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TOURNAMENTS WITH KERNELS BY MONOCHROMATIC PATHS

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ABSTRACT. In this paper we prove the existence of kernels by monochromatic paths in m-coloured tournaments in which every cyclic tournament of order 3 is at most 2-coloured in addition to other restrictions on the colouring of certain subdigraphs. We point out that in all previous results on kernels by monochromatic paths in arc coloured tournaments, certain small substructures are required to be monochromatic or monochromatic with at most one exception, whereas here we allow up to three colours in two small substructures.

1. INTRODUCTION

For general concepts, we refer the reader to [1, 4]. Throughout this paper all paths and cycles will be directed paths and directed cycles. A digraph D is an m-coloured digraph if its arcs are coloured with m colours, and a subdigraph H is: *monochromatic* whenever all of its arcs are coloured alike, *j-amc* or *j-almost monochromatic* if with at most j exceptions all of its arcs are coloured with the same colour, and *2-wamc* or *2-weakly almost monochromatic* if it is 2-amc and the two possible exceptions do not form a directed path.

A tournament is said to be a *cyclic tournament* whenever it has at least one Hamiltonian directed cycle. In a cyclic tournament K of order 4 with (u, v, w, x, u) a Hamiltonian directed cycle, we denote by $K_t(u, v, w, x, u)$ (or simply K_t) any subdivision of K where at least three arcs of the Hamiltonian directed cycle are not subdivided (i.e. at most one arc of the Hamiltonian directed cycle is subdivided and the two diagonals can be subdivided or not). We say that $K_t(u, v, w, x, u)$ is *2-samc* or a *2-subdivision almost monochromatic* if it is a subdivision of a 2-amc cyclic tournament of order 4 (i.e. at least 4 of the arcs or directed paths joining the vertices u, v, w, and x are coloured with the same colour). Finally, if T is an m-coloured tournament a *kmp* or a *kernel by monochromatic paths* of T is a vertex $v \in V(T)$ such that for every other vertex x of T there is an xv monochromatic directed path in T.

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The topic of domination in graphs has been widely studied by several authors, and a complete study of this topic is presented in [16, 17]. A special class of domination is the domination in digraphs, and it is defined as follows. Let D be a digraph. A set of vertices $S \subseteq V(D)$ is *dominating* whenever for every $w \in (V(D) \setminus S)$ there exists a wS-arc in D. Dominating independent sets in digraphs (kernels in digraphs) have found many applications in different topics of mathematics (see, for instance, [2, 3, 7, 8]) and they have been studied by several authors. Interesting surveys of kernels in digraphs can be found in [6, 9]. A digraph D is called *kernel-perfect* if every one of its induced subdigraphs has a kernel.

Let D be an m-*coloured* digraph. A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths* (kmp) if it satisfies the following two conditions:

(1) for every pair of different vertices $u, v \in N$ there is no monochromatic path between them, and
(2) for every vertex $x \in (V(D) \setminus N)$ there is a vertex $y \in N$ such that there is a directed xy-monochromatic path.

Clearly the concepts of domination, independence, and kernel by monochromatic paths in edge-coloured digraphs are a generalisation of those of domination, independence, and kernel in digraphs. The study of the existence of kernels by monochromatic paths in edge-coloured digraphs starts with the theorem of Sands, Sauer, and Woodrow, proved in [18], which asserts that every 2-coloured digraph has a kernel by monochromatic paths. In several papers (see [10, 12, 13]), sufficient conditions for the existence of kernels by monochromatic paths in edge-coloured digraphs have been obtained mainly for tournaments and near tournaments, and require monochromaticity or 1-amc of small subdigraphs (due to the difficulty of the problem). Other interesting results can be found in [14]. In [10] (resp. [15]) it was proved that if D is an m-coloured tournament (resp. bipartite tournament) such that every directed cycle of length 3 (resp. every directed cycle of length 4) is monochromatic, then D has a kernel by monochromatic paths.

In 1982, Sands, Sauer, and Woodrow [18] proved that every 2-coloured tournament has a kmp, and they posed the following problem: Let T be a 3-coloured tournament which does not contain C_3 (the 3-coloured cyclic tournament of order 3). Then, must T contain a kmp?

In 1988 Shen [19] proved that if T is an m-coloured tournament which does not contain C_3 or T_3 (the 3-coloured transitive tournament of order 3) then T has a kmp. He also proved that the situation is best possible for $m \geq 5$. In 2004, Galeana-Sánchez and Rojas-Monroy [11] found a family of counterexamples to this question for $m = 4$. The question for $m = 3$ is still open, that is, does every 3-coloured tournament which does not contain C_3 have a kmp?

