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$2L$-CONVEX POLYOMINOES: DISCRETE TOMOGRAPHICAL ASPECTS

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Abstract. This paper uses the theoretical material developed in a previous article by the authors in order to reconstruct a subclass of $2L$-convex polyominoes. The main idea is to control the shape of these polyominoes by combining 4 types of geometries. Some modifications are made in the reconstruction algorithm of Chrobak and Dürr for HV-convex polyominoes in order to impose these geometries.

1. Introduction

The present paper uses the theoretical material developed in a previous article by the authors [14] in order to reconstruct a sub-class of $2L$-convex polyominoes. Indeed, $2L$-convex polyominoes are the first difficult class of polyominoes in terms of tomographical reconstruction in the hierarchy of kL-polyominoes and in this article we design an algorithm of reconstruction for a sub-class of $2L$-convex which is the first step in the whole comprehension of the hierarchy of kL-polyominoes.

One main problem in discrete tomography consists on the reconstruction of discrete objects according to their horizontal and vertical projection vectors. In order to restrain the number of solutions, we could add convexity constraints to these discrete objects. There are many notions of discrete convexity of polyominoes (namely HV-convex [2], Q-convex [3], L-convex polyominoes [6]) and each one leads to interesting studies. One natural notion of convexity on the discrete plane is the class of HV-convex polyominoes, that is, polyominoes with consecutive cells in rows and columns. Following the work of Del Lungo, Nivat, Barcucci, and Pinzani [2], we are able to use discrete tomography to reconstruct polyominoes that are HV-convex according to their horizontal and vertical projections. In addition to that, for any HV-convex polyomino P, every pair of cells of P can be reached using a path included in P with only two kinds of unit steps (such a path is called monotone). A polyomino is called kL-convex if for every two cells we find a monotone path with at most k changes in direction. Obviously a kL-convex polyomino is an HV-convex polyomino. Thus, the

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set of kL-convex polyominoes for $k \in \mathbb{N}$ forms a hierarchy of HV-convex polyominoes according to the number of changes in direction of monotone paths. This notion of L-convex polyominoes has been considered from several points of view. In [4], combinatorial aspects of L-convex polyominoes are analyzed, giving the enumeration according to the semi-perimeter and the area. In [5], we are given an algorithm that reconstructs an L-convex polyomino from the set of its maximal L-polyominoes. Similarly in [6], we are given another way to reconstruct an L-convex polyomino from the size of some special paths, called bordered L-paths.

In fact $2L$-convex polymoninoes are more geometrically complex and there was no result for their direct reconstruction. We could notice that Duchi, Rinaldi, and Schaeffer are able to enumerate this class in an interesting and technical article (see [9]), but the enumeration technique gives no idea for the tomographical reconstruction.

The first subclass that creates the link with $2L$-convex polyominoes is the class of HV-centered polyominoes. In [14], it is shown that if P is an HV-centered polyomino, then P is $2L$-convex. Note that the tomographical properties of this subclass have been studied in [7], and its reconstruction algorithm is well known.

The main contribution of this paper is an $O(m^3n^3)$-time algorithm for reconstructing a subclass of $2L$-convex polyominoes using the geometrical properties studied in [14], and the algorithm of Chrobak and Dürr (see [7]). In particular, we add well chosen clauses to the original construction of Chrobak and Dürr in order to control the $2L$-convexity using a 2SAT satisfaction problem.

This paper is divided into 5 sections. After basics on polyominoes, section 3 talks about the geometrical properties of a subclass of $2L$-convex polyominoes (see [14]). In section 4, the algorithm of Chrobak and Dürr for the reconstruction of the HV-convex polyominoes is given ([7]). Section 5 describes the reconstruction of different subclasses of $2L$-convex polyominoes starting by the classes γ and $\Im_{2L}^{0,0}$, and ending by the other classes using an horizontal reflexion called S_H.

2. Definition and notation

A planar discrete set is a finite subset of the integer lattice $\mathbb{N}^2$ defined up to translation. A discrete set can be represented either by a set of cells, i.e. unitary squares of the cartesian plane, or by a binary matrix, where the 1's determine the cells of the set (see Figure 1). A polyomino P is a finite connected set of adjacent cells (in the sequel, we use a 4-neighborhood, that is two cells are adjacent if they are sharing a segment), defined up to translation, in the cartesian plane. A polyomino is said to be *column-convex* (resp. *row-convex*) if every column (resp. row) is connected (see [8, 13]). Finally, a polyomino is said to be *convex* (or HV-convex) if it is both column and row-convex (see Figure 2).

FIGURE 1. A finite set of $\mathbb{N} \times \mathbb{N}$, and its representation in terms of a binary matrix and a set of cells.

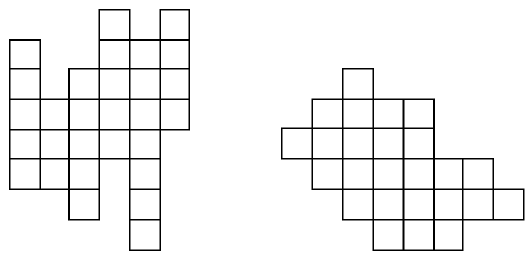

FIGURE 2. A column convex and an HV-convex polyomino.

To each discrete set S, represented as a $m \times n$ binary matrix, we associate two integer vectors $H = (h_1, \ldots, h_m)$ and $V = (v_1, \ldots, v_n)$ such that, for each $1 \leq i \leq m$, and $1 \leq j \leq n$, h_i and v_j are the number of cells of S (elements 1 of the matrix) which lie on row i and column j, respectively. The vectors H and V are called the *horizontal* and *vertical* projections of S, respectively (see Figure 3). Moreover if S has H and V as horizontal and vertical projections, respectively, then we say that S satisfies (H, V). Using the usual matrix notations, the element (i, j) denotes the entry in row i and column j.

