $$\boldsymbol{w}_t = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle} \left\{ \sum_{\ell=0}^{t} \frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A)\boldsymbol{v}^{*\downarrow} + \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \sum_{\ell=t+1}^{d} \frac{\eta_\ell^*(\theta_0^*)}{\phi_{d-\ell+1}\cdots\phi_{d-t}} \tau_\ell(A)\boldsymbol{v}^{*\downarrow} \right\}.$$

(ii) $$\boldsymbol{w}_t = E_0\boldsymbol{v}^* + \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} \times \sum_{j=t+1}^{d} \frac{\phi_{d-j+1}\cdots\phi_d}{\varphi_2\cdots\varphi_j(\theta_j-\theta_0)} \left(\sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta_{\ell-1}^*(\theta_0^*)\vartheta_\ell}{\phi_{d-\ell+1}\cdots\phi_{d-t}} \right) E_j\boldsymbol{v}^*.$$

[7]The Leonard systems with $d \geqslant 3$ that do *not* satisfy this assumption are precisely those of *Bannai/Ito* type [27, Example 5.14] with d odd, and those of *Orphan* type [27, Example 5.15].

(iii) $$\boldsymbol{w}_t = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{||\boldsymbol{v}||^2} \left\{ \frac{\eta_d^*(\theta_0^*)\eta_{d-t}(\theta_0)}{\phi_1 \cdots \phi_{d-t}\eta_t^*(\theta_0^*)} E_0^* \boldsymbol{v} + \sum_{i=d-t+1}^{d} \frac{\phi_{d-t+1} \cdots \phi_i}{\varphi_2 \cdots \varphi_i(\theta_i^* - \theta_0^*)} \left(\sum_{\ell=d-t+1}^{i} \frac{\tau_\ell^*(\theta_i^*)\eta_{\ell-1}(\theta_0)\vartheta_\ell}{\phi_{d-t+1} \cdots \phi_\ell} \right) E_i^* \boldsymbol{v} \right\}.$$

Proof. (i): By Lemma 2.4 (v) and since $E_0\boldsymbol{w}_t = E_0\boldsymbol{v}^*$, we have

$$\frac{\langle \boldsymbol{w}_t, E_0\boldsymbol{v}^{*\downarrow} \rangle}{||E_0\boldsymbol{v}^{*\downarrow}||^2} = \frac{\langle \boldsymbol{w}_t, E_0\boldsymbol{v}^{*} \rangle}{||E_0\boldsymbol{v}^{*}||^2} \cdot \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle}. \tag{12}$$

Combining this with (11), it follows that

$$\begin{aligned} E_0^*\boldsymbol{w}_t &= \frac{\langle \boldsymbol{w}_t, E_0^*\boldsymbol{v} \rangle}{||E_0^*\boldsymbol{v}||^2} E_0^*\boldsymbol{v} \\ &= \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle \langle \boldsymbol{w}_t, E_0\boldsymbol{v}^{*\downarrow} \rangle}{||E_0^*\boldsymbol{v}||^2 ||E_0\boldsymbol{v}^{*\downarrow}||^2} \cdot \frac{\phi_{d-t+1} \dots \phi_d \eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} E_0^*\boldsymbol{v} \\ &= \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{||E_0^*\boldsymbol{v}||^2} \cdot \frac{\phi_{d-t+1} \dots \phi_d \eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)} E_0^*\boldsymbol{v}, \end{aligned} \tag{13}$$

from which it follows that

$$\frac{\langle \boldsymbol{w}_t, E_0^*\boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle} \cdot \frac{\phi_{d-t+1} \dots \phi_d \eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_t^*(\theta_0^*)}. \tag{14}$$

Now the result follows from (8)–(10), (12), and (14).

(ii): Immediate from Proposition 3.2 (i) and $E_0\boldsymbol{w}_t = E_0\boldsymbol{v}^*$.

(iii): Follows from Proposition 3.2 (ii), (5), and (13). □

Corollary 3.10. *Let $\{\boldsymbol{w}_t^*\}_{t=0}^d$ be the Φ^*-EKR basis of V normalized so that $E_0^*\boldsymbol{w}_t^* = E_0^*\boldsymbol{v}$ $(0 \leqslant t \leqslant d)$. Then*

$$\boldsymbol{w}_t^* = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{||\boldsymbol{v}^*||^2} \cdot \frac{\eta_d(\theta_0)\eta_{d-t}^*(\theta_0^*)}{\phi_{t+1} \dots \phi_d \eta_t(\theta_0)} \boldsymbol{w}_{d-t} \quad (0 \leqslant t \leqslant d).$$

Proof. By (6), $\boldsymbol{w}_t^*$ is a scalar multiple of $\boldsymbol{w}_{d-t}$, and the scalar is found by looking at the coefficient of $E_0^*\boldsymbol{v}$ in $\boldsymbol{w}_{d-t}$ as given in Theorem 3.9 (iii), and by noting that $\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle^2 ||\boldsymbol{v}^*||^{-2} = ||E_0^*\boldsymbol{v}||^2 = \phi_1 \dots \phi_d \eta_d(\theta_0)^{-1}\eta_d^*(\theta_0^*)^{-1}||\boldsymbol{v}||^2$ in view of (5). □

Our next goal is to compute the transition matrix from the EKR basis $\{\boldsymbol{w}_t\}_{t=0}^d$ to each of the three bases $\{\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\}_{\ell=0}^d$, $\{E_j\boldsymbol{v}^*\}_{j=0}^d$, and $\{E_i^*\boldsymbol{v}\}_{i=0}^d$. Let $G_t : V \to W_t$ $(0 \leqslant t \leqslant d)$ be the projection map onto W_t with respect to $V = \bigoplus_{t=0}^d W_t$.