In this paper we prove:

(1) If T is an m-coloured tournament such that every cyclic tournament of order 3 is 1-amc and every K_t is 2-samc then T has a kmp, and

(2) If T is an m-coloured tournament such that every cyclic tournament of order 3 is 1-amc and every cyclic tournament of order 4 is 2-wamc, then T has a kmp.

Notice that a 2-wamc cyclic tournament of order 4 may have three colours. In all previous results on the existence of kmp, certain small substructures are required to be monochromatic or 1-amc, that is, they are allowed to have at most two colours.

2. Preliminaries

The set of vertices of D will be denoted by $V(D)$, and the arcs of D will be $A(D)$. An arc $(u,v) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(v,u) \notin A(D)$ (resp. if $(v,u) \in A(D)$). The asymmetrical part of D (resp. symmetrical) which is denoted by $\mathrm{Asym}(D)$ (resp. $\mathrm{Sym}(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. If S is a non-empty subset of $V(D)$, then the subdigraph $D[S]$ of D induced by S is the digraph having vertex set S, and whose arcs are the arcs of D joining vertices of S. If there is a monochromatic path from x to y we use the notation $x \rightarrow_{mp} y$.

If D is an m-coloured digraph, then the *closure* of D, denoted by $\mathcal{C}(D)$, is the m-coloured multidigraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, and $A(\mathcal{C}(D)) = A(D) \cup \{(u,v)_i|$ in D there exists a uv-monochromatic path with colour $i\}$, where $(u,v)_i$ denotes the arc (u,v) coloured with colour i. Notice that for any digraph D, $\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D)$ and K is a kmp of D if and only if K is a kernel of $\mathcal{C}(D)$.

Finally, a *tournament* T (resp. *semicomplete* digraph) is a digraph such that between any two vertices there is one and only one arc (resp. at least one arc).

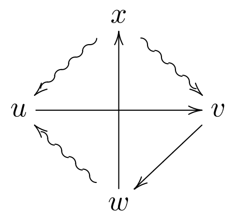
3. Results

We begin this section with a well known result which will be useful in our work.

Theorem 1 (Berge-Duchet [5])**.** *Let D be a semicomplete digraph. Then D is kernel-perfect if and only if every directed cycle of D has at least one symmetric arc.*

Recall from the Introduction that a key question for this work is: What restrictions do we need to impose on an m-coloured tournament so that it has a kernel by monochromatic paths?

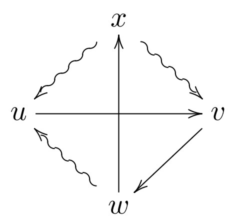
We have defined the configuration K_t, where u, v, w, and x are vertices of D, the straight arrows are arcs in D, and the wiggly arrows can be either arcs or monochromatic paths in D.



We denote this K_t by $K_t(u, v, w, x, u)$.

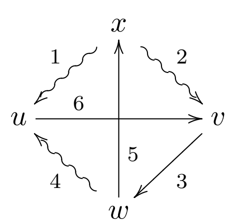
Lemma 1. *Let T be an m-coloured tournament such that every K_t is 2-samc. Then for at least one of the three arcs not subdivided in every K_t there is a monochromatic path in the opposite direction.*

Proof. Let T be an m-coloured tournament with every K_t 2-samc, and let $K_t(u, v, w, x, u)$ be a subgraph of T, where the straight arrows are arcs and the wiggly arrows are either arcs or monochromatic paths.



By hypothesis, K_t is 2-samc, so with at most two exceptions all the arcs or paths have the same colour. We will number the arcs (and monochromatic paths), and show case by case that no matter which two are the exceptions, there is always a monochromatic path in the opposite direction of at least one of the arcs.

First, we number the arrows of K_t so that 1 is (x, u), 2 is (x, v), 3 is (v, w), 4 is (w, u), 5 is (w, x), and 6 is (u, v).



Notice that if 4 and 6 have the same colour, then there is a monochromatic path from w to v, which is the opposite direction of 3, which is an arc. This happens if the exceptions are $\{1, 2\}$, $\{1, 3\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 5\}$, and $\{3, 5\}$.