For any two cells A and B in a polyomino, a *path* $\prod_{AB}$, from A to B, is a sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ of adjacent disjoint cells belonging in P, with $A = (i_1, j_1)$, and $B = (i_r, j_r)$. For each $1 \leq k \leq r - 1$, we say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form

- an *east* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$,
- a *north* step if $i_{k+1} = i_k - 1$ and $j_{k+1} = j_k$,
- a *west* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k - 1$,
- a *south* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

Finally, we define a path to be *monotone* if it is entirely made of only two of the four types of steps defined above.

Proposition 2.1 (Castiglione, Restivo [5]). *A polyomino P is HV-convex if and only if every pair of cells is connected by a monotone path.*

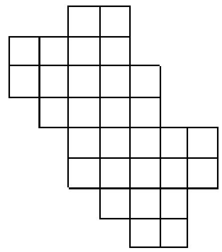

FIGURE 3. A polyomino P with $H = (2, 4, 5, 4, 5, 5, 3, 2)$ and $V = (2, 3, 6, 7, 6, 4, 2)$.

Let us consider a polyomino P. A path in P has a change of direction in the cell (i_k, j_k), for $2 \leq k \leq r-1$, if

$$i_k \neq i_{k-1} \Longleftrightarrow j_{k+1} \neq j_k.$$

Definition 2.2. *We call an HV-convex polyomino kL-convex if every pair of its cells can be connected by a monotone path with at most k changes of direction.*

In [5], a hierarchy is proposed on convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. For $k = 1$, we have the first level of hierarchy, i.e. the class of $1L$-convex polyominoes, also denoted as L-convex polyominoes for the typical shape of each path having at most one single change of direction. In the present study, we focus our attention on the next level of the hierarchy, i.e. the class of $2L$-convex polyominoes, whose tomographical properties turn out to be more interesting and substantially harder to investigate than those of L-convex polyominoes (see Figure 4).

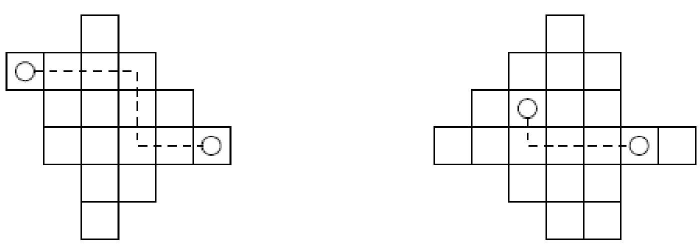

FIGURE 4. The convex polyomino on the left is $2L$-convex, while the one on the right is L-convex. For each polyomino, two cells and a monotone path connecting them are shown.

3. $2L$-convex polyominoes

Let H, V be two projection vectors and let P be an HV-convex polyomino that satisfies (H, V). By a classical argument P is contained in a rectangle R (called the minimal bounding box) where in this box no projection gives a zero. Let $[\min(S), \max(S)]$ (resp. $[\min(E), \max(E)]$, $[\min(N), \max(N)]$, $[\min(W), \max(W)]$) be the intersection of the boundary of P with the lower (resp. right, upper, left) side of R (see [2]). By abuse of notation, we call $\min(S)$ (resp. $\min(E)$, $\min(N)$, $\min(W)$) the cell at the position $(m, \min(S))$ [resp. $(\min(E), n)$, $(1, \min(N))$, $(\min(W), 1)$] and $\max(S)$ (resp. $\max(E)$, $\max(N)$, $\max(W)$) the cell at the position $(m, \max(S))$ (resp. $(\max(E), n)$, $(1, \max(N))$, $(\max(W), 1)$) (see Figure 5).

Definition 3.1. *The segment $[min(S), max(S)]$ is called the S-foot. Similarly, the segments $[\min(E), \max(E)]$, $[\min(N), \max(N)]$ and $[\min(W), \max(W)]$ are called the E-foot, N-foot and W-foot.*

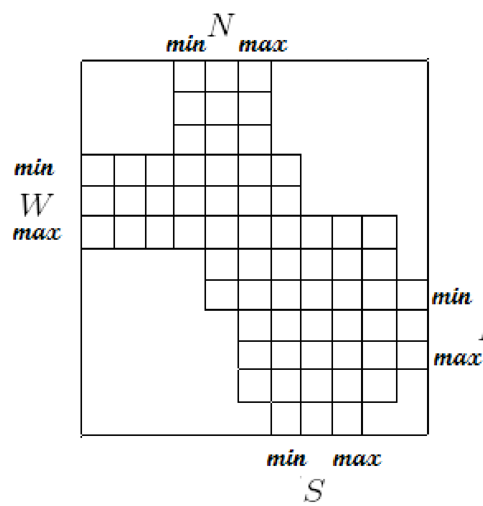

Figure 5. Min and max of the four feet in the rectangle R.

For a bounding rectangle R and for a given polyomino P, let us define the following sets:

$$
\begin{aligned}
WN &= \{(i, j) \in P \mid i < \min(W) \textit{ and } j < \min(N)\}, \\
SE &= \{(i, j) \in P \mid i > \max(E) \textit{ and } j > \max(S)\}, \\
NE &= \{(i, j) \in P \mid i < \min(E) \textit{ and } j > \max(N)\}, \\
WS &= \{(i, j) \in P \mid i > \max(W) \textit{ and } j < \min(S)\}.
\end{aligned}
$$

Let $\mathcal{C}$ (resp. $\mathcal{C}_{2L}$) be the class of HV-convex polyominoes (resp. $2L$-convex polyominoes). We have the following classes of polyominoes regarding the position of the non-intersecting feet.

$$\begin{aligned}
\Im^{0,0} &= \{P \in \mathcal{C} \mid \operatorname{card}(WN) = 0 \text{ and } \operatorname{card}(SE) = 0, \\
&\qquad \max(W) < \min(E) \text{ and } \max(N) < \min(S)\}, \\
\Im^{0,0}_{2L} &= \{P \in \mathcal{C}_{2L} \mid \operatorname{card}(WN) = 0 \text{ and } \operatorname{card}(SE) = 0, \\
&\qquad \max(W) < \min(E) \text{ and } \max(N) < \min(S)\}, \\
\Im'^{\,0,0} &= \{P \in \mathcal{C} \mid \operatorname{card}(NE) = 0 \text{ and } \operatorname{card}(SW) = 0, \\
&\qquad \max(S) < \min(N) \text{ and } \max(E) < \min(W)\}, \\
\Im'^{\,0,0}_{2L} &= \{P \in \mathcal{C}_{2L} \mid \operatorname{card}(NE) = 0 \text{ and } \operatorname{card}(SW) = 0, \\
&\qquad \max(S) < \min(N) \text{ and } \max(E) < \min(W)\}, \\
\gamma &= \{P \in \mathcal{C} \mid \max(N) < \min(S) \text{ and } \max(E) < \min(W)\}, \\
\gamma' &= \{P \in \mathcal{C} \mid \max(S) < \min(N) \text{ and } \max(W) < \min(E)\}.
\end{aligned}$$