Lemma 3.11. *The following hold.*

(i) $G_tE_i^* = 0$ *if* $t > d - i + 1$, *or* $t > 0$ *and* $i = 0$ $\quad (0 \leqslant i, t \leqslant d)$.

(ii) $G_tE_j = 0$ *if* $t < j - 1$, *or* $t < d$ *and* $j = 0$ $\quad (0 \leqslant j, t \leqslant d)$.

Proof. Immediate from Corollary 3.6 (i). □

For the moment, we write $\boldsymbol{u} = \boldsymbol{u}_\ell = \tau_\ell(A)\boldsymbol{v}^{*\downarrow} \in U_\ell^\downarrow$. Then it follows that

$$G_t\boldsymbol{u} = \sum_{i=d-\ell}^{d} G_t E_i^* \boldsymbol{u} = \sum_{j=\ell}^{d} G_t E_j \boldsymbol{u} \quad (0 \leqslant t \leqslant d).$$

Hence it follows from Lemma 3.11 that

$$G_t\boldsymbol{u} = \begin{cases} G_{\ell+1}E_{d-\ell}^*\boldsymbol{u} & \text{if } t = \ell+1, \\ G_\ell E_\ell \boldsymbol{u} + G_\ell E_{\ell+1}\boldsymbol{u} & \text{if } t = \ell, \\ G_{\ell-1}E_\ell \boldsymbol{u} & \text{if } t = \ell-1, \\ 0 & \text{if } t \leqslant \ell-2 \text{ or } t \geqslant \ell+2. \end{cases} \tag{15}$$

In particular:

$$\boldsymbol{u} = G_{\ell-1}\boldsymbol{u} + G_\ell \boldsymbol{u} + G_{\ell+1}\boldsymbol{u}. \tag{16}$$

By Lemma 2.4 (iv) and (v), we have

$$E_\ell \boldsymbol{u} = \tau_\ell(\theta_\ell)E_\ell \boldsymbol{v}^{*\downarrow} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\phi_{d-\ell+1}\cdots\phi_d \tau_\ell(\theta_\ell)}{\varphi_1\cdots\varphi_\ell} E_\ell \boldsymbol{v}^*, \tag{17}$$

$$E_{\ell+1} \boldsymbol{u} = \tau_\ell(\theta_{\ell+1})E_{\ell+1} \boldsymbol{v}^{*\downarrow} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\phi_{d-\ell}\cdots\phi_d \tau_\ell(\theta_{\ell+1})}{\varphi_1\cdots\varphi_{\ell+1}} E_{\ell+1} \boldsymbol{v}^*. \tag{18}$$

Likewise, by Lemma 2.4 (ii) and Lemma 2.3 (ii),

$$\begin{aligned} E_{d-\ell}^*\boldsymbol{u} &= E_\ell^{*\downarrow}\boldsymbol{u} \\ &= \langle \boldsymbol{v}^\downarrow, \boldsymbol{v}^{*\downarrow}\rangle \cdot \frac{\varphi_1^\downarrow \cdots \varphi_\ell^\downarrow}{\tau_\ell^{*\downarrow}(\theta_\ell^{*\downarrow})||E_\ell^{*\downarrow}\boldsymbol{v}^\downarrow||^2} E_\ell^{*\downarrow}\boldsymbol{v}^\downarrow \\ &= \langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle \cdot \frac{\phi_{d-\ell+1}\cdots\phi_d}{\eta_\ell^*(\theta_{d-\ell}^*)||E_{d-\ell}^*\boldsymbol{v}||^2} E_{d-\ell}^*\boldsymbol{v}. \end{aligned} \tag{19}$$

Notice that the transition matrix from the basis $E_1\boldsymbol{v}^*, \dots, E_d\boldsymbol{v}^*, E_0\boldsymbol{v}^*$ to the EKR basis $\boldsymbol{w}_0, \dots, \boldsymbol{w}_d$ is lower triangular. Hence, for fixed t with $0 \leqslant t \leqslant d-2$, if we write

$$\begin{aligned} (E_{t+1}+E_{t+2})\boldsymbol{w}_t &= aE_{t+1}\boldsymbol{v}^* + bE_{t+2}\boldsymbol{v}^*, \\ (E_{t+1}+E_{t+2})\boldsymbol{w}_{t+1} &= cE_{t+2}\boldsymbol{v}^*, \end{aligned}$$

then it follows that

$$(G_t + G_{t+1})E_{t+1}\boldsymbol{v}^* = a^{-1}\boldsymbol{w}_t - a^{-1}c^{-1}b\boldsymbol{w}_{t+1}, \tag{20}$$

$$(G_t + G_{t+1})E_{t+2}\boldsymbol{v}^* = c^{-1}\boldsymbol{w}_{t+1}. \tag{21}$$

By Theorem 3.9 (ii), we routinely obtain

$$a^{-1} = -\frac{\varphi_2 \dots \varphi_{t+1}\eta_d(\theta_0)}{\phi_{d-t+1}\dots\phi_d\tau_{t+1}(\theta_{t+1})\eta_{d-t-1}(\theta_0)\vartheta_{t+1}}, \tag{22}$$

$$c^{-1} = -\frac{\varphi_2 \dots \varphi_{t+2}\eta_d(\theta_0)}{\phi_{d-t}\dots\phi_d\tau_{t+2}(\theta_{t+2})\eta_{d-t-2}(\theta_0)\vartheta_{t+2}}, \tag{23}$$

$$\begin{aligned} -a^{-1}c^{-1}b = {} & \frac{\varphi_2 \dots \varphi_{t+1}\eta_d(\theta_0)(\theta_0-\theta_{t+1})}{\phi_{d-t}\dots\phi_d\tau_{t+1}(\theta_{t+1})\eta_{d-t-1}(\theta_0)} \\ & \times\left(\frac{\phi_{d-t-1}}{(\theta_{t+2}-\theta_{t+1})\vartheta_{t+2}} + \frac{\theta^*_0-\theta^*_{d-t}}{\vartheta_{t+1}}\right). \end{aligned} \tag{24}$$

