Contributions to Discrete Mathematics

Volume 8, Number 1, Pages 31–37
ISSN 1715-0868

QUASI-HERMITIAN VARIETIES IN $PG(r, q^2)$, q EVEN

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Abstract. In this paper a new example of quasi–Hermitian variety $\mathcal{V}$ in $PG(r, q^2)$ is provided, where q is an odd power of 2. In higher-dimensional spaces, $\mathcal{V}$ can be viewed as a generalization of the Buekenhout-Tits unital in the desarguesian projective plane; see [9].

1. Introduction

In the r-dimensional projective space $PG(r, q^2)$ over a finite field $GF(q^2)$ of order q^2, a *quasi–Hermitian variety* is a set of points which has the same intersection numbers with hyperplanes as a (non–degenerate) Hermitian variety does. Therefore quasi-Hermitian varieties are two-character sets with respect to hyperplanes, where the characters, that is the intersection numbers, are

$$\frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1},$$

and

$$\frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} + (-1)^{r-1}q^{r-1}.$$

Quasi–Hermitian varieties other than Hermitian varieties are known to exist; see [1] and [6]. The interest for quasi-Hermitian varieties arose from coding theory. Delsarte [8] proved indeed that a two–character set gives rise to a projective linear two weights code and a strongly regular graph. Recent papers on this subject are [4, 5, 7].

We construct a new family of non–trivial quasi–Hermitian varieties $\mathcal{V}$ in $PG(r, q^2)$ with $q = 2^e$ and e odd, using a procedure similar to that developed in [1]. The essential idea is to keep a Hermitian variety $\mathcal{H} = \mathcal{H}(r, q^2)$ invariant but modify the ambient space $\mathrm{PG}(r, q^2)$ by a birational transformation so that $\mathcal{H}$ becomes a quasi–Hermitian variety of the r-dimensional projective space $PG(r, q^2)$ represented by a (non-standard) model Π of $PG(r, q^2)$ where

Received by the editors July 1, 2011, and in revised form July 5, 2012.
2010 *Mathematics Subject Classification.* 51E30, 05B25.
Key words and phrases. Hermitian variety; hypersurface.

(i) points of Π are those of $\mathrm{PG}(r, q^2)$;
(ii) hyperplanes of Π are certain hyperplanes and hypersurfaces of $\mathrm{PG}(r, q^2)$.

We give the equations of these hypersurfaces later in Section 2 where we also extend some results obtained in [2] from $r = 2$ to any $r > 2$. Interestingly, some planar sections of $\mathcal{V}$ are Buekenhout-Tits unitals, in particular $\mathcal{V}$ is a Buekenhout-Tits unital for $r = 2$.

For generalities on Hermitian varieties and unitals in projective spaces, the reader is referred to [14, 11, 13, 10, 3]. Basic facts on rational transformations of projective spaces are found in [12, Section 3.3].

2. A non-standard model of $\mathrm{PG}(r, q^2)$

Fix a projective frame in $\mathrm{PG}(r, q^2)$, where q is an odd power of 2. Let $(X_0, X_1, \ldots, X_r)$ denote homogeneous coordinates, and consider the affine plane $\mathrm{AG}(r, q^2)$ whose infinite hyperplane Σ_∞ has equation $X_0 = 0$. Then $\mathrm{AG}(r, q^2)$ has affine coordinates $(x_1, x_2, \ldots, x_r)$ where $x_i = X_i/X_0$ for $i \in \{1, \ldots, r\}$.

Take $\varepsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ such that $\varepsilon^2 + \varepsilon + \delta = 0$ for some $\delta \in \mathrm{GF}(q) \setminus \{1\}$ with $\mathrm{Tr}\,(\delta) = 1$. Here, Tr stands for the trace function $\mathrm{GF}(q) \to \mathrm{GF}(2)$. Then $\varepsilon^{2q} + \varepsilon^q + \delta = 0$. Therefore, $(\varepsilon^q + \varepsilon)^2 + (\varepsilon^q + \varepsilon) = 0$, whence $\varepsilon^q + \varepsilon + 1 = 0$. Moreover, if $q = 2^e$, with e an odd integer, then

$$\sigma : x \mapsto x^{2^{(e+1)/2}}$$

is an automorphism of $\mathrm{GF}(q)$. Set

$$\Delta_\varepsilon(x) = \varepsilon^{\sigma+2} x^{q(\sigma+2)} + (\varepsilon^\sigma + \varepsilon^{\sigma+2}) x^{q\sigma+2} + x^\sigma + (1 + \varepsilon) x^2.$$