Theorem 3.2 (Tawbe, Vuillon [14])**.** *Let P be an HV-convex polyomino in the class $\Im^{0,0}$. P is $2L$-convex if and only if one of the following holds:*

(1) *There exist L-paths from* $\max(N)$ *to* $\max(E)$ *and from* $\max(W)$ *to* $\max(S)$*,*

(2) *There exist L-paths from* $\min(N)$ *to* $\min(E)$ *and from* $\min(W)$ *to* $\min(S)$*,*

(3) *There exist L-paths from* $\min(N)$ *to* $\min(E)$ *and from* $\max(W)$ *to* $\max(S)$*, as well as a $2L$-path from* $\min(W)$ *to* $\max(E)$*,*

(4) *There exist L-paths from* $\max(N)$ *to* $\max(E)$ *and from* $\min(W)$ *to* $\min(S)$*, as well as a $2L$-path from* $\min(N)$ *to* $\max(S)$*.*

Corollary 3.3. *If P satisfies the conditions of Theorem 3.2, then P is in the class $\Im^{0,0}_{2L}$.*

The visualisation of the paths is shown below.

4. HV-Convex Polyominoes

Assume that H, V denote strictly positive row and column sum vectors. We also assume that $\sum_i h_i = \sum_j v_j$, since otherwise (H, V) does not have a realization.

The idea of Chrobak and Dürr (see [7]) for the control of the HV-convexity is in fact to impose convexity on the four corner regions outside of the polyomino.

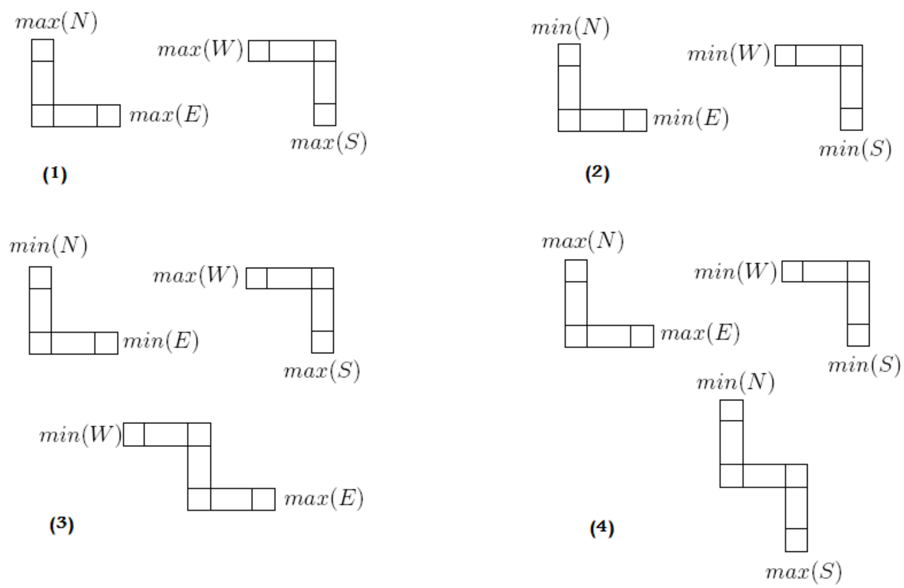

FIGURE 6. L-paths and $2L$-paths on (1): first geometry, (2): second geometry, (3): third geometry, (4): fourth geometry.

An object A is called an *upper-left corner region* if $(i+1, j) \in A$ or $(i, j+1) \in A$ implies $(i, j) \in A$. In an analogous fashion we can define the other corner regions. Let $\overline{P}$ be the complement of P. The definition of HV-convex polyominoes directly implies the following lemma.

Lemma 4.1. *P is an HV-convex polyomino if and only if $\overline{P} = A \cup B \cup C \cup D$, where A, B, C, D are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively) such that*

(i) $(i-1, j-1) \in A$ *implies* $(i, j) \notin D$, *and*
(ii) $(i-1, j+1) \in B$ *implies* $(i, j) \notin C$.

Given an HV-convex polyomino P and two row indices $1 \leq k, l \leq m$, P is said to be *anchored* at (k, l) if $(k, 1), (l, m) \in P$. The idea of Chrobak and Dürr is, given (H, V), to reconstruct a 2SAT expression $F_{k,l}(H, V)$ (a boolean expression in conjunctive normal form with at most two literals in each clause) with the property that $F_{k,l}(H, V)$ is satisfiable if and only if there is an HV-convex polyomino realization P of (H, V) that is anchored at (k, l). $F_{k,l}(H, V)$ consists of several sets of clauses, each set expressing a certain property: "Corners" (Cor), "Disjointness" (Dis), "Connectivity" (Con), "Anchors" (Anc), "Lower bound on column sums" (LBC) and "Upper bound on row sums" (UBR).

$$\mathrm{Cor} = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

The set of clauses Cor means that the corners are convex, that is for the corner A if the cell (i,j) belongs to A then cells $(i-1,j)$ and $(i,j-1)$ belong also to A. Similarly for corners B, C, and D.

$$\mathrm{Dis} = \bigwedge_{i,j} \left\{ X_{i,j} \Rightarrow \overline{Y}_{i,j} \mid \text{for symbols } X, Y \in \{A,B,C,D\}, X \neq Y \right\}$$

The set of clauses Dis means that all four corners are pairwise disjoint, that is $X \cap Y = \emptyset$ for $X, Y \in \{A,B,C,D\}$.

