Contributions to Discrete Mathematics

Volume 15, Number 1, Pages 42–51
ISSN 1715-0868

THE POLAR OF CONVEX LATTICE SETS

JIE WANG AND LIN SI

Abstract. Let K be a convex lattice set in $\mathbb{Z}^n$ containing the origin as the interior of its convex hull. In this paper, the definition of the polar of a convex lattice set K is given both in $\mathbb{Q}^n$ and $\mathbb{Z}^n$. Some properties and inequalities about the convex lattice sets and their polar are established.

1. Introduction

For a compact convex set C that contains the origin as its interior in Euclidean n-space $\mathbb{R}^n$, its polar body C^* is defined by

$$C^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \text{ in } C\},$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. The polar body plays an important role in geometric inequalities and it is a powerful tool in convex geometry; some excellent results are provided in the monographs [5, 7].

Analogous to convex bodies in convex geometry are convex lattice sets in discrete geometry. Throughout this paper, a convex lattice set K in the integer lattice $\mathbb{Z}^n$ is a convex lattice set containing the origin as the interior of the convex hull of K.

Inequalities about convex lattice sets have been also widely studied recently. Gardner and Gronchi obtained a close discrete analog of the classical Brunn–Minkowski inequality [2]. In 2005, Gardner, Gronchi, and Zong proved a discrete version of the dual Loomis–Whitney inequality in $\mathbb{Z}^2$ [3]. Berg and Henk studied upper bounds on the number of lattice points for convex bodies having their centroid at the origin and established the inequality between the number of lattice points and the first successive minimum [1]. For more information about theory of lattice polytopes and their numerous applications, one can see Barvinok [4].

The very first notion relating the polar of convex lattice sets may be reflexive polytopes, whose polar are also lattice polytopes [6]. Notice that

Received by the editors February 27, 2018, and in revised form March 25, 2019.

2010 *Mathematics Subject Classification.* 11H06, 52C07.

Key words and phrases. convex lattice sets, polar, inequality.

This research was supported by the Young Talent Program of Beijing (YETP0770), the Fundamental Research Funds for the Central Universities (2017ZY44,2015ZCQ-LY-01) and the State Forestry Administration 948 Project (2013-4-66). Lin Si is the corresponding author.

not all polytopes are reflexive polytopes, and not all convex lattice sets have the polar in $\mathbb{Z}^n$.

In this paper, the definition of the polar of convex lattice sets is given both in $\mathbb{Q}^n$ and $\mathbb{Z}^n$. Under this definition, we find some special classes of convex lattice sets that have the dual property. The upper bound of the number of points of polar in $\mathbb{Z}^n$ is obtained. In the plane, the connection between the number of the lattice points in the convex lattice set K and the area of convex hull of K is established.

2. Definition of the polar of convex lattice sets

A convex lattice set K is a finite subset of $\mathbb{Z}^n$ such that $K = \mathrm{conv}(K) \cap \mathbb{Z}^n$, where conv denotes the convex hull. Let K be a convex lattice set in $\mathbb{Z}^n$ containing the origin O as the interior of its convex hull. Notice that for a convex body C containing the origin as its interior, its polar C^* is determined by its boundary. Further, a convex lattice set can be determined by vertices of its convex hull.

Let $K' = \{\bigcap \bar{K} : \bar{K} \subset K, \mathrm{conv}(\bar{K}) = \mathrm{conv}(K)\}$, then we have the following proposition.

Proposition 2.1. $K' = \mathrm{vert}(\mathrm{conv}(K))$, *where* vert *denotes the vertex set.*

Proof. Obviously, $\mathrm{conv}(\mathrm{vert}(\mathrm{conv}(K))) = \mathrm{conv}(K)$, so, by the definition of K', $K' \subset \mathrm{vert}(\mathrm{conv}(K))$.