For any $m = (m_1, \ldots, m_{r-1}, d) \in GF(q^2)^r$, let $\mathcal{D}(m)$ denote the algebraic hypersurface

$$x_r = \Delta_\varepsilon(x_1) + \cdots + \Delta_\varepsilon(x_{r-1}) + m_1 x_1 + \cdots + m_{r-1} x_{r-1} + d. \tag{2.1}$$

Consider the incidence structure $\Pi_\varepsilon = (\mathcal{P}, \Sigma)$ whose points are the points of $\mathrm{AG}(r, q^2)$ and whose hyperplanes are the hyperplanes through the point at infinity $P_\infty(0, 0, \ldots, 0, 1)$ together with the hypersurfaces $\mathcal{D}(m)$, where m ranges over $\mathrm{GF}(q^2)^r$.

Lemma 2.1. *The incidence structure $\Pi_\varepsilon = (\mathcal{P}, \Sigma)$ is an affine space isomorphic to $\mathrm{AG}(r, q^2)$.*

Proof. The birational transformation φ given by

$$\varphi : (x_1, \ldots, x_{r-1}, x_r) \mapsto (x_1, \ldots, x_{r-1}, x_r + \Delta_\varepsilon(x_1) + \cdots + \Delta_\varepsilon(x_{r-1})), \tag{2.2}$$

transforms the hyperplanes through $P_\infty(0, 0, \ldots, 0, 1)$ into themselves, whereas the hyperplane of equation $x_r = m_1 x_1 + \cdots + m_{r-1} x_{r-1} + d$ is mapped into the hypersurface $\mathcal{D}(m)$. Therefore, φ determines an isomorphism

$$\Pi_\varepsilon \simeq \mathrm{AG}(r, q^2),$$

and the assertion is proven. □

Completing Π_ε with its points at infinity in the usual way gives a projective space isomorphic to $\mathrm{PG}(r, q^2)$.

3. Main result

Let $\mathcal{H}$ be the Hermitian variety of $PG(r, q^2)$. $\mathcal{H}$ is assumed to be in an affine canonical form

$$x_r^q + x_r = x_1^{q+1} + \cdots + x_{r-1}^{q+1}. \tag{3.1}$$

The set of the infinity points of $\mathcal{H}$ is

$$\mathcal{F} = \{(0, x_1, \ldots, x_r) \mid x_1^{q+1} + \cdots + x_{r-1}^{q+1} = 0\} \tag{3.2}$$

and it can be viewed as a Hermitian cone of $PG(r-1, q^2)$ projecting a Hermitian variety of $PG(r-2, q^2)$. Set

$$\Gamma_\varepsilon(x) = [x + (x^q + x)\varepsilon]^{\sigma+2} + (x^q + x)^\sigma + (x^{2q} + x^2)\varepsilon + x^{q+1} + x^2.$$

Theorem 3.1. *The affine algebraic variety of equation*

$$x_r^q + x_r = \Gamma_\varepsilon(x_1) + \cdots + \Gamma_\varepsilon(x_{r-1}), \tag{3.3}$$

together with the infinity points (3.2) of $\mathcal{H}$ is a quasi-Hermitian variety $\mathcal{V}$ of $\mathrm{PG}(r, q^2)$.

Proof. Let $P = (\xi_1, \ldots, \xi_r)$ be an affine point in Π_ε. This point, viewed as an element of $\mathrm{AG}(r, q^2)$, has coordinates $x_i = \xi_i$, for $i = 1, \ldots, r-1$, and $x_r = \xi_r + \Delta_\varepsilon(\xi_1) + \cdots + \Delta_\varepsilon(\xi_{r-1})$. Therefore, $\mathcal{H}$ and $\mathcal{V}$ coincide in the projective closure of Π_ε thus, we just have to prove the following lemma. Let $\mathcal{D}(m)$ be the hypersurface with equation (2.1).

Lemma 3.2. *The hypersurface $\mathcal{D}(m)$ and $\mathcal{H}$ have either*

$$N_1 = \frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} - \left|\mathcal{H}(r-2, q^2)\right|$$

or

$$N_2 = \frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} + (-1)^{r-1} q^{r-1} - \left|\mathcal{H}(r-2, q^2)\right|$$

common points in $\mathrm{AG}(r, q^2)$.