$$\mathrm{Con} = \bigwedge_{i,j} \left\{ A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \right\}$$

The set of clauses Con means that if the cell (i,j) belongs to A then the cell $(i+1,j+1)$ does not belong to D, and similarly if the cell (i,j) belongs to B then the cell $(i+1,j-1)$ does not belong to C.

$$\mathrm{Anc} = \left\{ \overline{A}_{k,1} \wedge \overline{B}_{k,1} \wedge \overline{C}_{k,1} \wedge \overline{D}_{k,1} \wedge \overline{A}_{l,n} \wedge \overline{B}_{l,n} \wedge \overline{C}_{l,n} \wedge \overline{D}_{l,n} \right\}$$

The set of clauses Anc means that we fix two cells on the west and east feet of the polyomino P, for $k,l = 1,\ldots,m$; the first one at the position $(k,1)$ and the second one at the position (l,n).

$$\mathrm{LBC} = \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} & A_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \\ B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} & B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_{j} \left\{ \overline{C}_{v_j,j} \quad \overline{D}_{v_j,j} \right\}$$

The set of clauses LBC implies that for each column j, we have that $\sum_i P_{i,j} \geq v_j$.

$$\mathrm{UBR} = \bigwedge_{j} \left\{ \begin{array}{ll} \wedge_{i \leq min\{k,l\}} \overline{A}_{i,j} \Rightarrow B_{i,j+h_i} & \wedge_{k \leq i \leq l} \overline{C}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{l \leq i \leq k} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} & \wedge_{max\{k,l\} \leq i} \overline{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

The set of clauses UBR implies that for each row i, we have that $\sum_j P_{i,j} \leq h_i$.

Define $F_{k,l}(H,V) = \mathrm{Cor} \wedge \mathrm{Dis} \wedge \mathrm{Con} \wedge \mathrm{Anc} \wedge \mathrm{LBC} \wedge \mathrm{UBR}$. All literals with indices outside the set $\{1,\ldots,m\} \times \{1,\ldots,n\}$ are assumed to have value 1.

Algorithm 1
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.L.O.G. assume: $\forall i : h_i \in [1, n],\ \forall j : v_j \in [1, m],\ \sum_i h_i = \sum_j v_j$ and $m \leq n$.
for $k, l = 1, \ldots, m$ **do begin**
  if $F_{k,l}(H, V)$ is satisfiable,
  then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
Output: "failure".

The following theorem allows us to link the existence of HV-convex solution and the evaluation of $F_{k,l}(H, V)$. The crucial part of this algorithm comes from the constraints on the two sets of clauses LBC and UBR.

Theorem 4.2 (Chrobak, Dürr)**.** *$F_{k,l}(H, V)$ is satisfiable if and only if (H, V) has a realization P that is an HV-convex polyomino anchored at (k, l).*

Each formula $F_{k,l}(H, V)$ has size $O(mn)$ and can be computed in time $O(mn)$. Since 2SAT can be solved in linear time (see [1, 10]), Chrobak and Dürr give the following result.

Theorem 4.3 (Chrobak, Dürr)**.** *Algorithm 1 solves the reconstruction problem for HV-convex polyominoes in time $O(mn \min(m^2, n^2))$.*

5. Reconstruction of $2L$-convex polyominoes in γ and $\Im_{2L}^{0,0}$

The present section uses the theoretical material developed in the above sections in order to reconstruct $2L$-convex polyominoes in γ and $\Im_{2L}^{0,0}$. Some modifications are made to the reconstruction algorithm of Chrobak and Dürr for HV-convex polyominoes in order to impose our geometries. All the clauses that have been added and the modifications of the original algorithm are explained in the proofs of each subclass. Finally, by defining a horizontal symmetry S_H, we show how to reconstruct P in the class γ' and $\Im'^{0,0}_{2L}$.

5.1. Clauses for the class γ.

In this section, we add the clause Pos and we modify the clause Anc of the original Chrobak and Dürr algorithm in order to reconstruct, if possible, all polyominoes in the subclass γ.

$$\mathrm{Pos} = \left\{ \ A_{(\max(E),1)} \wedge C_{(m,\max(N))} \ \right\}$$

$$\mathrm{Cor} = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$\mathrm{Dis} = \bigwedge_{i,j} \left\{ \ X_{i,j} \Rightarrow \overline{Y}_{i,j} \mid \text{for symbols } X, Y \in \{A, B, C, D\}, X \neq Y \ \right\}$$

$$\mathrm{Con} = \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} & B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \end{array} \right\}$$

$$\mathrm{Anc} = \left\{ \begin{array}{l} \overline{A}_{\min(W),1} \wedge \overline{A}_{\min(E),n} \wedge \overline{B}_{\min(W),1} \wedge \overline{B}_{\min(E),n} \wedge \\ \overline{C}_{\min(W),1} \wedge \overline{C}_{\min(E),n} \wedge \overline{D}_{\min(W),1} \wedge \overline{D}_{\min(E),n} \wedge \\ \overline{A}_{1,\min(N)} \wedge \overline{A}_{m,\min(S)} \wedge \overline{B}_{1,\min(N)} \wedge \overline{B}_{m,\min(S)} \wedge \\ \overline{C}_{1,\min(N)} \wedge \overline{C}_{m,\min(S)} \wedge \overline{D}_{1,\min(N)} \wedge \overline{D}_{m,\min(S)} \wedge \\ \overline{A}_{\max(W),1} \wedge \overline{A}_{\max(E),n} \wedge \overline{B}_{\max(W),1} \wedge \overline{B}_{\max(E),n} \wedge \\ \overline{C}_{\max(W),1} \wedge \overline{C}_{\max(E),n} \wedge \overline{D}_{\max(W),1} \wedge \overline{D}_{\max(E),n} \wedge \\ \overline{A}_{1,\max(N)} \wedge \overline{A}_{m,\max(S)} \wedge \overline{B}_{1,\max(N)} \wedge \overline{B}_{m,\max(S)} \wedge \\ \overline{C}_{1,\max(N)} \wedge \overline{C}_{m,\max(S)} \wedge \overline{D}_{1,\max(N)} \wedge \overline{D}_{m,\max(S)} \end{array} \right\}$$