On the other hand, if $\mathrm{vert}(\mathrm{conv}(K))$ is not a subset of K', then denote $\mathrm{vert}(\mathrm{conv}(K))$ by $\{x_1, x_2, \ldots, x_k\}$. Without loss of generality, we may assume $x_1 \notin K'$, so there must exist $\bar{K}$, $\bar{K} \subset K$ and $\mathrm{conv}(\bar{K}) = \mathrm{conv}(K)$, but $x_1 \notin \bar{K}$. Denote $\bar{K}$ by $\{a_1, a_2, \ldots, a_m\}$, so, $x_1 \neq a_i, i = 1, 2, \ldots, m$. Since $x_1 \in \mathrm{conv}(K) = \mathrm{conv}(\bar{K})$, then x_1 is the convex combination of $a_1, a_2, \ldots, a_m$. Thus $\mathrm{conv}(\bar{K}) \setminus \{x_1\}$ is not convex, $x_1 \notin \mathrm{vert}(\mathrm{conv}(\bar{K})) = \mathrm{vert}(\mathrm{conv}(K))$, a contradiction. □

For each $A' \in K'$, define a coordinate A which satisfies:

- A is on the ray OA',
- $\max\{OA \cdot OB', \text{ for } B' \in K'\} = 1$.

The coordinate A has the following property.

Proposition 2.2. *The coordinate of A is in $\mathbb{Q}^n$.*

Proof. We may assume that OB' has the maximum projection onto OA'. Let the coordinates A' and B' be $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively, then the coordinate of A is

$$\left(\frac{a_1}{\sum_{i=1}^n a_i b_i}, \ldots, \frac{a_n}{\sum_{i=1}^n a_i b_i} \right),$$

which indicates that A belongs to $\mathbb{Q}^n$. □

So, for all $A' \in K'$, we get the corresponding A's in $\mathbb{Q}^n$. Thus the collection of all A is the polar in $\mathbb{Q}^n$. In view of these facts, we give the definition of the polar of the convex lattice sets in $\mathbb{Q}^n$.

Definition 2.3. *The polar of the convex lattice sets in* $\mathbb{Q}^n$ *is defined by* $\bigcup A$, *and denoted by* $K^*_{\mathbb{Q}^n}$.

Next, we extend this notion to $\mathbb{Z}^n$. Let $K^*_{\mathbb{Q}^n}$ be denoted by

$$\left\{ \left(\frac{q_{11}}{p_{11}}, \ldots, \frac{q_{1n}}{p_{1n}} \right), \ldots, \left(\frac{q_{k1}}{p_{k1}}, \ldots, \frac{q_{kn}}{p_{kn}} \right) \right\}.$$

For $A \in K^*_{\mathbb{Q}^n}$, consider the transformation

$$\mathscr{F}(A) = \frac{\operatorname{lcm}(p_{11}, \ldots, p_{1n}, \ldots, p_{kn})}{\gcd(q_{11}, \ldots, q_{1n}, \ldots, q_{kn})} A, \tag{2.1}$$

where lcm and gcd stand for the least common multiple and the greatest common divisor respectively.

Then, the definition of the polar of the convex lattice sets in $\mathbb{Z}^n$ is as follows:

Definition 2.4. *The polar of the convex lattice sets in* $\mathbb{Z}^n$ *is defined by* $\operatorname{conv}(\mathscr{F}(K^*_{\mathbb{Q}^n})) \cap \mathbb{Z}^n$, *and denoted by* $K^*_{\mathbb{Z}^n}$.

For example in the plane, $P = \{(-1,0), (0,1), (0,0), (0,-1), (1,2)\}$ is a convex lattice set, $P^*_{\mathbb{Q}^2} = \{(1/5, 2/5), (-1,0), (0,-1)\}$, and $P^*_{\mathbb{Z}^2} = \operatorname{conv}(\{(1,2), (-5,0), (0,-5)\}) \cap \mathbb{Z}^2$. See Figure 1.

