



ANTICHAINS AND COUNTERPOINT DICHOTOMIES

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ABSTRACT. We construct a special type of antichain (i.e., a family of subsets of a set, such that no subset is contained in another) using group-theoretical considerations, and obtain an upper bound on the cardinality of such an antichain. We apply the result to bound the number of strong counterpoint dichotomies up to affine isomorphisms.

1. INTRODUCTION

Sperner systems, antichains or clutters (families of subsets of an n -set such that no one is contained in another) have been important combinatorial objects since the foundational theorem of Sperner was published, which states that they cannot contain more than $\binom{n}{\lfloor n/2 \rfloor}$ elements. Concerning their size, it is possible to establish upper bounds on their cardinality using elegant elementary methods and any additional properties they may have. Sometimes it is even possible to exhibit maximal antichains that attain such bounds.

In particular, some antichains can be studied from the standpoint of group theory as in [3], to obtain bounds (in the spirit of the Erdős-Ko-Rado theorem) that rely on the isotropy groups of the members of the antichain. However, particular types of antichains that are of interest for mathematical musicology (namely, pairwise non-isomorphic counterpoint dichotomies) are not encompassed in such an approach, for the isotropy group of their members is trivial.

In Section 2 we define such antichains in a slightly more general setting than the original formulation given in [5, Part VII], and we obtain a somewhat crude upper bound for their cardinality. Then, in Section 3, we apply the result to bound the number of non-isomorphic counterpoint dichotomies.

2. STRONG ANTICHAINS

Let G be a subgroup of the symmetric group of S . The group G acts obviously on S ; it also acts on the powerset 2^S of S , with an action defined

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by

$$\begin{aligned} \cdot : G \times 2^S &\rightarrow 2^S, \\ (g, A) &\mapsto gA := \{ga : a \in A\}. \end{aligned}$$

Two sets $A, B \in 2^S$ are said to be *isomorphic* if there exists $g \in G$ such that $A = gB$. In this paper we will always consider a set S of even cardinality and the subset

$$\binom{S}{|S|/2} := \{A \subseteq S : |A| = |S|/2\},$$

of its power set, whose elements will be called *dichotomies*.

Definition 2.1. The dichotomy $D \in \binom{S}{|S|/2}$ is:

- (1) *G-rigid* when $gD = D$ for $g \in G$ implies that $g = e$,
- (2) *G-autocomplementary* if there exists a $p \in G$ (called *autocomplementarity function*) such that

$$pD = \mathbb{C}D := S \setminus D,$$

- (3) *G-strong* if it is *G-rigid* and *G-autocomplementary*.

Remark 2.2.

- (1) If a dichotomy $D \in \binom{S}{|S|/2}$ is *G-strong* and $p \in G$ is such that $pD = \mathbb{C}D$, then p is unique. We call it the *polarity* of D . For any $g \in G$, the set gD is also *G-strong* (with polarity gpg^{-1}).
- (2) Since S is finite, any polarity is an involution that does not fix points of S , i.e., it is an involutive derangement.

Definition 2.3. An involutive derangement $q \in G$ is called a *quasipolarity*.

Since any quasipolarity q decomposes only in 2-cycles, we can construct a dichotomy $U_q \in \binom{S}{|S|/2}$ that contains exactly one element from each cycle. It is clear that U_q is autocomplementary and $qU_q = \mathbb{C}U_q$.

Consider now the set of dichotomies whose autocomplementarity function is q , which is

$$\begin{aligned} M_q &= \left\{ D \in \binom{S}{|S|/2} : qD = S \setminus D \right\} \\ &= \{A \cup B : A \subseteq U_q, B = (S \setminus U_q) \setminus qA\}, \end{aligned}$$

where the last equality follows from

$$qA = (S \setminus U_q) \setminus A \quad \text{and} \quad q((S \setminus U_q) \setminus qA) = U_q \setminus A.$$

We see that $|M_q| = 2^{|U_q|} = 2^{|S|/2}$.