$$\mathrm{LBC} = \bigwedge_{i} \left\{ \begin{array}{l} \wedge_{j<\min(N)} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(N)\le j\le\max(N)} C_{i+v_j,j} \Rightarrow \overline{A}_{i,j} \\ \wedge_{\max(N)<j<\min(S)} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(S)\le j\le\max(S)} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{j>\max(S)} B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_{j} \left\{ \begin{array}{ll} \overline{C}_{v_j,j} & \overline{D}_{v_j,j} \end{array} \right\}$$

$$\mathrm{UBR} = \bigwedge_{j} \left\{ \begin{array}{l} \wedge_{i<\min(E)} \overline{A}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{\min(E)\le i\le\max(E)} \overline{B}_{i,j+h_i} \Rightarrow A_{i,j} \\ \wedge_{\max(E)<i<\min(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{\min(W)\le i\le\max(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{i>\max(W)} \overline{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

Define $\gamma(H,V) = \mathrm{Pos} \wedge \mathrm{Cor} \wedge \mathrm{Dis} \wedge \mathrm{Con} \wedge \mathrm{Anc} \wedge \mathrm{LBC} \wedge \mathrm{UBR}$. All literals with indices outside the set $\{1,\dots,m\} \times \{1,\dots,n\}$ are assumed to have value 1.

Proposition 5.1. *If P is an HV-convex polyomino in γ, then P is a $2L$-convex polyomino.*

Proof. The proof is straightforward by using the L-paths between each pair of feet (see [14]). □

Algorithm 2
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.L.O.G. assume: $\forall i : h_i \in [1, n],\ \forall j : v_j \in [1, m],\ \sum_i h_i = \sum_j v_j$.
for $\min(W), \min(E) = 1, \ldots, m$ and
$\min(S), \min(N) = 1, \ldots, n$ **do begin**
if $\gamma(H, V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
Output: "failure".

Proof of Algorithm 2. We make the following modifications of the original algorithm of Chrobak and Durr [7] in order to add the geometrical constraints of the class γ. The set Anc gives the feet of suitable size by fixing 8 cells outside the corners A, B, C, D. Thus these cells of the extremities of the feet are in the interior of the polyomino. The set Pos imposes the constraint of the relative positions of feet in the class γ. In particular the cell $A_{(\max(E),1)}$ implies that $\min(W) > \max(E)$ and the cell $C_{(m,\max(N))}$ implies that $\max(N) < \min(S)$ (see Figure 7) . Using the combination of the whole set of clauses, if $\gamma(H, V)$ is satisfiable then we are able to reconstruct an HV-convex with the constraints of the class γ. By Proposition 2 this HV-convex polyomino must be also $2L$-convex. $\square$

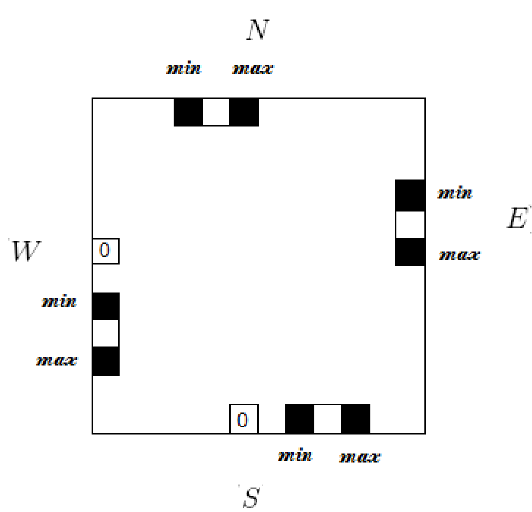

FIGURE 7. Relative position and anchors of the feet in the class γ

5.2. **Clauses for the class $\Im_{2L}^{0,0}$.** We code the four geometries that characterize all $2L$-convex polyominoes in the class $\Im_{2L}^{0,0}$ using a 2SAT formula, in order to reconstruct them.

$$\text{Pos} = \left\{ \; C_{(\min(E),1)} \wedge C_{(m,\max(N))} \wedge A_{1,1} \wedge D_{m,n} \; \right\}$$

$$\text{Cor} = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$\text{Dis} = \bigwedge_{i,j} \left\{ \; X_{i,j} \Rightarrow \overline{Y}_{i,j} \mid \text{for symbols } X, Y \in \{A, B, C, D\}, X \neq Y \; \right\}$$

$$\text{Con} = \bigwedge_{i,j} \left\{ \; A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \; \right\}$$

$$\text{Anc} = \left\{ \begin{array}{l} \overline{A}_{\min(W),1} \wedge \overline{A}_{\min(E),n} \wedge \overline{B}_{\min(W),1} \wedge \overline{B}_{\min(E),n} \wedge \\ \overline{C}_{\min(W),1} \wedge \overline{C}_{\min(E),n} \wedge \overline{D}_{\min(W),1} \wedge \overline{D}_{\min(E),n} \wedge \\ \overline{A}_{1,\min(N)} \wedge \overline{A}_{m,\min(S)} \wedge \overline{B}_{1,\min(N)} \wedge \overline{B}_{m,\min(S)} \wedge \\ \overline{C}_{1,\min(N)} \wedge \overline{C}_{m,\min(S)} \wedge \overline{D}_{1,\min(N)} \wedge \overline{D}_{m,\min(S)} \wedge \\ \overline{A}_{\max(W),1} \wedge \overline{A}_{\max(E),n} \wedge \overline{B}_{\max(W),1} \wedge \overline{B}_{\max(E),n} \wedge \\ \overline{C}_{\max(W),1} \wedge \overline{C}_{\max(E),n} \wedge \overline{D}_{\max(W),1} \wedge \overline{D}_{\max(E),n} \wedge \\ \overline{A}_{1,\max(N)} \wedge \overline{A}_{m,\max(S)} \wedge \overline{B}_{1,\max(N)} \wedge \overline{B}_{m,\max(S)} \wedge \\ \overline{C}_{1,\max(N)} \wedge \overline{C}_{m,\max(S)} \wedge \overline{D}_{1,\max(N)} \wedge \overline{D}_{m,\max(S)} \end{array} \right\}$$