From (15), (17), (18), and (20)–(24), it follows that

$$\begin{aligned} G_{\ell-1}\boldsymbol{u} &= \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\phi_{d-\ell+1}\dots\phi_d\tau_\ell(\theta_\ell)}{\varphi_1\dots\varphi_\ell} G_{\ell-1}E_\ell\boldsymbol{v}^* \\ &= \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\phi_{d-\ell+1}\eta_d(\theta_0)(\theta_\ell-\theta_0)}{\varphi_1\eta_{d-\ell+1}(\theta_0)\vartheta_\ell}\boldsymbol{w}_{\ell-1} \end{aligned} \tag{25}$$

when $1 \leqslant \ell \leqslant d$, and that

$$\begin{aligned} G_\ell\boldsymbol{u} = {} & \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle}\Bigg(\frac{\phi_{d-\ell+1}\dots\phi_d\tau_\ell(\theta_\ell)}{\varphi_1\dots\varphi_\ell}G_\ell E_\ell \boldsymbol{v}^* \\ & + \frac{\phi_{d-\ell}\dots\phi_d\tau_\ell(\theta_{\ell+1})}{\varphi_1\dots\varphi_{\ell+1}}G_\ell E_{\ell+1}\boldsymbol{v}^*\Bigg) \\ = {} & \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\eta_d(\theta_0)}{\varphi_1\eta_{d-\ell}(\theta_0)}\left(\frac{\phi_{d-\ell}}{\vartheta_{\ell+1}} + \frac{(\theta_0-\theta_\ell)(\theta^*_0-\theta^*_{d-\ell+1})}{\vartheta_\ell}\right)\boldsymbol{w}_\ell \\ = {} & \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\eta_d(\theta_0)}{\varphi_1\eta_{d-\ell}(\theta_0)}\left(\frac{\phi_{d-\ell}}{\vartheta_{\ell+1}} + \frac{\phi_{d-\ell+1}}{\vartheta_\ell} - \varphi_1\right)\boldsymbol{w}_\ell \end{aligned} \tag{26}$$

when $1 \leqslant \ell \leqslant d-1$, where the last line follows from (3) and (4). When $\ell = 0$ or $\ell = d$, we interpret $\phi_0/\vartheta_{d+1} = \phi_{d+1}/\vartheta_0 = \varphi_1$ in (26). Indeed, when $\ell = 0$, since $G_0E_0\boldsymbol{u}_0 = 0$ by Lemma 3.11 (ii), it follows from (15), (18), (20), and (22) that

$$G_0\boldsymbol{u}_0 = G_0E_1\boldsymbol{u}_0 = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\phi_d}{\varphi_1}G_0E_1\boldsymbol{v}^* = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^{*}\rangle} \cdot \frac{\phi_d}{\varphi_1}\boldsymbol{w}_0.$$

When $\ell = d$, since

$$\begin{aligned} (E_d+E_0)\boldsymbol{w}_{d-1} &= -\frac{\phi_2\dots\phi_d\tau_d(\theta_d)}{\varphi_2\dots\varphi_d\eta_d(\theta_0)}E_d\boldsymbol{v}^* + E_0\boldsymbol{v}^*, \\ (E_d+E_0)\boldsymbol{w}_d &= E_0\boldsymbol{v}^* \end{aligned}$$

by Theorem 3.9 (ii), it follows that

$$(G_{d-1}+G_d)E_d\boldsymbol{v}^* = \frac{\varphi_2\dots\varphi_d\eta_d(\theta_0)}{\phi_2\dots\phi_d\tau_d(\theta_d)}(-\boldsymbol{w}_{d-1}+\boldsymbol{w}_d),$$

so that by (15) and (17) we have

$$G_d \boldsymbol{u}_d = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \frac{\phi_1 \dots \phi_d \tau_d(\theta_d)}{\varphi_1 \dots \varphi_d} G_d E_d \boldsymbol{v}^* = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \frac{\phi_1 \eta_d(\theta_0)}{\varphi_1} \boldsymbol{w}_d.$$

Notice that the transition matrix from the basis $E_0^* \boldsymbol{v}, E_d^* \boldsymbol{v}, \dots, E_1^* \boldsymbol{v}$ to the EKR basis $\boldsymbol{w}_0, \dots, \boldsymbol{w}_d$ is upper triangular. Hence, for $1 \leqslant t \leqslant d$, since

$$E^*_{d-t+1} \boldsymbol{w}_t = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle}{||\boldsymbol{v}||^2} \cdot \frac{\tau^*_{d-t+1}(\theta^*_{d-t+1}) \eta_{d-t}(\theta_0) \vartheta_t}{\varphi_2 \dots \varphi_{d-t+1}(\theta^*_{d-t+1} - \theta^*_0)} E^*_{d-t+1} \boldsymbol{v}$$

by Theorem 3.9 (iii) and (4), it follows that

$$G_t E^*_{d-t+1} \boldsymbol{v} = \frac{||\boldsymbol{v}||^2}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \frac{\varphi_2 \dots \varphi_{d-t+1}(\theta^*_{d-t+1} - \theta^*_0)}{\tau^*_{d-t+1}(\theta^*_{d-t+1}) \eta_{d-t}(\theta_0) \vartheta_t} \boldsymbol{w}_t,$$