Similarly, if 5 and 2 are the same colour, then again there is a monochromatic path from w to v, in the opposite direction of 3. This happens if the exceptions are $\{1, 3\}$, $\{1, 4\}$, $\{1, 6\}$, $\{3, 4\}$, $\{3, 6\}$, and $\{4, 6\}$.

We only need to check now the cases where the exceptions are $\{2, 4\}$, $\{2, 6\}$, $\{4, 5\}$, and $\{5, 6\}$. In the first case, when 2 and 4 are the exceptions, 1, 6, and 3 have the same colour, so there is a monochromatic path from x to w, which is the opposite direction of the arc 5.

In the second case, where the exceptions are 2 and 6, 3 and 4 have the same colour, so there is a monochromatic path from v to u, which is the opposite direction of the asymmetric arc 6.

In the third and fourth cases when the exceptions are 4 and 5, and 5 and 6 respectively, there is a monochromatic path from x to w, so the arc 5 is symmetric. This completes the proof of the lemma. □

Lemma 2. *If D is an m-coloured semicomplete digraph such that every directed cycle γ in $\mathcal{C}(D)$ has a symmetric arc, then D has a kernel by monochromatic paths.*

Proof. Let D be an m-coloured semicomplete digraph such that every directed cycle γ in $\mathcal{C}(D)$ has a symmetric arc. This by Theorem 1 is equivalent to $\mathcal{C}(D)$ being kernel-perfect, which in turn implies D has a kernel by monochromatic paths. □

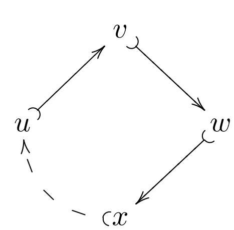
Theorem 2. *Let T be an m-coloured tournament such that all directed triangles are at most 2-coloured, and all K_t are 2-samc. Then T has a kmp.*

Proof. We will prove that every directed cycle γ in $\mathcal{C}(T)$ has a symmetric arc. This by Lemma 2 implies T has a kernel by monochromatic paths.

Let T be an m-coloured tournament such that every directed triangle is at most two coloured, and every K_t is 2-samc, and suppose there is a directed cycle in $\mathcal{C}(T)$ in which every arc is asymmetric, and let γ be such a cycle of minimum length.

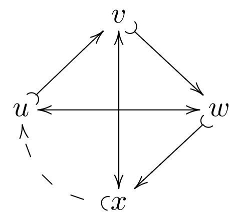
Suppose $l(\gamma) = 3$. Then γ is a directed triangle, say, (x, y, z), so all the arcs in γ are also in T. Since every directed triangle in T is 1-amc, there are two arcs of the same colour, say, (x, y) and (y, z), that is, a monochromatic path from x to z which in $\mathcal{C}(T)$ becomes an arc (x, z). This contradiction implies $l(\gamma) \geq 4$.

γ :

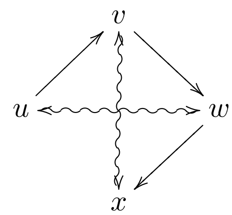


Since all the arcs are asymmetric, they are arcs in T.

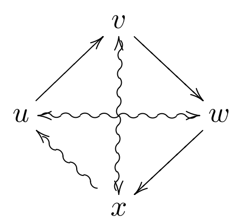
Notice that in $\mathcal{C}(T)$ the arcs between any two non consecutive vertices must be symmetric, otherwise there would be a shorter asymmetric cycle, which is a contradiction, so in $\mathcal{C}(T)$ we have:



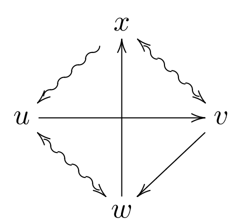
This implies in T we have the following, where the wiggly arrows are monochromatic paths.



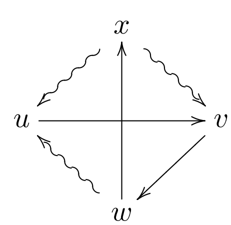
If $l(\gamma) = 4$, then in $\mathcal{C}(T)$ there is an asymmetric arc (x, u), which is therefore also in T. If $l(\gamma) > 4$, then in $\mathcal{C}(T)$ the arc (x, u) must be symmetric, which implies there is a monochromatic path in T from x to u, so in any case, in T, we have:



This is the same as



which has the following K_t embedded.