$$\text{LBC} = \bigwedge_{i} \left\{ \begin{array}{l} \wedge_{j<\min(N)} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(N)\leq j\leq\max(N)} C_{i+v_j,j} \Rightarrow \overline{A}_{i,j} \\ \wedge_{\max(N)<j<\min(S)} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{\min(S)\leq j\leq\max(S)} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} \\ \wedge_{j>\max(S)} B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_{j} \left\{ \; \overline{C}_{v_j,j} \quad \overline{D}_{v_j,j} \; \right\}$$

$$\text{UBR} = \bigwedge_{j} \left\{ \begin{array}{l} \wedge_{i<\min(E)} \overline{A}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{\min(E)\leq i\leq\max(E)} \overline{B}_{i,j+h_i} \Rightarrow A_{i,j} \\ \wedge_{\max(E)<i<\min(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{\min(W)\leq i\leq\max(W)} \overline{A}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{i>\max(W)} \overline{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

$$\text{REC} = \left\{ \; A_{\min(W)-1,\min(N)-1} \wedge D_{\max(E)+1,\max(S)+1} \; \right\}$$

$$\mathrm{GEO1} = \left\{ \begin{array}{l} \overline{A}_{\max(W),\max(S)} \wedge \overline{B}_{\max(W),\max(S)} \\ \wedge \overline{C}_{\max(W),\max(S)} \wedge \overline{D}_{\max(W),\max(S)} \\ \wedge \overline{A}_{\max(E),\max(N)} \wedge \overline{B}_{\max(E),\max(N)} \\ \wedge \overline{C}_{\max(E),\max(N)} \wedge \overline{D}_{\max(E),\max(N)} \end{array} \right\}$$

$$\mathrm{GEO2} = \left\{ \begin{array}{l} \overline{A}_{\min(W),\min(S)} \wedge \overline{B}_{\min(W),\min(S)} \\ \wedge \overline{C}_{\min(W),\min(S)} \wedge \overline{D}_{\min(W),\min(S)} \\ \wedge \overline{A}_{\min(E),\min(N)} \wedge \overline{B}_{\min(E),\min(N)} \\ \wedge \overline{C}_{\min(E),\min(N)} \wedge \overline{D}_{\min(E),\min(N)} \end{array} \right\}$$

$$\mathrm{LGEO3} = \left\{ \begin{array}{l} \overline{A}_{\max(W),\max(S)} \wedge \overline{B}_{\max(W),\max(S)} \\ \wedge \overline{C}_{\max(W),\max(S)} \wedge \overline{D}_{\max(W),\max(S)} \\ \wedge \overline{A}_{\min(E),\min(N)} \wedge \overline{B}_{\min(E),\min(N)} \\ \wedge \overline{C}_{\min(E),\min(N)} \wedge \overline{D}_{\min(E),\min(N)} \end{array} \right\}$$

$$\mathrm{2LGEO3} = \left\{ \begin{array}{l} B_{\min(W),j} \Rightarrow \overline{C}_{\max(E),j-1} \wedge_{j>1} \\ \wedge \overline{X}_{\min(W),\max(N)+1} \\ \wedge \overline{X}_{\max(E),\min(S)-1}, \forall X \in \{A, B, C, D\} \end{array} \right\}$$

$$\mathrm{LGEO4} = \left\{ \begin{array}{l} \overline{A}_{\min(W),\min(S)} \wedge \overline{B}_{\min(W),\min(S)} \\ \wedge \overline{C}_{\min(W),\min(S)} \wedge \overline{D}_{\min(W),\min(S)} \\ \wedge \overline{A}_{\max(E),\max(N)} \wedge \overline{B}_{\max(E),\max(N)} \\ \wedge \overline{C}_{\max(E),\max(N)} \wedge \overline{D}_{\max(E),\max(N)} \end{array} \right\}$$

$$\mathrm{2LGEO4} = \left\{ \begin{array}{l} C_{i,\min(N)} \Rightarrow \overline{B}_{i-1,\max(S)} \wedge_{i>1} \\ \overline{X}_{\max(W)+1,\min(N)} \\ \wedge \overline{X}_{\min(E)-1,\max(S)}, \forall X \in \{A, B, C, D\} \end{array} \right\}$$

In order to reconstruct and to obtain the uniqueness of all $2L$-convex polyominoes in the class $\Im^{0,0}_{2L}$, we use all the combinations of the whole set of clauses that impose the union (or the sub-union) of the 4 geometries starting from all geometries and leading to each single one (see [14]).

$$\begin{aligned}
\Im^{0,0}_{2L,geo1,geo2,geo3,geo4}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO1}\wedge\mathrm{GEO2}\wedge\mathrm{LGEO3}\wedge\mathrm{LGEO4},\\
\Im^{0,0}_{2L,geo2,geo4}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO2}\wedge\mathrm{LGEO4},\\
\Im^{0,0}_{2L,geo2,geo3}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO2}\wedge\mathrm{LGEO3},\\
\Im^{0,0}_{2L,geo1,geo4}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO1}\wedge\mathrm{LGEO4},\\
\Im^{0,0}_{2L,geo1,geo3}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO1}\wedge\mathrm{LGEO3},\\
\Im^{0,0}_{2L,geo4}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{LGEO4}\wedge\mathrm{2LGEO4},\\
\Im^{0,0}_{2L,geo3}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{LGEO3}\wedge\mathrm{2LGEO3},\\
\Im^{0,0}_{2L,geo2}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO2},\\
\Im^{0,0}_{2L,geo1}(H,V) &= \mathrm{Pos}\wedge\mathrm{Cor}\wedge\mathrm{Dis}\wedge\mathrm{Con}\wedge\mathrm{Anc}\wedge\mathrm{LBC}\wedge\mathrm{UBR}\wedge\\
&\quad \mathrm{REC}\wedge\mathrm{GEO1}\,.
\end{aligned}$$

Algorithm 3
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.L.O.G. assume: $\forall i : h_i \in [1,n],\ \forall j : v_j \in [1,m],\ \sum_i h_i = \sum_j v_j$.
for $\min(W), \min(E) = 1, \ldots, m$ and
    $\min(N), \min(S) = 1, \ldots, n$ **do begin**
  if $\Im^{0,0}_{2L,geo1,geo2,geo3,geo4}(H,V)$ or $\Im^{0,0}_{2L,geo2,geo4}(H,V)$ or $\cdots$
    or $\Im^{0,0}_{2L,geo1}(H,V)$ is satisfiable,
  then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
Output: "failure".