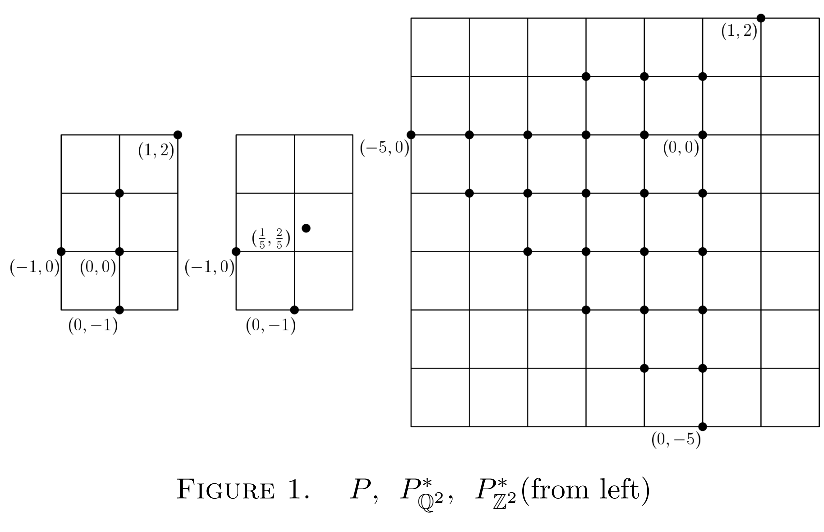

FIGURE 1. P, $P^*_{\mathbb{Q}^2}$, $P^*_{\mathbb{Z}^2}$(from left)

The nonsingular transformation $\mathscr{F}$ in (2.1) has the following property.

Proposition 2.5. *Among all the dilatations of* $K^*_{\mathbb{Q}^n}$, $\mathscr{F}(A)$ *is in* $\mathbb{Z}^n$, *and* $\operatorname{conv}(\mathscr{F}(K^*_{\mathbb{Q}^n})) \cap \mathbb{Z}^n$ *has the minimum number of points.*

Proof. Let K be a convex lattice set in $\mathbb{Z}^n$. By Proposition 2.1, K can be determined by K', i.e., $K = \mathrm{conv}(K') \cap \mathbb{Z}^n$, so we may assume that

$$K' = \{(a_{11}, \dots, a_{1n}), \dots, (a_{k1}, \dots, a_{kn})\},$$

where $a_{ij} \in \mathbb{N}$, and for any i, $(a_{i1}, \dots, a_{in}) \neq (0, \dots, 0)$.

Then $K^*_{\mathbb{Q}^n}$ can be represented as

$$K^*_{\mathbb{Q}^n} = \left\{ \left(\frac{q_{11}}{p_{11}}, \dots, \frac{q_{1n}}{p_{1n}} \right), \dots, \left(\frac{q_{k1}}{p_{k1}}, \dots, \frac{q_{kn}}{p_{kn}} \right) \right\},$$

where $p_{ij}, q_{ij} \in \mathbb{N}$, and for any i, $(q_{i1}, \dots, q_{in}) \neq (0, \dots, 0)$.

By the property of the greatest common divisor and the least common multiple, we know that $\mathscr{F}$ maps $K^*_{\mathbb{Q}^n}$ to $\mathbb{Z}^n$.

If $\mathscr{F}$ is not the transformation in (2.1), suppose that there exists a map $\mathscr{G}(A) = aA$, where $A \in K^*_{\mathbb{Q}^n}$, $a \in \mathbb{Q}$ and $\mathscr{G}(K^*_{\mathbb{Q}^n}) \subset \mathbb{Z}^n$. So we get $p_{ij} \mid aq_{ij}$,$i = 1, 2, \dots, k$; $j = 1, 2, \dots, n$. Assume

$$a = \frac{\mathrm{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn})}{\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})} m, \qquad m \in \mathbb{R},$$

then, $p_{ij} \mid aq_{ij}$ can be written as

$$p_{ij} \mid m \cdot \mathrm{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn}) \frac{q_{ij}}{\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})}.$$

This implies

$$m \cdot q_{ij} \frac{\mathrm{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn})}{\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})}$$

must be a positive integer. Since $(q_{ij}, p_{ij}) = 1$, we have

$$\gcd(\mathrm{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn}), \gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})) = 1,$$

so

$$\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn}) \mid q_{ij} \cdot m, i = 1, 2, \dots, k.j = 1, 2, \dots, n.$$

By the property of the greatest common divisor, we get that $m \geq 1$, which means

$$a \geq \frac{\mathrm{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn})}{\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})}.$$

Thus $\mathscr{F}$ is the map in Proposition 2.5. □

For a convex lattice set P, another way to define the polar is by $\mathrm{conv}(P)^* \cap \mathbb{Z}^n$, where $\mathrm{conv}(P)^*$ is the usual polar of a convex body. However the vertices of $\mathrm{conv}(\mathrm{conv}(P)^* \cap \mathbb{Z}^n)$ may not coincide with the vertices of $\mathrm{conv}(P)^*$, unless $\mathrm{conv}(P)$ is a reflexive polytope [6]. The polar of convex lattice sets in Definition 2.4 is quite different from $\mathrm{conv}(P)^* \cap \mathbb{Z}^n$ in some cases, see Figure 2. The difference is caused by the absence of some extreme points of $\mathrm{conv}(P)^*$ and the dilatation from $\mathbb{Q}^n$ to $\mathbb{Z}^n$.