Thus, $K_t(u, v, w, x, u)$ is a K_t, which by hypothesis is 2-samc, so by Lemma 1 one of the arcs has a monochromatic path in the opposite direction. This implies one of the arcs of γ (in $\mathcal{C}(T)$) is symmetric, which is a contradiction. □

Theorem 3. *Let T be an m-coloured tournament such that all directed triangles are 1-amc, and every cyclic tournament of order four is 2-wamc. Then T has a kmp.*

Proof. We will prove that every directed cycle γ in $\mathcal{C}(T)$ has a symmetric arc. This by Lemma 2 implies T has a kernel by monochromatic paths.

Suppose then that there is a directed cycle in $\mathcal{C}(T)$ which does not have a symmetric arc, and choose γ to be such a cycle of minimum length.

First suppose γ is a directed triangle, say $(x, y), (y, z), (z, x)$. Since all arrows are asymmetric they are all in T, so the triangle is 1-amc, which implies there are two consecutive arcs of the same colour, say $(x, y), (y, z)$, that is, in T there is a monochromatic path $x \to_{mp} z$. This forces the arc (x, z) to be in $\mathcal{C}(T)$, so γ has a symmetric arc, a contradiction, therefore $l(\gamma) \geq 4$.

We point out a few remarks:

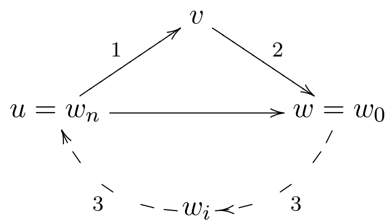
(1) γ is also contained in T. If an arc of γ is not in the arcs of T, then its inverse is in the arcs of T, so it is also in the arcs of $\mathcal{C}(T)$, which implies γ in $\mathcal{C}(T)$ has a symmetric arc, which is a contradiction.

(2) There is at least one change of colour in γ, otherwise the monochromatic path in T would yield a symmetric arc in $\mathcal{C}(T)$.

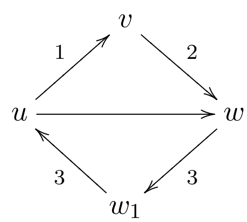
(3) Finally, in $\mathcal{C}(T)$ there is a symmetric arc between any two non-consecutive vertices of γ, otherwise there would be a shorter directed cycle without symmetric arcs.

We will focus our attention on a vertex where there is a change of colour in γ, so say γ has the arcs (u, v) and (v, w), with (u, v) of colour 1, (v, w) of colour 2. We will also study the arcs in T, rather than $\mathcal{C}(T)$. Note that since u and w are not consecutive vertices, by the above observations the arc between them is symmetric. This implies in T there is a monochromatic path from w to u, so consider a minimal such monochromatic path, say $(w = w_0, w_1, \ldots, w_{n-1}, u = w_n)$, and we claim it is neither colour 1 nor colour 2. If it were colour 1, then there would be a monochromatic path, in T, from w to v which would make the arc (v, w) symmetric. Similarly, if it were colour 2, there would be a monochromatic path from v to u, making the arc (u, v) symmetric, another contradiction. Therefore, in T, the monochromatic path $(w_0, \ldots, w_n = u)$ is of, say, colour 3.

Notice (w, u) is not in the arcs of T, for if it is, it must be colour 1 or colour 2, since every directed triangle in T has at most two colours. If it is colour 1, then (w, v) becomes a symmetric arc in $\mathcal{C}(T)$, and if it is colour 2, then (v, u) is a symmetric arc in $\mathcal{C}(T)$, both of which are contradictions. Hence in T we have the subdigraph γ', where $(w = w_0, \ldots, w_n = u)$ is a minimal $w \to_{mp} u$ monochromatic path.



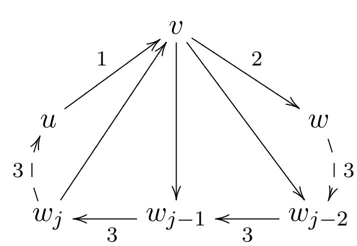
We claim $n > 2$. Assume for a contradiction $n = 2$. Then we have the following.



Regardless of the direction of the arc between v and w_1 and whether most arcs are coloured with colour 1, 2, or 3, this is a cyclic tournament of order four which is not 2-wamc, a contradiction. Hence $n > 2$.

Observe (w, w_1) is in T, but (w_n, w) is in T, so let k be the first subindex such that (w_k, w) is in T, that is, (w, w_j) is in T for all $j < k$. Notice from the above observation that $k > 1$.