Proof of Algorithm 3. By Theorem 3.2, all $2L$-convex polyominoes of the class $\Im_{2L}^{0,0}$ are given by combining the 4 geometries. Thus we combine all geometries using a suitable set of clauses in order to try to reconstruct a polyomino in the class $\Im_{2L}^{0,0}$. We make the following modifications of the original algorithm of Chrobak and Durr (see [7]) in order to add the geometrical constraints of the class $\Im_{2L}^{0,0}$. The set Pos imposes the constraint of the relative positions of feet in $\Im_{2L}^{0,0}$ (see Figure 8). The set GEO1 implies that we put a cell in the interior of the polyomino at the position $(\max(W), \max(S))$ (resp. $(\max(E), \max(N))$) and then by convexity an L-path between $\max(W)$ and $\max(S)$ (resp. $\max(N)$ and $\max(E)$). Thus we have exactly the definition of the first geometry. The set GEO2 (resp. LGEO3, LGEO4) gives the L-paths of the second (resp. third and fourth) geometry. The set 2LGEO3 (resp. 2LGEO4) controls the $2L$-paths of the third (resp. fourth) geometry (see Figure 9).

In particular, 2LGEO3 gives the $2L$-path between $\min(W)$ and $\max(E)$ by using the clause $B_{\min(W),j} \Rightarrow \overline{C}_{\max(E),j-1}$. This clause says that if the cell $(\min(W), j)$ is in the corner B, then the cell $(\max(E), j-1)$ is in the interior of the polyomino. By contraposition, we have the clause $\wedge_j C_{\max(E),j-1} \Rightarrow \overline{B}_{\min(W),j}$, meaning that $(\max(E), j-1)$ is in the corner C, while the cell $(\min(W), j)$ is in the interior of the polyomino. We would like to have the $2L$-path between $\min(W)$ and $\max(E)$, and thus we add two limit cases: $\overline{X}_{\min(W),\max(N)+1} \wedge \overline{X}_{\max(E),\min(S)-1}, \forall X \in \{A, B, C, D\}$ which impose that the cells $(\min(W), \max(N)+1)$ and $(\max(E), \min(S)-1)$ are in the interior of the polyomino (see Figure 10). Thus we have a $2L$-path between $\min(W)$ and $\max(E)$. The same technique is applied for the clauses in 2LGEO4. We remark that the clauses in 2LGEO3 (resp. 2LGEO4) are used only to determine the third (resp. the fourth) geometry because all other geometry combinations give L-paths between the feet and thus there is no reason to satisfy 2LGEO3 and 2LGEO4.

Using the conjunction of the whole set of clauses, if one of $\Im_{2L,geo1,geo2,geo3,geo4}^{0,0}(H,V), \Im_{2L,geo2,geo4}^{0,0}(H,V), \ldots, \Im_{2L,geo1}^{0,0}(H,V)$ is satisfiable, then we are able to reconstruct an HV-convex polyomino with the constraints of the class $\Im_{2L}^{0,0}$. $\square$

In order to compute the complexity of this algorthim, one can see that the possible positions of the four feet is $(n-h_m+1)(n-h_1+1)(m-v_1+1)(m-v_n+1) \leq n^2m^2$ (see [2]). And so by imposing the paths in the interior of the polyominoes using the algortihm of Chrobak and Dürr, we obtain the following result.

Theorem 5.2. *Algorithms 2 and 3 solve the reconstruction problem for $2L$-convex polyominoes in γ or $\Im_{2L}^{0,0}$ in time $O(n^3m^3)$.*

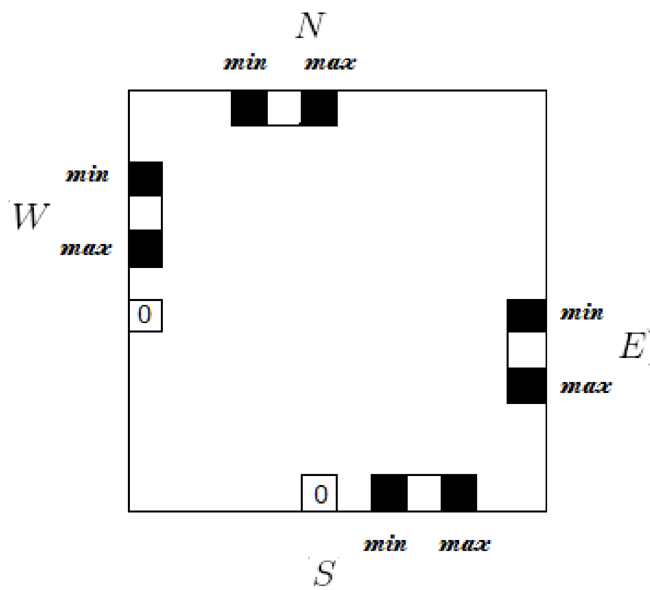

FIGURE 8. Relative position and anchors of the feet in the class $\Im_{2L}^{0,0}$

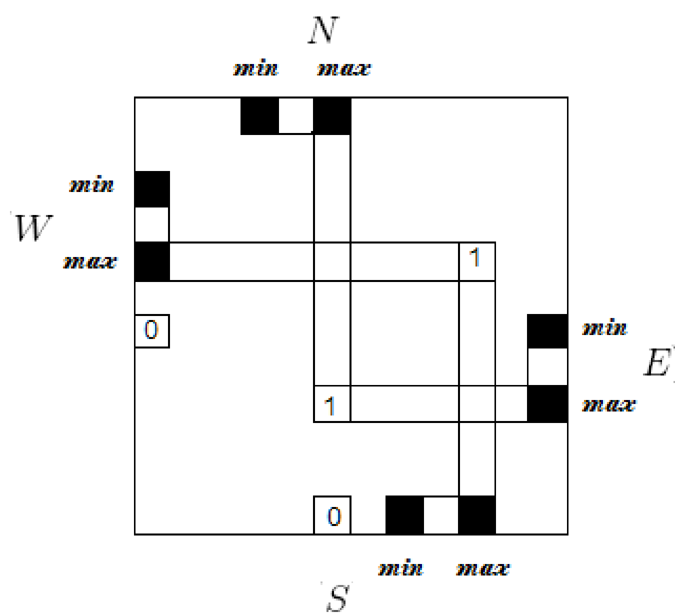

FIGURE 9. First geometry.