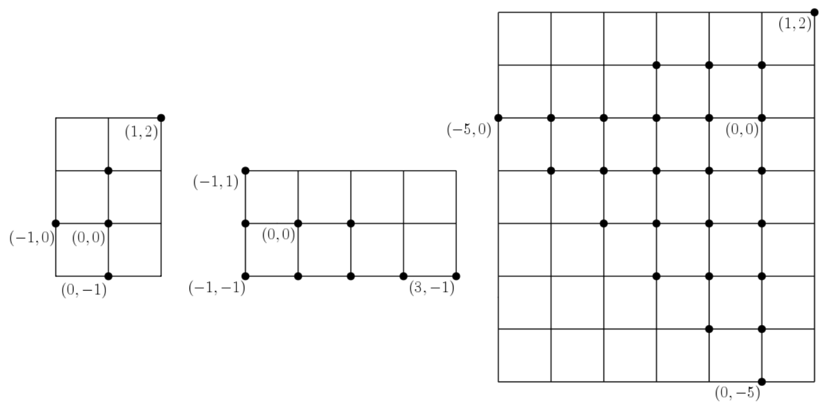

FIGURE 2. P, $\text{conv}(P)^* \cap \mathbb{Z}^2$, and $P^*_{\mathbb{Z}^2}$, respectively

3. Duality and Inequalities

In the following, we mainly consider the duality of the polar of convex lattice sets in $\mathbb{Z}^n$. Meanwhile, some inequalities are established. It turns out that only some special classes of convex lattice sets have the dual property, i.e., $(K^*)^* = K$. A counterexample is $P = \text{conv}\{(2,0),(0,2),(-2,0),(0,-2)\} \cap \mathbb{Z}^2$, $(P^*_{\mathbb{Z}^2})^*_{\mathbb{Z}^2} = \text{conv}\{(1,0),(0,1),(-1,0),(0,-1)\} \cap \mathbb{Z}^2 \neq P$.

Theorem 3.1. *Let K be a convex lattice set in $\mathbb{Z}^n$ with $K' = \{(k_1, 0, \ldots, 0),$ $(0, k_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, k_n),\ (-m_1, 0, \ldots, 0), (0, -m_2, 0, \ldots, 0), \ldots, (0,$ $\ldots, 0, -m_n)\}$, $k_i, m_i \in \mathbb{N}_+$. Then*

$$(K^*_{\mathbb{Z}^n})^*_{\mathbb{Z}^n} = \frac{K}{k},$$

where $k = \gcd(k_1, \ldots, k_n, m_1, \ldots, m_n)$.

Proof. By the definition of $K^*_{\mathbb{Q}^n}$, we get

$$K^*_{\mathbb{Q}^n} = \left\{ \left(\frac{1}{k_1}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \frac{1}{k_n}\right), \left(-\frac{1}{m_1}, 0, \ldots, 0\right), \ldots, \right.$$
$$\left. \left(0, \ldots, 0, -\frac{1}{m_n}\right) \right\}.$$

Let $m = \text{lcm}(k_1, \ldots, k_n, m_1, \ldots, m_n)$, by Proposition 2.5, $\mathscr{F}(A) = mA$. Thus

$$K^*_{\mathbb{Z}^n} = \text{conv}(mK^*_{\mathbb{Q}^n}) \cap \mathbb{Z}^n.$$

Let $W = K^*_{\mathbb{Z}^n}$, we get

$$W' = mK^*_{\mathbb{Q}^n}, W^*_{\mathbb{Q}^n} = \frac{1}{m}K'.$$

Obviously, $g(x) = mx/k$ is the nonsingular linear transformation as in Definition 2.4, so

$$(K^*_{\mathbb{Z}^n})^*_{\mathbb{Z}^n} = W^*_{\mathbb{Z}^n} = \mathrm{conv}\left(\frac{m}{k}W^*_{\mathbb{Q}^n}\right) \cap \mathbb{Z}^n = \mathrm{conv}\left(\frac{K'}{k}\right) \cap \mathbb{Z}^n = \frac{K}{k}.$$