We will first assume the arc (v, w_1) is in T, and consider the arcs in T between v and the w_i for $i = 1, \ldots, n$. Since (w_n, v) is in T, there is $j \leq n$ such that (w_j, v) is in T, and (v, w_i) is in T for all $0 \leq i < j$. (Notice that $j \geq 2$ as $(v, w_0) \in A(T)$ and we are assuming $(v, w_1) \in A(T)$).

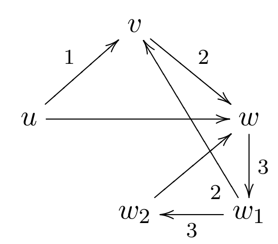


The vertices v, w_{j-2}, w_{j-1}, and w_j form a cyclic tournament, which must therefore be 2-wamc. Since (w_{j-2}, w_{j-1}) and (w_{j-1}, w_j) are consecutive and coloured with 3, these cannot be the exceptions, so the tournament in these four vertices is coloured mostly with colour 3.

If (w_j, v) has colour 3, then there is a monochromatic path from w to v, so the arc (v, w) is symmetric in $\mathcal{C}(T)$, a contradiction. If, on the other hand, either the arc (v, w_{j-1}) or the arc (v, w_{j-2}) has colour 3, then there is a monochromatic path from v to u, forcing the arc (u, v) in $\mathcal{C}(T)$ to be symmetric, another contradiction. This forces three exceptions to the colour 3, a contradiction. We conclude the arc (w_1, v) is in T.

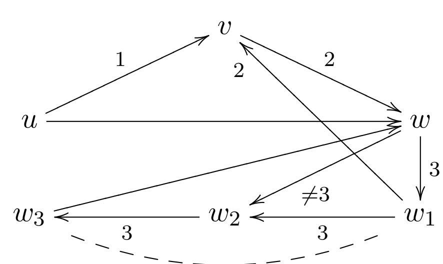
Notice the directed triangle (v, w, w_1) has colours 2 and 3. If (w_1, v) is colour 3, then there is a monochromatic path from w to v, making the arc (v, w) symmetric in $\mathcal{C}(T)$, a contradiction which forces the arc (w_1, v) to have colour 2 in T.

Going back to (w_k, w), where k is the minimum subindex such that (w_k, w) is in T, suppose first that $k = 2$.



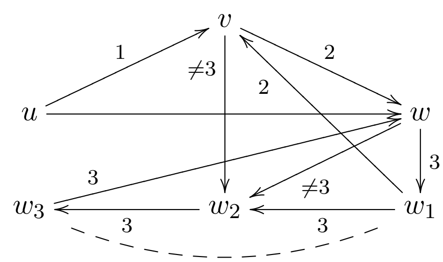
Regardless of the direction of the arc between v and w_2, there is a cyclic tournament on the vertices v, w, w_1 and w_2 which has two consecutive arcs of colour 2 and two consecutive arcs of colour 3, which is a contradiction.

If $k = 3$ we have the following (and note (w, w_2) cannot have colour 3 as this would shorten the monochromatic path from w to u).

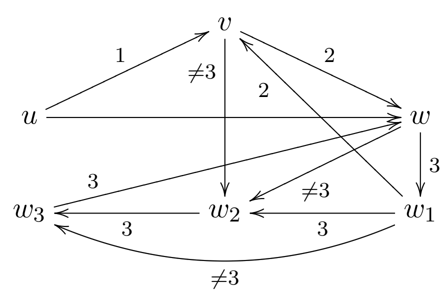


The vertices w, w_1, w_2, and w_3 form a cyclic tournament with three arcs coloured with 3, therefore with at most two exceptions which are not consecutive, all arcs between these vertices have colour 3. As noted above, (w, w_2) cannot have colour 3, so this forces (w_3, w) to have colour 3, otherwise there would be two consecutive arcs coloured differently.

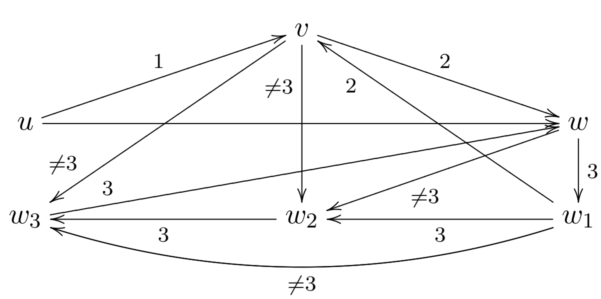
If $(w_2, v) \in A(T)$ then the vertices v, w, w_1 and w_2 form a directed C_4, and the tournament in these four vertices has two consecutive arcs of colour 2 and two consecutive arcs of colour 3, a contradiction which forces $(v, w_2) \in A(T)$. Note also that if (v, w_2) has colour 3, then there is a monochromatic path from v to u, which makes the arc (u, v) symmetric in $\mathcal{C}(T)$, a contradiction, so (v, w_2) has colour different from 3.