5.3. **Reconstruction of the classes γ' and $\Im_{2L}^{\prime 0,0}$ using the horizontal reflexion S_H.** We are given two integer vectors $H = (h_1, \ldots, h_m)$ and

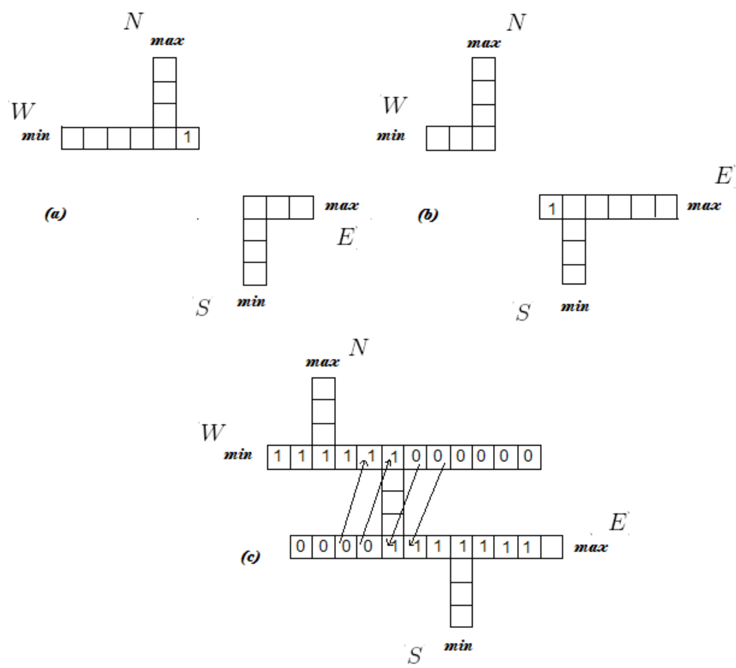

FIGURE 10. 2LGEO3: (a) and (b) are the limit cases for the third geometry in the class $\Im_{2L}^{0,0}$

$V = (v_1, \ldots, v_n)$. To reconstruct a polyomino P in the class γ' (resp. $\Im_{2L}^{\prime 0,0}$), one can see that the horizontal reflexion $S_H : (i,j) \longrightarrow (m-i+1, j)$, for every $i, j \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ sends the projection vectors (H, V) to $(\tilde{H}, V)$, where $\tilde{H} = (h_m, \ldots, h_1)$. Now from the two vectors of projections $(\tilde{H}, V)$, one can reconstruct the polyomino P in the class γ (resp. $\Im_{2L}^{0,0}$) and then by the horizontal reflexion S_H, we reconstruct P in the class γ' (resp. $\Im_{2L}^{\prime 0,0}$).

N, S	W, E
$S \longrightarrow N$	$W \longrightarrow W$
$N \longrightarrow S$	$E \longrightarrow E$
	$\leq \Longleftrightarrow \geq$
$\min \longrightarrow \min$	$\min \longrightarrow \max$
$\max \longrightarrow \max$	$\max \longrightarrow \min$

References

1. B. Aspvall, M. F. Plass, and R. E. Tarjan, *A linear-time algorithm for testing the truth of certain quantified boolean formulas*, Information Processing Letters **8** (1979), no. 3, 121–123.
2. E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani, *Reconstructing convex polyominoes from horizontal and vertical projections*, Theoretical Computer Science **155** (1996).
3. S. Brunetti and A. Daurat, *Random generation of Q-convex sets*, Theoretical Computer Science **347** (2005), 393–414.
4. G. Castiglione, A. Frosini, E. Munarini, A. Restivo, and S. Rinaldi, *Combinatorial aspects of L-convex polyominoes*, European Journal of Combinatorics **28** (2007), 1724–1741.
5. G. Castiglione and A. Restivo, *Reconstruction of L-convex polyominoes*, Electronic Notes in Discrete Mathematics, vol. 12, Elsevier Science, 2003.
6. G. Castiglione, A. Restivo, and R. Vaglica, *A reconstruction algorithm for L-convex polyominoes*, Theoretical Computer Science **356** (2006), 58–72, available online at www.sciencedirect.com.
7. M. Chrobak and C. Dürr, *Reconstructing hv-convex polyominoes from orthogonal projections*, Information Processing Letters **69** (1999), 283–289.
8. M. -P. Delest and G. Viennot, *Algebraic language and polyominoes [sic] enumerations*, Theoretical Computer Science **34** (1984), 169–206.
9. E. Duchi, S. Rinaldi, and G. Schaeffer, *The number of Z-convex polyominoes*, Advances in applied mathematics **40** (2001), no. 1, 54–72.
10. S. Even, A. Itai, and A. Shamir, *On the complexity of timetable and multicommodity flow problems*, SIAM Journal on Computing **5** (1976), no. 4, 691–703.
11. G. T. Herman and A. Kuba (Eds.), *Discrete tomography: Foundations, algorithms and applications*, Birkhäuser, Boston, Cambridge, MA, 1999.
12. ———, *Advances in discrete tomography and its applications*, Birkhäuser, Boston, Cambridge, MA, 2007.
13. D. Klarner, *Some results concerning polyominoes*, Fibonacci Quart. **3** (1965), no. 1, 9–20.
14. K. Tawbe and L. Vuillon, *2L-convex polyominoes: geometrical aspects*, Contributions to Discrete Mathematics **6** (2011), no. 1.

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