$\square$

Corollary 3.2. *Let K be a convex lattice set in $\mathbb{Z}^n$ with $K' = \{(k_1, 0, \dots, 0), (0, k_2, 0, \dots, 0), \dots, (0, \dots, 0, k_n), (-m_1, 0, \dots, 0), (0, -m_2, 0, \dots, 0), \dots, (0, \dots, 0\ , -m_n)\}$, $k_i, m_i \in \mathbb{N}_+$, and $\gcd(k_1, \dots, k_n, m_1, \dots, m_n) = 1$. Then*

$$(K^*_{\mathbb{Z}^n})^*_{\mathbb{Z}^n} = K.$$

Remark 3.3. Another class of convex lattice sets that has the dual property is of the form $K = \mathrm{conv}(\{(\pm x_1, \dots, \pm x_n)\}) \cap \mathbb{Z}^n$, $x_i \in \mathbb{N}_+$, and $\gcd(x_1, \dots, x_n) = 1$. We can calculate that

$$\begin{aligned}(K^*_{\mathbb{Z}^n})^*_{\mathbb{Z}^n} &= \mathrm{conv}\left(\frac{\mathrm{lcm}(x_1, \dots, x_n)}{\gcd(x_1, \dots, x_n)}(K^*_{\mathbb{Z}^n})^*_{\mathbb{Q}^n}\right) \cap \mathbb{Z}^n \\ &= \mathrm{conv}\left(\frac{K'}{\gcd(x_1, \dots, x_n)}\right) \cap \mathbb{Z}^n \\ &= \mathrm{conv}(K') \cap \mathbb{Z}^n \\ &= K.\end{aligned}$$

Recall that the first successive minimum $\lambda_1(K)$ of a convex body K with $O \in \mathrm{int}(K)$ is

$$\lambda_1(K) = \min\{\lambda \in \mathbb{R}_{>0} : \lambda K \cap \mathbb{Z}^n \neq \{O\}\}.$$

For any point P belong to K', the corresponding $P^* \in K^*_{\mathbb{Q}^n}$ is obtained on the ray OP, so we have $P^* \in \mathrm{conv}\{\sum_{i=1}^n \pm e_i\}$. It is easy to find that $\lambda_1(\mathrm{conv}(K^*_{\mathbb{Q}^n})) \geq 1$ while the upper bound does not exist.

Based on $\mathrm{conv}(K^*_{\mathbb{Q}^n}) \subset \mathrm{conv}\{\sum_{i=1}^n \pm e_i\}$, we get the following inequality.

Theorem 3.4. *Let*

$$a = \frac{\mathrm{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn})}{\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})},$$

then

$$|K^*_{\mathbb{Z}^n}| \leq (2a+1)^n. \tag{3.1}$$

Proof. Since $\operatorname{conv}(K^*_{\mathbb{Q}^n}) \subset \operatorname{conv}\{\sum_{i=1}^n \pm e_i\}$, $\operatorname{conv}(K^*_{\mathbb{Q}^n})/2$ is a packing set with respect to $\mathbb{Z}^n$. Then, we have

$$\begin{aligned}
|K^*_{\mathbb{Z}^n}| &= \frac{V(K^*_{\mathbb{Z}^n} + \frac{1}{2}\operatorname{conv}(K^*_{\mathbb{Q}^n}))}{V(\frac{1}{2} conv(K^*_{\mathbb{Q}^n}))} \\
&\le \frac{V(\operatorname{conv}(K^*_{\mathbb{Z}^n}) + \frac{1}{2}\operatorname{conv}(K^*_{\mathbb{Q}^n}))}{V(\frac{1}{2}\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&= \frac{V(a\operatorname{conv}(K^*_{\mathbb{Q}^n}) + \frac{1}{2}\operatorname{conv}(K^*_{\mathbb{Q}^n}))}{V(\frac{1}{2}\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&= \frac{V((a+\frac{1}{2})\operatorname{conv}(K^*_{\mathbb{Z}^n}))}{V(\frac{1}{2}\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&= \left(\frac{a+\frac{1}{2}}{\frac{1}{2}}\right)^n \\
&= (2a+1)^n.
\end{aligned}$$

□

In fact, for any point $P \in K'$, $P \notin \operatorname{lin}(e_i)$ where lin denotes the linear span, then $K^*_{\mathbb{Q}^n} \subseteq \operatorname{conv}\{\sum_{i=1}^n \pm e_i\}/2$, and inequality (3.1) can be improved.