Lemma 2.4. *If $g \in G_q$, then $gM_q = M_q$.*

Proof. Since $g \in G_q$ is equivalent to $gq = qg$, we have

$$q(gD) = g(qD) = g(S \setminus D) = S \setminus gD,$$

which means that $gD \in M_q$, for $|gD| = |S|/2$. □

Lemma 2.5. *If $h \in G$, then $M_{hqh^{-1}} = hM_q$.*

Proof. First, we have

$$hM_q = \{hD : qD = S \setminus D\}.$$

Now take any $hD \in hM_q$, then

$$hqh^{-1}hD = hqD = h(S \setminus D) = S \setminus hD,$$

thus $hD \in M_{hqh^{-1}}$ and $hM_q \subseteq M_{hqh^{-1}}$. Since both sets in the inclusion have the same cardinality, the lemma follows. \square

Let \mathcal{T} be a traversal of $\binom{S}{|S|/2}/G$ (i.e., a set consisting of exactly one element from each G -orbit on $\binom{S}{|S|/2}/G$) and $\mathcal{R}_G \subseteq \mathcal{T}$ a subset such that all of its members are G -strong. For any $D_1, D_2 \in \mathcal{T}$, it is impossible that $D_1 \subseteq D_2$, for then $D_1 = D_2$ and they would represent the same orbit. Thus \mathcal{T} and \mathcal{R}_G are antichains. In particular, \mathcal{R}_G is a *complement-free* antichain, since $D \in \mathcal{R}_G$ implies that $\mathcal{C}D \notin \mathcal{R}_G$. Indeed, if $D_1 \cap D_2 = \emptyset$, it would imply that $D_1 = \mathcal{C}D_2 = pD_2$ for some $p \in G$, hence D_1 would be in the orbit of D_2 .

Definition 2.6. Let G be a subgroup of the symmetric group of a finite set S , with $|S|$ even. A *G -strong antichain* is a subset \mathcal{R}_G of a traversal \mathcal{T} of the orbit set $\binom{S}{|S|/2}/G$, such that all its members are G -strong.

It is obvious that the cardinality of a G -strong antichain \mathcal{R}_G is not greater than the number of $(|S|/2)$ -subsets of S , so

$$|\mathcal{R}_G| \leq \binom{|S|}{|S|/2}.$$

Being \mathcal{R}_G a complement-free antichain, a theorem by Purdy [2, p. 139] tells us that

$$|\mathcal{R}_G| \leq \binom{|S|}{|S|/2 - 1},$$

and using the Erdős-Ko-Rado theorem [4, Theorem 1], we can improve this slightly:

$$|\mathcal{R}_G| \leq \binom{|S| - 1}{|S|/2 - 1}.$$

These upper bounds, however, do not make full use of the structure of \mathcal{R}_G derived from the action of G on S . In order to exploit it, note first that G -strong dichotomies have orbits of maximum cardinality, namely $|G|$.

Let G_q be the isotropy group of a quasipolarity $q \in G$ under the conjugation action. The number of non-isomorphic G_q -strong dichotomies for a

given quasipolarity q is bounded by

$$(1) \quad |\mathcal{R}_{G_q}| \leq \frac{|M_q|}{|G_q|} = \frac{2^{|S|/2}}{|G_q|},$$

since, by Lemmas 2.4 and 2.5, the set M_q of dichotomies whose autocomplementary function is q coincide with $M_{q'}$, for any conjugate q' of q .

Every G -strong dichotomy is a G_q -strong dichotomy because G_q is a subgroup of G , hence $|\mathcal{R}_G| \leq \sum_{[q] \in \mathcal{Q}} |\mathcal{R}_{G_q}|$, where \mathcal{Q} is the set of conjugacy classes of the quasipolarities of G . Now summing (1) over \mathcal{Q} , we prove the following result.