so that by (15), (19), and (5), we have

$$\begin{aligned} G_{\ell+1} \boldsymbol{u} &= \langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle \cdot \frac{\phi_{d-\ell+1} \dots \phi_d}{\eta^*_\ell(\theta^*_{d-\ell}) ||E^*_{d-\ell} \boldsymbol{v}||^2} G_{\ell+1} E^*_{d-\ell} \boldsymbol{v} \\ &= \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \frac{\eta_d(\theta_0)(\theta^*_{d-\ell} - \theta^*_0)}{\varphi_1 \eta_{d-\ell-1}(\theta_0) \vartheta_{\ell+1}} \boldsymbol{w}_{\ell+1} \end{aligned} \tag{27}$$

when $0 \leqslant \ell \leqslant d-1$.

Theorem 3.12. *Setting $\boldsymbol{w}_{-1} = \boldsymbol{w}_{d+1} = 0$, the following hold.*[8]

(i) $$\begin{aligned} \tau_\ell(A) \boldsymbol{v}^{*\downarrow} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^{*\downarrow} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v}^* \rangle} \cdot \frac{\eta_d(\theta_0)}{\varphi_1} \Bigg\{ &- \frac{\phi_{d-\ell+1}}{\eta_{d-\ell}(\theta_0) \vartheta_\ell} \boldsymbol{w}_{\ell-1} \\ &+ \frac{1}{\eta_{d-\ell}(\theta_0)} \left(\frac{\phi_{d-\ell}}{\vartheta_{\ell+1}} + \frac{\phi_{d-\ell+1}}{\vartheta_\ell} - \varphi_1 \right) \boldsymbol{w}_\ell \\ &+ \frac{\theta^*_{d-\ell} - \theta^*_0}{\eta_{d-\ell-1}(\theta_0) \vartheta_{\ell+1}} \boldsymbol{w}_{\ell+1} \Bigg\} \end{aligned}$$
for $0 \leqslant \ell \leqslant d$, where we interpret $\phi_0 / \vartheta_{d+1} = \phi_{d+1} / \vartheta_0 = \varphi_1$.

(ii) $$\begin{aligned} E_j \boldsymbol{v}^* = \frac{\varphi_2 \cdots \varphi_j \eta_d(\theta_0)}{\phi_{d-j+1} \cdots \phi_d \tau_j(\theta_j) \eta_{d-j}(\theta_j)} \Bigg\{ &- \frac{\phi_{d-j+1} \eta_{d-j}(\theta_j)}{\eta_{d-j}(\theta_0) \vartheta_j} \boldsymbol{w}_{j-1} \\ &+ (\theta_j - \theta_0) \sum_{t=j}^{d-1} \frac{\eta_{d-t-1}(\theta_j)}{\eta_{d-t}(\theta_0)} \Bigg(\frac{\phi_{d-t}}{\vartheta_{t+1}} \\ &\qquad + \frac{(\theta_j - \theta_{t+1})(\theta^*_{d-t+1} - \theta^*_0)}{\vartheta_t} \Bigg) \boldsymbol{w}_t \\ &+ \big(\varphi_1 + (\theta^*_1 - \theta^*_0)(\theta_j - \theta_0) \big) \boldsymbol{w}_d \Bigg\} \end{aligned}$$
for $1 \leqslant j \leqslant d$, and $E_0 \boldsymbol{v}^* = \boldsymbol{w}_d$.

[8] We also interpret the coefficients of $\boldsymbol{w}_{-1}$ and $\boldsymbol{w}_{d+1}$ as zero (or indeterminates), whenever these terms appear.

$$\text{(iii)}\quad E_i^*\boldsymbol{v}=\frac{\langle\boldsymbol{v},\boldsymbol{v}^*\rangle}{||\boldsymbol{v}^*||^2}\cdot\frac{\varphi_2\cdots\varphi_i\eta_d(\theta_0)\eta_d^*(\theta_0^*)}{\phi_1\cdots\phi_i\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)}\Biggl\{\frac{\varphi_1+(\theta_1-\theta_0)(\theta_i^*-\theta_0^*)}{\eta_d(\theta_0)}\boldsymbol{w}_0$$
$$+\ (\theta_i^*-\theta_0^*)\sum_{t=1}^{d-i}\frac{\eta_{t-1}^*(\theta_i^*)}{\phi_{d-t+1}\cdots\phi_d\eta_{d-t}(\theta_0)}\biggl(\frac{\phi_{d-t+1}}{\vartheta_t}+\frac{(\theta_i^*-\theta_{d-t+1}^*)(\theta_{t+1}-\theta_0)}{\vartheta_{t+1}}\biggr)\boldsymbol{w}_t$$
$$+\ \frac{\eta_{d-i}^*(\theta_i^*)(\theta_i^*-\theta_0^*)}{\phi_{i+1}\cdots\phi_d\eta_{i-1}(\theta_0)\vartheta_i}\boldsymbol{w}_{d-i+1}\Biggr\}$$

for $1\leqslant i\leqslant d$, *and* $E_0^*\boldsymbol{v}=\langle\boldsymbol{v},\boldsymbol{v}^*\rangle||\boldsymbol{v}^*||^{-2}\boldsymbol{w}_0$.

Proof. (i): Immediate from (16), (25), (26), and (27).