Corollary 3.5. *Let*

$$a = \frac{\operatorname{lcm}(p_{11}, \dots, p_{1n}, \dots, p_{kn})}{\gcd(q_{11}, \dots, q_{1n}, \dots, q_{kn})}.$$

For any point $P \in K'$, $P \notin \operatorname{lin}(e_i)$, then $|K^*_{\mathbb{Z}^n}| \le (a+1)^n$.

Proof. Let $P(x_1, \dots, x_n) \in K'$, then we have $P^*(y_1, \dots, y_n) \in K^*_{\mathbb{Q}^n}$ on the ray OP such that

$$|y_i| \le \left|\frac{x_i}{\sum_{i=1}^n x_i^2}\right|.$$

$P \notin \operatorname{lin}(e_i)$ indicates that there exist i_1, i_2 such that $x_{i_1}^2 \ge 1$, $x_{i_2}^2 \ge 1$, $i_1, i_2 \in \{1, 2, \dots, n\}$. If $x_i > 0$, then

$$\begin{aligned}
&\left|\frac{x_i}{\sum_{j=1}^n x_j^2}\right| = \frac{x_i}{\sum_{j=1}^n x_j^2} \le \frac{1}{2} \\
\iff & \sum_{j=1}^n x_j^2 - 2x_i \ge 0 \\
\iff & x_i^2 - 2x_i + \sum_{j\ne i, j=1}^n x_j^2 \ge 0 \\
\iff & (x_i - 1)^2 + \left(\sum_{j\ne i, j=1}^n x_j^2 - 1\right) \ge 0,
\end{aligned}$$

with equality if and only if $x_i = 1, x_k = \pm 1 (k \neq i)$, $x_j = 0$ $(j \neq i \neq k)$.

If $x_i < 0$, then

$$\left|\frac{x_i}{\sum_{j=1}^n x_j^2}\right| = -\frac{x_i}{\sum_{j=1}^n x_j^2} \leq \frac{1}{2}$$

$$\iff (x_i+1)^2 + \left(\sum_{j\neq i, j=1}^{n} x_j^2 - 1\right) \geq 0,$$

with equality if and only if $x_i = -1, x_k = \pm 1 (k \neq i)$, $x_j = 0$ $(j \neq i \neq k)$. Thus

$$|y_i| \leq \left|\frac{x_i}{\sum_{i=1}^n x_i^2}\right| \leq \frac{1}{2}.$$

This implies $K^*_{\mathbb{Q}^n} \subseteq \frac{1}{2}\operatorname{conv}\{\sum_{i=1}^n \pm e_i\}$, with equality if and only if $n = 2$ and $K = \operatorname{conv}\{(1,1),(-1,1),(-1,-1),(1,-1)\} \cap \mathbb{Z}^2$.

So, $\operatorname{conv}(K^*_{\mathbb{Q}^n})$ is a packing set with respect to $\mathbb{Z}^n$. Then, we have

$$\begin{aligned}
|K^*_{\mathbb{Z}^n}| &= \frac{V(K^*_{\mathbb{Z}^n} + \operatorname{conv}(K^*_{\mathbb{Q}^n}))}{V(\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&\leq \frac{V(\operatorname{conv}(K^*_{\mathbb{Z}^n}) + \operatorname{conv}(K^*_{\mathbb{Q}^n}))}{V(\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&= \frac{V(a\operatorname{conv}(K^*_{\mathbb{Q}^n}) + \operatorname{conv}(K^*_{\mathbb{Q}^n}))}{V(\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&= \frac{V((a+1)\operatorname{conv}(K^*_{\mathbb{Z}^n}))}{V(\operatorname{conv}(K^*_{\mathbb{Q}^n}))} \\
&= (a+1)^n.
\end{aligned}$$

□

Remark 3.6. In the plane, $K = \operatorname{conv}\{(1,1),(-1,1),(-1,-1),(1,-1)\} \cap \mathbb{Z}^2$ satisfies the equality in Corollary 3.5.