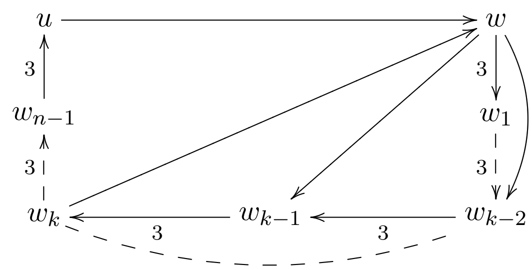
If $(w_3, w_1) \in A(T)$, then (w_1, v, w_2, w_3) is a directed C_4, and the tournament in these vertices has two consecutive arcs of colour 3, and two consecutive arcs of colour different from 3, which is a contradiction. Therefore $(w_1, w_3) \in A(T)$, and it cannot have colour 3 as this would shorten the monochromatic path between w and u.



The arc (w_3, v) cannot be in T, as (v, w, w_2, w_3) would form a directed C_4, and the tournament in these four vertices has two consecutive arcs of colour 3, and two consecutive arcs of colour different from 3, a contradiction. Therefore (v, w_3) is in T, and cannot have colour 3 as this would produce a monochromatic path from v to u, with which the arc (u, v) would be symmetric in $\mathcal{C}(T)$, a contradiction.

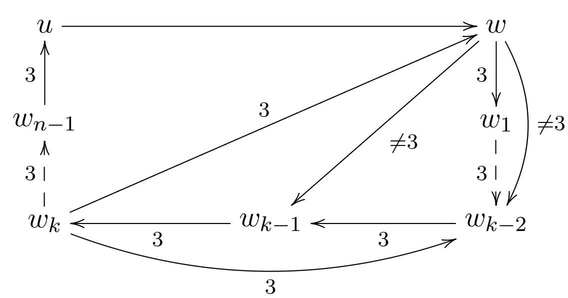


However (v, w_3, w, w_1) is a directed C_4 with two consecutive arcs of colour 3 and two consecutive arcs of colour different from 3, a contradiction. Therefore $k \geq 4$, so $k - 2 \geq 2$.



Regardless of the direction of the arc between w_k and w_{k-2}, there is a cyclic tournament on the four vertices w, w_{k-2}, w_{k-1} and w_k containing the Hamiltonian cycle $(w, w_{k-2}, w_{k-1}, w_k)$, which must therefore be 2-wamc.

Since (w_{k-2}, w_{k-1}) and (w_{k-1}, w_k) are consecutive and of colour 3, most arcs must be coloured 3. However if either (w, w_{k-2}) or (w, w_{k-1}) have colour 3, there is a shorter directed monochromatic path between w and u, a contradiction, hence these two arcs have colour different from 3 and all other arcs between these four vertices must be coloured with 3. This forces $(w_k, w_{k-2}) \in A(T)$, otherwise, again, there would be a shorter directed monochromatic path between w and u. That is, we have:



Now consider the arc between v and w_k. If $(w_k, v) \in A(T)$ then it cannot have colour 3, as this would produce a monochromatic path $w \to_{mp} v$. There is a cyclic tournament of order four with cycle (w_k, v, w, w_{k-1}). There are two arcs with colour different from 3, one arc with colour two, and one arc with colour 3, therefore the arcs coloured different from 3 must be coloured 2 and this must be the predominant colour, with 3 being the exception, however the two arcs with colour 3 are consecutive, a contradiction. We conclude $(v, w_k) \in A(D)$.

Observe (v, w_k) cannot have colour 3, as this would produce a monochromatic path $v \to_{mp} u$, a contradiction. Now there is a cyclic tournament of order four with cycle (v, w_k, w, w_1). Recall (w_1, v) has colour 2, as does (v, w), and these two arcs form a directed path. On the other hand, (w_k, w) and (w, w_1) are consecutive arcs of colour 3, a contradiction.

This completes the proof of the theorem. $\square$

Acknowledgements

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