(ii): By (i) above, Lemma 2.4 (iii) and (v), and Lemma 2.3 (ii), we have

$$E_j\boldsymbol{v}^*=\frac{\langle\boldsymbol{v},\boldsymbol{v}^*\rangle}{\langle\boldsymbol{v},\boldsymbol{v}^{*\downarrow}\rangle}\cdot\frac{\varphi_1\ldots\varphi_j}{\phi_{d-j+1}\ldots\phi_d}\sum_{\ell=j}^{d}\frac{\eta_{d-\ell}(\theta_j)}{\tau_j(\theta_j)\eta_{d-j}(\theta_j)}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}$$
$$=\frac{\varphi_2\ldots\varphi_j\eta_d(\theta_0)}{\phi_{d-j+1}\ldots\phi_d\tau_j(\theta_j)\eta_{d-j}(\theta_j)}\sum_{\ell=j}^{d}\eta_{d-\ell}(\theta_j)\Biggl\{\frac{\phi_{d-\ell+1}(\theta_\ell-\theta_0)}{\eta_{d-\ell+1}(\theta_0)\vartheta_\ell}\boldsymbol{w}_{\ell-1}$$
$$+\frac{1}{\eta_{d-\ell}(\theta_0)}\biggl(\frac{\phi_{d-\ell}}{\vartheta_{\ell+1}}+\frac{\phi_{d-\ell+1}}{\vartheta_\ell}-\varphi_1\biggr)\boldsymbol{w}_\ell+\frac{\theta_{d-\ell}^*-\theta_0^*}{\eta_{d-\ell-1}(\theta_0)\vartheta_{\ell+1}}\boldsymbol{w}_{\ell+1}\Biggr\}$$

for $1\leqslant j\leqslant d$. Now simplify the last line using (3) and (4).

(iii): Apply "$*$" to (ii) above with respect to the Φ^*-EKR basis $\{\boldsymbol{w}_t^*\}_{t=0}^d$ with $E_0^*\boldsymbol{w}_t^*=E_0^*\boldsymbol{v}$ $(0\leqslant t\leqslant d)$, and then use Corollary 3.10, Lemma 2.3 (i), and (4). □

Finally, we shall describe the matrices representing A and A^* with respect to the EKR basis $\{\boldsymbol{w}_t\}_{t=0}^d$. We use the following notation:

$$\Delta_s=\frac{\eta_{s-1}^*(\theta_0^*)\bigl((\theta_{d-s+1}^*-\theta_0^*)\vartheta_{s+1}-(\theta_{d-s}^*-\theta_0^*)\vartheta_s\bigr)}{\phi_{d-s+1}\ldots\phi_d\eta_{d-s-1}(\theta_0)\vartheta_{s+1}}\quad(1\leqslant s\leqslant d-1).$$

Notice that

$$\Delta_s^*=\frac{\eta_{s-1}(\theta_0)\bigl((\theta_{d-s+1}-\theta_0)\vartheta_{s+1}-(\theta_{d-s}-\theta_0)\vartheta_s\bigr)}{\phi_1\ldots\phi_s\eta_{d-s-1}^*(\theta_0^*)\vartheta_{s+1}}\quad(1\leqslant s\leqslant d-1),$$

by virtue of Theorem 2.3 (i) and (4).

Theorem 3.13. *With the above notation, the following hold.*

$$\text{(i)}\quad A\boldsymbol{w}_t=\theta_{t+1}\boldsymbol{w}_t+\biggl(\frac{\phi_{d-t+1}\cdots\phi_d\eta_{d-t}(\theta_0)}{\eta_t^*(\theta_0^*)}\Delta_{t+1}-(\theta_{t+1}-\theta_0)\biggr)\boldsymbol{w}_{t+1}$$
$$+\ \frac{\phi_{d-t+1}\cdots\phi_d\eta_{d-t}(\theta_0)}{\eta_t^*(\theta_0^*)}\Biggl\{\sum_{s=t+2}^{d-1}(\Delta_s-\Delta_{s-1})\boldsymbol{w}_s-\Delta_{d-1}\boldsymbol{w}_d\Biggr\}$$

for $0\leqslant t\leqslant d-2$, $A\boldsymbol{w}_{d-1}=\theta_d\boldsymbol{w}_{d-1}-(\theta_d-\theta_0)\boldsymbol{w}_d$, *and* $A\boldsymbol{w}_d=\theta_0\boldsymbol{w}_d$.

(ii)
$$\begin{aligned}A^*\boldsymbol{w}_t = &-\frac{\phi_1\cdots\phi_d}{\eta_d(\theta_0)}\Delta^*_{d-1}\boldsymbol{w}_0\\ &+\sum_{s=1}^{t-2}\frac{\phi_1\cdots\phi_{d-s}\eta^*_s(\theta^*_0)}{\eta_{d-s}(\theta_0)}(\Delta^*_{d-s}-\Delta^*_{d-s-1})\boldsymbol{w}_s\\ &+\left(\frac{\phi_1\cdots\phi_{d-t+1}\eta^*_{t-1}(\theta^*_0)}{\eta_{d-t+1}(\theta_0)}\Delta^*_{d-t+1}-\frac{\phi_{d-t+1}}{\theta_t-\theta_0}\right)\boldsymbol{w}_{t-1}\\ &+\theta^*_{d-t+1}\boldsymbol{w}_t\end{aligned}$$

for $2\leqslant t\leqslant d$, $A^*\boldsymbol{w}_1=\theta^*_d\boldsymbol{w}_1-(\theta^*_d-\theta^*_0)\boldsymbol{w}_0$, *and* $A^*\boldsymbol{w}_0=\theta^*_0\boldsymbol{w}_0$.