Theorem 2.7. *Let G be a subgroup of the symmetric group of the set S (where $|S|$ is an even number) and \mathcal{R}_G be a G -strong antichain. Then*

$$(2) \quad |\mathcal{R}_G| \leq \sum_{[q] \in \mathcal{Q}} \frac{2^{|S|/2}}{|G_q|},$$

where \mathcal{Q} is the set of conjugacy classes of quasipolarities of G .

Remark 2.8. Observe that if we have better estimations of the number of quasipolarities of G and of the sizes of their isotropy groups, we can improve the bound of Theorem 2.7. But even with those refinements, (2) might be far from optimal.

Example 2.9. If $|S| = 2k > 2$ and $G = \text{Sym}(S)$ (the symmetric group), then there are no $\text{Sym}(S)$ -strong dichotomies. Considering that the number of conjugacy classes of quasipolarities in this case is 1, and the cardinality of the isotropy group of any quasipolarity q satisfies

$$|\text{Sym}(S)_q| = 2^k \cdot k!,$$

we conclude using (2) that

$$|\mathcal{R}_{\text{Sym}(S)}| \leq \frac{2^k}{2^k \cdot k!} = \frac{1}{k!},$$

which is a nice confirmation.

Example 2.10. If $|S| = 2k$ and $G = D_{2k}$ is the dihedral group of order $4k$, there are $k + 1$ quasipolarities: k reflections and one rotation. The reflections constitute a whole conjugacy class, each having isotropy groups of cardinality 2, whilst the rotation has D_{2k} as its isotropy group. This, together with (2), gives us

$$|\mathcal{R}_{D_{2k}}| \leq \frac{2^k}{2} + \frac{2^k}{4k} = 2^{k-1} + \frac{2^{k-2}}{k}.$$

3. THE CASE OF COUNTERPOINT DICHOTOMIES

As we stated in the introduction, strong G -antichains are of great interest for the mathematical theory of counterpoint, for they represent alternative ways of understanding consonances and dissonances (see [1] and [5], for example). In that context, we take $S = \mathbb{Z}_{2k}$ and G is the semidirect product

$$G = \overrightarrow{GL}(\mathbb{Z}_{2k}) := \mathbb{Z}_{2k} \ltimes \mathbb{Z}_{2k}^\times$$

where \mathbb{Z}_{2k}^\times is the set of units of \mathbb{Z}_{2k} . The elements of G can be seen as the *affine* functions

$$\begin{aligned} e^u v : \mathbb{Z}_{2k} &\rightarrow \mathbb{Z}_{2k}, \\ x &\mapsto vx + u, \end{aligned}$$

the element u is the *affine part* and v is the *linear part*. Note that

$$|\overrightarrow{GL}(\mathbb{Z}_{2k})| = 2k\varphi(2k)$$

where φ is the Euler totient function.

Our aim now is to characterize the quasipolarities of $\overrightarrow{GL}(\mathbb{Z}_{2k})$ in terms of the involutions of \mathbb{Z}_{2k}^\times . We will use this information to obtain a bound on the number of strong $\overrightarrow{GL}(\mathbb{Z}_{2k})$ -dichotomies (which we will call *counterpoint dichotomies*) up to isomorphism.

In order to state the characterization, for $v \in \mathbb{Z}_{2k}^\times$ define

$$\begin{aligned} \sigma_{2k}(v) &:= \gcd(v - 1, 2k), \\ \tau_{2k}(v) &:= \gcd(v + 1, 2k), \end{aligned}$$

where ν is taken as the minimum non-negative element in the residue class of v .

Theorem 3.1. *The function $e^u v \in \overrightarrow{GL}(\mathbb{Z}_{2k})$ is a quasipolarity if, and only if*

- (i) v is an involution,
- (ii) $2 \frac{2k}{\tau_{2k}(v)} = \sigma_{2k}(v)$, and
- (iii) $u = \sigma_{2k}(v)q + \frac{2k}{\tau_{2k}(v)}$ for some integer q .

If k is odd, the second condition can be omitted.