In the following, the connection between the number of the lattice points in the convex lattice set K and the area of $\operatorname{conv}(K)$ is given in the plane. The main ingredient is Pick's theorem.

Lemma 3.7 (Pick's theorem)**.** *Let A be the area of a simply closed lattice polygon. Let B denote the number of lattice points on the polygon edges and I the number of points in the interior of the polygon. Then*

$$A = I + \frac{1}{2}B - 1.$$

Theorem 3.8. *Suppose that K is a convex lattice set in $\mathbb{Z}^2$.*

If $\dim(\operatorname{conv}(K)) = 1$,

$$|K| \leq P(\operatorname{conv}(K)) + 1,$$

with equality if and only if K lies in the x-axis or the y-axis, where $P(\mathrm{conv}(K))$ *denotes the perimeter of* $\mathrm{conv}(K)$.

If $\dim(\mathrm{conv}(K)) = 2$,

$$|K| \geq A(\mathrm{conv}(K)) + \frac{5}{2},$$

with equality if and only if $B(\mathrm{conv}(K)) = 3$.

Here $|K|$ *denotes the number of points in* K *and* $A(\mathrm{conv}(K))$ *denotes the area of* $\mathrm{conv}(K)$.

Proof.

Case 1: $\dim(\mathrm{conv}(K)) = 1$.

According to Blichfeldt's theorem [4], $|K| \leq P(\mathrm{conv}(K)) + 1$, with equality if and only if $K \subset \mathrm{lin}(e_1)^{\perp}$, *or* $K \subset \mathrm{lin}(e_2)^{\perp}$, where $^{\perp}$ denotes the orthogonal complement space.

Case 2: $\dim(\mathrm{conv}(K)) = 2$.

In this case, $B(\mathrm{conv}(K)) \geq 3$, thus,

$$B(\mathrm{conv}(K)) \geq \frac{1}{2}B(\mathrm{conv}(K)) + \frac{3}{2}.$$

By Pick's theorem,

$$A(\mathrm{conv}(K)) = I(\mathrm{conv}(K)) + \frac{1}{2}B(\mathrm{conv}(K)) - 1.$$

Then,

$$\begin{aligned}
\mathrm{vert}\, K\, \mathrm{vert} &= I(\mathrm{conv}(K)) + B(\mathrm{conv}(K)) \\
&\geq I(\mathrm{conv}(K)) + \frac{1}{2}B(\mathrm{conv}(K)) + \frac{3}{2} \\
&= I(\mathrm{conv}(K)) + \frac{1}{2}B(\mathrm{conv}(K)) - 1 + \frac{5}{2} \\
&= A(\mathrm{conv}(K)) + \frac{5}{2},
\end{aligned}$$

with equality if and only if $B(\mathrm{conv}(K)) = 3$.

□

Acknowledgement

We are very grateful for the anonymous referee for their careful corrections and valuable suggestions that improved the paper.

References

1. S. Berg and M. Henk, *Lattice point inequalities for centered convex bodies*, SIAM J. Discrete Math. **30** (2016), 1148–1158.
2. R. Gardner and P. Gronchi, *A Brunn–Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc. **353** (2001), 3995–4024.
3. R. Gardner, P. Gronchi, and C. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram*, Discrete Comput. Geom. **34** (2005), 391–409.
4. J. Goodman, J. O'Rourke, and C. Tóth (eds.), *Handbook of discrete and computational geometry*, 3rd ed., CRC Press, Boca Raton, FL, 2017.

5. P. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematis-chen Wissenschaften, vol. 336, Springer, Berlin, 2007.
6. C. Haase and I. Melnikov, *The reflexive dimension of a lattice polytope*, Ann. Comb. **10** (2006), 211–217.
7. R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, 2nd ed., Cambridge Univ. Press, Cambridge, 2014.

College of Science, Beijing Forestry University, Beijing 100083, P. R. China
E-mail address: wangjiebulingbuling@foxmail.com

College of Science, Beijing Forestry University, Beijing 100083, P. R. China
E-mail address: silin@bjfu.edu.cn