Proof. (i): By Theorem 3.9 (i), (3), (4), and since $A\tau_\ell(A)=\tau_{\ell+1}(A)+\theta_\ell\tau_\ell(A)$, we obtain

$$\begin{aligned}A\boldsymbol{w}_t &= \frac{\langle\boldsymbol{v},\boldsymbol{v}^*\rangle}{\langle\boldsymbol{v},\boldsymbol{v}^{*\downarrow}\rangle}\left\{\sum_{\ell=1}^{t}\frac{\eta_{d-\ell+1}(\theta_0)}{\eta_d(\theta_0)}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}+\sum_{\ell=0}^{t}\frac{\eta_{d-\ell}(\theta_0)\theta_\ell}{\eta_d(\theta_0)}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\right.\\ &\qquad+\frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)}\sum_{\ell=t+1}^{d}\frac{\eta^*_{\ell-1}(\theta^*_0)}{\phi_{d-\ell+2}\cdots\phi_{d-t}}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\\ &\qquad\left.+\frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)}\sum_{\ell=t+1}^{d}\frac{\eta^*_\ell(\theta^*_0)\theta_\ell}{\phi_{d-\ell+1}\cdots\phi_{d-t}}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\right\}\\ &=\frac{\langle\boldsymbol{v},\boldsymbol{v}^*\rangle}{\langle\boldsymbol{v},\boldsymbol{v}^{*\downarrow}\rangle}\left\{\theta_0\sum_{\ell=0}^{t}\frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\right.\\ &\qquad+\frac{\eta_{d-t}(\theta_0)\theta_0}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)}\sum_{\ell=t+1}^{d}\frac{\eta^*_\ell(\theta^*_0)}{\phi_{d-\ell+1}\cdots\phi_{d-t}}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\\ &\qquad\left.+\frac{\varphi_1\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)}\sum_{\ell=t+1}^{d}\frac{\eta^*_{\ell-1}(\theta^*_0)\vartheta_{d-\ell+1}}{\phi_{d-\ell+1}\cdots\phi_{d-t}}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}\right\}\\ &=\theta_0\boldsymbol{w}_t+\frac{\langle\boldsymbol{v},\boldsymbol{v}^*\rangle}{\langle\boldsymbol{v},\boldsymbol{v}^{*\downarrow}\rangle}\cdot\frac{\varphi_1\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta^*_t(\theta^*_0)}\sum_{\ell=t+1}^{d}\frac{\eta^*_{\ell-1}(\theta^*_0)\vartheta_\ell}{\phi_{d-\ell+1}\cdots\phi_{d-t}}\tau_\ell(A)\boldsymbol{v}^{*\downarrow}.\end{aligned}$$

Now apply Theorem 3.12 (i) and simplify the result using (3) and (4).

(ii): Apply "$*$" to (i) above with respect to the Φ^*-EKR basis $\{\boldsymbol{w}^*_t\}_{t=0}^d$ such that $E^*_0\boldsymbol{w}^*_t=E^*_0\boldsymbol{v}$ $(0\leqslant t\leqslant d)$, and then use Corollary 3.10, Lemma 2.3 (i), and (4). □

We end this section with an attractive formula for Δ_s.

Lemma 3.14. *For* $1\leqslant s\leqslant d-1$, *we have*

$$\begin{aligned}&(\theta_{d-s+1}-\theta_0)\vartheta_{s+1}-(\theta_{d-s}-\theta_0)\vartheta_s\\ &\qquad=\frac{\big(\theta_{d-\lfloor s/2\rfloor}-\theta_{\lfloor s/2\rfloor}\big)\big(\theta_{d-\lfloor (s-1)/2\rfloor}-\theta_{\lfloor (s+1)/2\rfloor}\big)}{\theta_d-\theta_0}.\end{aligned}$$

Proof. This is verified case by case using [23, Lemma 10.2]. □

Corollary 3.15. *For $1 \leqslant s \leqslant d-1$, we have*

$$\Delta_s = \frac{\eta^*_{s-1}(\theta^*_0)\big(\theta^*_{d-\lfloor s/2\rfloor} - \theta^*_{\lfloor s/2\rfloor}\big)\big(\theta^*_{d-\lfloor (s-1)/2\rfloor} - \theta^*_{\lfloor (s+1)/2\rfloor}\big)}{\phi_{d-s+1}\cdots\phi_d \eta_{d-s-1}(\theta_0)(\theta^*_d - \theta^*_0)\vartheta_{s+1}}.$$

Proof. Immediate from Lemma 3.14 and (4). □

4. Applications to the Erdős–Ko–Rado theorems

The Erdős–Ko–Rado type theorems for various families of Q-polynomial distance-regular graphs provide one of the most successful applications of Delsarte's linear programming method [4].[9]

Let Γ be a Q-polynomial distance-regular graph with vertex set X. (We refer the reader to [2, 3, 21] for background material.) Pick a "base vertex" $x \in X$ and let $\Phi = \Phi(\Gamma)$ be the Leonard system (over $\mathbb{K} = \mathbb{R}$) afforded on the primary module of the Terwilliger algebra $\boldsymbol{T}(x)$; cf. [19, Example (3.5)].[10] The second eigenmatrix $Q = (Q_{ij})_{i,j=0}^d$ of Γ is defined by[11]

$$E_j \boldsymbol{v}^* = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}||^2} \sum_{i=0}^d Q_{ij} E^*_i \boldsymbol{v} \quad (0 \leqslant j \leqslant d).$$