Before proving the theorem, let us make a series of remarks. Let $g = e^u v \in \overrightarrow{GL}(\mathbb{Z}_{2k})$. The function g is an affine involution if, and only if,

$$(3) \quad v^2 = 1,$$

$$(4) \quad u(v + 1) = 0.$$

For g to be a derangement, it is necessary and sufficient that the equation

$$(5) \quad (v - 1)x = -u$$

has no solutions. It is easy to see that the only values of u that satisfy (4) are the multiples of

$$u_0 = \frac{2k}{\tau_{2k}(v)}.$$

Lemma 3.2. *Let $v \in \mathbb{Z}_{2k}^\times$ be an involution. Then*

$$u_0 = \frac{2k}{\tau_{2k}(v)} \quad \text{divides} \quad \sigma_{2k}(v).$$

Proof. By Bezout's identity, there are integers a, b, c, d such that

$$\sigma_{2k}(v) = a(\nu - 1) + 2kb, \quad \tau_{2k}(v) = c(\nu + 1) + 2kd.$$

We have the product

$$\sigma_{2k}(v)\tau_{2k}(v) = ac(\nu^2 - 1) + 2k(\nu(bc + ad) + bc - ad + 2kbd).$$

Since $\nu^2 - 1 = 0$, there exists an integer m such that $\nu^2 - 1 = 2km$, hence

$$\sigma_{2k}(v)\tau_{2k}(v) = 2k(acm + \nu(bc + ad) + bc - ad + 2kbd)$$

and the result is immediate. \square

Equation (5) does not hold if, and only if, $\sigma_{2k}(v) = \gcd(\nu - 1, n)$ does not divide $-u$ (or, equivalently, it does not divide u). By the division algorithm, u must be a multiple of u_0 of the form

$$(6) \quad u = \sigma_{2k}(v)q + r, \quad 0 < r < \sigma_{2k}(v), q \in \mathbb{Z}$$

From this and Lemma 3.2 we deduce that r must also be a multiple of u_0 . It is clear that $u_0 \leq \sigma_{2k}(v)$.

Proposition 3.3. *If $u_0 = \frac{2k}{\tau_{2k}(v)} = \sigma_{2k}(v)$, there are no quasipolarities with v as its linear part.*

Proof. For if $u_0 = \sigma_{2k}(v)$, then $\sigma_{2k}(v)$ divides u . \square

Proof of Theorem 3.1. Suppose $e^u v$ is a quasipolarity. Then equations (3) and (4) hold, and (5) has no solutions. Consequently, u is a multiple of u_0 . The integer u guarantees that (5) has no solutions if, and only if,

$$u = \sigma_{2k}(v)q + r$$

for some integer q and $0 < r < \sigma_{2k}(v)$. This happens if, and only if, $u_0 < \sigma_{2k}(v)$. The inequality is sufficient because u_0 divides $\sigma_{2k}(v)$ and Proposition 3.3; it is necessary because u_0 divides r and $u_0 \leq r < \sigma_{2k}(v)$.

For the converse, let v be an involution, let $u_0 = \frac{2k}{\tau_{2k}(v)}$, and let u be given by (6) with $r = u_0$. Thus u is a multiple of u_0 , hence it is a solution of (4). It also ensures that (5) has no solutions since $u_0 < \sigma_{2k}(v)$.

In particular, $u_0 < \sigma_{2k}(v)$ is always true when k is odd. Keeping in mind that ν is odd because v is an involution, let λ and μ be such that

$$(7) \quad \nu - 1 = 2\lambda, \quad \nu + 1 = 2\mu.$$

Note that λ and μ are coprime because

$$\mu - \lambda = \frac{\nu + 1}{2} - \frac{\nu - 1}{2} = 1.$$

We have

$$2km = \nu^2 - 1 = 4\lambda\mu.$$

and thus

$$km = 2\lambda\mu,$$

and therefore 2 divides m . It follows that $\lambda\mu$ is a multiple of k . Using this we can show that $\frac{k}{\gcd(\mu, k)} = \gcd(\lambda, k)$, which follows from the examination of the product

$$\begin{aligned} \gcd(\lambda, k) \gcd(\mu, k) &= \gcd(\lambda\mu, k) \\ &= \gcd\left(k\frac{m}{2}, k\right) \\ &= k \gcd\left(\frac{m}{2}, 1\right) \\ &= k. \end{aligned}$$

Now

$$(8) \quad \sigma_{2k}(v) = 2 \gcd(\lambda, k) = 2 \frac{k}{\gcd(\mu, k)} = 2 \frac{2k}{\tau_{2k}(v)} = 2u_0.$$

It remains to prove that $\sigma_{2k}(v) = 2u_0$ when k is even and $\frac{2k}{\tau_{2k}(v)} < \sigma_{2k}(v)$. The number $\frac{k}{\gcd(\lambda, k) \gcd(\mu, k)}$ is a non-zero integer and, by hypothesis,

$$\frac{k}{\gcd(\lambda, k) \gcd(\mu, k)} = \frac{4k}{(2 \gcd(\lambda, k))(2 \gcd(\mu, k))} = 2 \frac{2k}{\sigma_{2k}(v) \tau_{2k}(v)} < 2,$$

which implies that $\frac{k}{\gcd(\lambda, k) \gcd(\mu, k)} = 1$ and (8) is true again. \square

Proposition 3.4. *The isotropy group of an affine quasipolarity $g = e^u v \in \overrightarrow{GL}(\mathbb{Z}_{2k})$ under the conjugation action has cardinality*

$$|\overrightarrow{GL}(\mathbb{Z}_{2k})_g| = \sigma_{2k}(v) \varphi(2k).$$

Equivalently, there are $\frac{2k}{\sigma_{2k}(v)}$ elements in the orbit of g under the conjugation action.

Proof. Let $h = e^t s \in \overrightarrow{GL}(\mathbb{Z}_{2k})$. We have

$$(9) \quad g' = hgh^{-1} = (e^t s)(e^u v)(e^{-s^{-1}t} s^{-1}) = e^{t(1-v)+su} v.$$

The members of the stabilizer of g are those whose parameters satisfy

$$t(1-v) = u(1-s).$$

This equation has a solution for each $s \in GL(\mathbb{Z}_{2k})$ since $\sigma_{2k}(v) = \gcd(1-v, 2k)$ divides $u(1-s)$ since $1-s$ is even. In particular, the equation above has

exactly $\sigma_{2k}(v)$ different solutions. Since this occurs for each $s \in GL(\mathbb{Z}_{2k})$, the cardinality of the stabilizer of g is

$$|\overrightarrow{GL}(\mathbb{Z}_{2k})_g| = \sigma_{2k}(v)\varphi(2k).$$

This means that each orbit has

$$\frac{|\overrightarrow{GL}(\mathbb{Z}_{2k})|}{\sigma_{2k}(v)\varphi(2k)} = \frac{2k}{\sigma_{2k}(v)}$$

elements. □

Corollary 3.5. *The group $\overrightarrow{GL}(\mathbb{Z}_{2k})$ acts transitively on the set of all affine quasipolarities with fixed linear part.*

For a $\overrightarrow{GL}(\mathbb{Z}_{2k})$ -strong antichain, we have by Proposition 3.4

$$|\overrightarrow{GL}(\mathbb{Z}_{2k})_g| = \sigma_{2k}(v)\varphi(2k) \geq 2\varphi(2k)$$

using the involution $v = -1$. The number of conjugacy classes of quasipolarities is bounded by $\varphi(2k)$, thus Theorem 2.7 gives us

$$|\mathcal{R}_{\overrightarrow{GL}(\mathbb{Z}_{2k})}| \leq \varphi(2k) \frac{2^k}{2\varphi(2k)} = 2^{k-1}.$$

This estimate can be improved further with a better bound on the number of involutions of $GL(\mathbb{Z}_{2k})$.

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