As summarized in [20], every "t-intersecting family" $Y \subseteq X$ is associated with a vector $\boldsymbol{e} = (e_0, \dots, e_d)$ (called the inner distribution of Y) satisfying

$$e_0 = 1, \quad e_1 \geqslant 0, \dots, e_{d-t} \geqslant 0, \quad e_{d-t+1} = \cdots = e_d = 0,$$
$$|Y| = (\boldsymbol{e}Q)_0, \quad \text{and} \quad (\boldsymbol{e}Q)_1 \geqslant 0, \dots, (\boldsymbol{e}Q)_d \geqslant 0.$$

Viewing these as forming a linear programming maximization problem with objective function $(\boldsymbol{e}Q)_0$, we are then to construct a vector $\boldsymbol{f} = (f_0, \dots, f_d)$ such that

$$f_0 = 1, \quad f_1 = \cdots = f_t = 0, \quad \text{and} \quad (\boldsymbol{f}Q^{\mathsf{T}})_1 = \cdots = (\boldsymbol{f}Q^{\mathsf{T}})_{d-t} = 0, \tag{28}$$

which turns out to give a feasible solution to the dual problem with objective value $(\boldsymbol{f}Q^{\mathsf{T}})_0$, provided that $f_{t+1} \geqslant 0, \dots, f_d \geqslant 0$.

Set $\boldsymbol{w} = \sum_{j=0}^d f_j E_j \boldsymbol{v}^*$. Then

$$\boldsymbol{w} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}||^2} \sum_{j=0}^d f_j \sum_{i=0}^d Q_{ij} E^*_i \boldsymbol{v} = \frac{\langle \boldsymbol{v}, \boldsymbol{v}^*\rangle}{||\boldsymbol{v}||^2} \sum_{i=0}^d (\boldsymbol{f}Q^{\mathsf{T}})_i E^*_i \boldsymbol{v}.$$

Hence it follows that $\boldsymbol{f}$ satisfies (28) if and only if $\boldsymbol{w} = \boldsymbol{w}_t$. In particular, such a vector $\boldsymbol{f}$ is unique and is given by Theorem 3.9 (ii).

[9] See, e.g., [5, 15] for more applications as well as extensions of this method.

[10] We remark that Φ is independent of $x \in X$ up to isomorphism.

[11] The matrix Q is denoted P^* in [26, p. 264].

We now give three examples. First, suppose Φ is of *dual Hahn* type [27, Example 5.12], i.e.,

$$\theta_i = \theta_0 + hi(i+1+s), \quad \theta^*_i = \theta^*_0 + s^* i$$

for $0 \leqslant i \leqslant d$, and

$$\varphi_i = hs^* i(i-d-1)(i+r), \quad \phi_i = hs^* i(i-d-1)(i+r-s-d-1)$$

for $1 \leqslant i \leqslant d$, where h, s^* are nonzero. Then it follows that

$$f_j = \frac{(1-j)_t (j+s+2)_t (s-r+1)_j (-1)^{j-1}}{(t-r+s+1)(s+2)_t t! (r+2)_{j-1}} \times {}_3F_2\left(\begin{matrix} t-j+1, t+j+s+2, 1 \\ t+1, t-r+s+2 \end{matrix} \,\middle|\, 1\right)$$

for $t+1 \leqslant j \leqslant d$, and

$$(\boldsymbol{f} Q^{\mathsf{T}})_0 = \frac{(-d-s-1)_{d-t}}{(r-s-d)_{d-t}}.$$

If Γ is the Johnson graph $J(v,d)$ [3, Section 9.1], then Φ is of dual Hahn type with $r = d-v-1$, $s = -v-2$, and $s^* = -v(v-1)/d(v-d)$; cf. [22, pp. 191–192]. In this case, the vector $\boldsymbol{f}$ was essentially constructed by Wilson [29] and was used to prove the original Erdős–Ko–Rado theorem [6] in full generality.

Suppose Φ is of *Krawtchouk* type [27, Example 5.13], i.e.,

$$\theta_i = \theta_0 + si, \quad \theta^*_i = \theta^*_0 + s^* i$$

for $0 \leqslant i \leqslant d$, and

$$\varphi_i = ri(i-d-1), \quad \phi_i = (r-ss^*)i(i-d-1)$$

for $1 \leqslant i \leqslant d$, where r, s, s^* are nonzero. Then it follows that

$$f_j = \frac{(1-j)_t}{t!}\left(\frac{r-ss^*}{r}\right)^{j-1} \cdot {}_2F_1\left(\begin{matrix} t-j+1, 1 \\ t+1 \end{matrix} \,\middle|\, \frac{ss^*}{ss^*-r}\right)$$

for $t+1 \leqslant j \leqslant d$, and

$$(\boldsymbol{f} Q^{\mathsf{T}})_0 = \left(\frac{ss^*}{ss^*-r}\right)^{d-t}.$$

If Γ is the Hamming graph $H(d,n)$ [3, Section 9.2], then Φ is of Krawtchouk type with $r = n(n-1)$ and $s = s^* = -n$; cf. [22, p. 195]. In this case, the vector $\boldsymbol{f}$ coincides (up to normalization) with the weight distribution of an *MDS code* [14, Chapter 11], i.e., a code attaining the Singleton bound.[12]

Finally, suppose Φ is of the most general *q-Racah* type [27, Example 5.3], i.e.,

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i}, \quad \theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$$

[12] In this regard, one may also wish to call $\{\boldsymbol{w}_t\}_{t=0}^d$ an *MDS* basis or a *Singleton* basis.

for $0 \leqslant i \leqslant d$, and

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1 q^i)(1-r_2 q^i),$$
$$\phi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^*$$

for $1 \leqslant i \leqslant d$, where $h, h^*, r_1, r_2, s, s^*, q$ are nonzero and $r_1 r_2 = ss^* q^{d+1}$. Then it follows that the f_j are expressed as balanced ${}_4\phi_3$ series:

$$f_j = \frac{s^{*j-1} q^{(d+1)(j-1)+t} (q^{1-j};q)_t (sq^{j+2};q)_t (sq/r_1;q)_j (sq/r_2;q)_j}{(1-sq^{t+1}/r_1)(1-sq^{t+1}/r_2)(q;q)_t (sq^2;q)_t (r_1 q^2;q)_{j-1} (r_2 q^2;q)_{j-1}} \times {}_4\phi_3\left(\begin{matrix} q^{t-j+1}, sq^{t+j+2}, q^{t-d-1}/s^*, q \\ q^{t+1}, sq^{t+2}/r_1, sq^{t+2}/r_2 \end{matrix} \,\middle|\, q; q \right)$$

for $t+1 \leqslant j \leqslant d$, and

$$(\boldsymbol{f} Q^{\mathsf{T}})_0 = \frac{(sq^{t+2};q)_{d-t} (s^* q^2;q)_{d-t}}{r_1^{d-t} q^{d-t} (sq^{t+1}/r_1;q)_{d-t} (s^* q/r_1;q)_{d-t}}.$$

References

1. R. Askey and J. Wilson, *A set of orthogonal polynomials that generalize the Racah coefficients or $6-j$ symbols*, SIAM J. Math. Anal. **10** (1979), 1008–1016.
2. E. Bannai and T. Ito, *Algebraic combinatorics I: Association schemes*, Benjamin/Cummings, Menlo Park, CA, 1984.
3. A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, Berlin, 1989.
4. P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. **No. 10** (1973).
5. P. Delsarte and V. I. Levenshtein, *Association schemes and coding theory*, IEEE Trans. Inform. Theory **44** (1998), 2477–2504.
6. P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320.
7. P. Frankl and R. M. Wilson, *The Erdős–Ko–Rado theorem for vector spaces*, J. Combin. Theory Ser. A **43** (1986), 228–236.
8. T. Ito, K. Tanabe, and P. Terwilliger, *Some algebra related to P- and Q-polynomial association schemes*, Codes and association schemes (A. Barg and S. Litsyn, eds.), American Mathematical Society, Providence, RI, 2001, pp. 167–192, arXiv:math/0406556.
9. T. Ito and P. Terwilliger, *The augmented tridiagonal algebra*, Kyushu J. Math. **64** (2010), 81–144, arXiv:0904.2889.
10. R. Koekoek and R. F. Swarttouw, *The Askey scheme of hypergeometric orthogonal polynomials and its q-analog*, report 98–17, Delft University of Technology, The Netherlands, 1998, available at `http://aw.twi.tudelft.nl/~koekoek/askey.html`.
11. J. H. Koolen, W. S. Lee, and W. J. Martin, *Characterizing completely regular codes from an algebraic viewpoint*, Combinatorics and graphs (R. Brualdi et al., eds.), Contemporary Mathematics, vol. 531, American Mathematical Society, Providence, RI, 2010, pp. 223–242, arXiv:0911.1828.
12. D. A. Leonard, *Orthogonal polynomials, duality and association schemes*, SIAM J. Math. Anal. **13** (1982), 656–663.
13. L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979), 1–7.
14. F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland, Amsterdam, 1977.

15. W. J. Martin and H. Tanaka, *Commutative association schemes*, European J. Combin. **30** (2009), 1497–1525, arXiv:0811.2475.
16. A. Schrijver, *A comparison of the Delsarte and Lovász bounds*, IEEE Trans. Inform. Theory **25** (1979), 425–429.
17. H. Tanaka, *Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs*, J. Combin. Theory Ser. A **113** (2006), 903–910.
18. ———, *A bilinear form relating two Leonard systems*, Linear Algebra Appl. **431** (2009), 1726–1739, arXiv:0807.0385.
19. ———, *Vertex subsets with minimal width and dual width in Q-polynomial distance-regular graphs*, Electron. J. Combin. **18** (2011), P167, arXiv:1011.2000.
20. ———, *The Erdős–Ko–Rado theorem for twisted Grassmann graphs*, Combinatorica **32** (2012), 735–740, arXiv:1012.5692.
21. P. Terwilliger, *The subconstituent algebra of an association scheme I*, J. Algebraic Combin. **1** (1992), 363–388.
22. ———, *The subconstituent algebra of an association scheme III*, J. Algebraic Combin. **2** (1993), 177–210.
23. ———, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other*, Linear Algebra Appl. **330** (2001), 149–203, arXiv:math/0406555.
24. ———, *Leonard pairs from* 24 *points of view*, Rocky Mountain J. Math. **32** (2002), 827–888, arXiv:math/0406577.
25. ———, *Introduction to Leonard pairs*, J. Comput. Appl. Math. **153** (2003), 463–475.
26. ———, *Leonard pairs and the q-Racah polynomials*, Linear Algebra Appl. **387** (2004), 235–276, arXiv:math/0306301.
27. ———, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array*, Des. Codes Cryptogr. **34** (2005), 307–332, arXiv:math/0306291.
28. ———, *An algebraic approach to the Askey scheme of orthogonal polynomials*, Orthogonal polynomials and special functions: Computation and applications (F. Marcellán and W. Van Assche, eds.), Lecture Notes in Mathematics, vol. 1883, Springer-Verlag, Berlin, 2006, pp. 255–330, arXiv:math/0408390.
29. R. M. Wilson, *The exact bound in the Erdős–Ko–Rado theorem*, Combinatorica **4** (1984), 247–257.

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