Proof. The intersection size of $\mathcal{H}$ and $\mathcal{D}(m)$ in $AG(r, q^2)$ is the number of solutions $(x_1, \ldots, x_r) \in \mathrm{GF}(q^2)^r$ of the following system

$$\begin{cases} x_r^q + x_r = x_1^{q+1} + \cdots + x_{r-1}^{q+1}. \\ \qquad x_r = \Delta_\varepsilon(x_1) + \cdots + \Delta_\varepsilon(x_{r-1}) + m_1 x_1 + \cdots + m_{r-1} x_{r-1} + d. \end{cases} \tag{3.4}$$

Substituting the value of x_r in the first equation gives

(3.5)
$$\begin{aligned} x_1^{q+1}+\cdots+x_{r-1}^{q+1} = &\Delta_\varepsilon(x_1)^q+\cdots+\Delta_\varepsilon(x_{r-1})^q+m_1^q x_1^q+\cdots+m_{r-1}^q x_{r-1}^q+\\ &\Delta_\varepsilon(x_1)+\cdots+\Delta_\varepsilon(x_{r-1})+m_1x_1+\cdots+m_{r-1}x_{r-1}+\\ &d^q+d. \end{aligned}$$

Consider now GF(q^2) as a vector space over GF(q). The set $\{1,\varepsilon\}$ is a basis of $GF(q^2)$, thus the elements in GF(q^2) can be written as linear combinations with respect to this basis, that is, $x_i=x_i^0+x_i^1\varepsilon$, with $x_i^0,x_i^1\in$ GF(q). Hence, (3.5) becomes an equation over GF(q),
(3.6)
$$\begin{aligned} 0 = &(x_1^0)^{\sigma+2}+x_1^0x_1^1+(x_1^1)^\sigma+\cdots+(x_{r-1}^0)^{\sigma+2}+x_{r-1}^0x_{r-1}^1+(x_{r-1}^1)^\sigma+\\ &m_1^1x_1^0+(m_1^0+m_1^1)x_1^1+\cdots+m_{r-1}^1x_{r-1}^0+(m_{r-1}^0+m_1^1)x_{r-1}^1+d^1. \end{aligned}$$

The solutions $(x_1^0,x_1^1,\dots,x_{r-1}^0,x_{r-1}^1)$ of (3.6) may be regarded as points of the affine space AG($2(r-1),q$) over GF(q). In fact, (3.6) turns out to be the equation of a (possibly degenerate) affine hypersurface $\mathcal{S}$ of AG($2(r-1),q$). The number N of points in AG($2(r-1),q$) which lie on $\mathcal{S}$ is the number of points in AG(r,q^2) on $\mathcal{H}\cap\mathcal{D}(m)$. We will show that N is either N_1 or N_2 by induction on r.

First, suppose $r=2$. In this case $\mathcal{S}$ can be viewed as an affine planar section of the Tits ovoid O of affine equation $(x_1^1)^\sigma+x_1^0x_1^1+(x_1^0)^{\sigma+2}=z$. Here (x_1^0,x_1^1,z) denote affine coordinates for points in the affine 3-space in which AG($2,q$) is embedded as a hyperplane. Therefore, $\mathcal{S}$ consists of 1 or $q+1$ points according as our plane of equation $z=m_1^1x_1^0+(m_1^0+m_1^1)x_1^1+d^1$ is tangent to O or not, and the assertion follows.

Now suppose $r>2$. Fix a $2(r-2)$-tuple $(\bar{x}_2^0,\bar{x}_2^1,\dots,\bar{x}_{r-1}^0,\bar{x}_{r-1}^1)$ of elements in $GF(q)$. For each such tuple, the number of $2(r-1)$-tuples
$$(\alpha,\beta,\bar{x}_2^0,\bar{x}_2^1,\dots,\bar{x}_{r-1}^0,\bar{x}_{r-1}^1)$$
satisfying (3.6) is 1 or $q+1$ according to whether

(3.7)
$$\begin{aligned} 0 = &(\bar{x}_2^0)^{\sigma+2}+\bar{x}_2^0\bar{x}_2^1+(\bar{x}_2^1)^\sigma+\cdots+(\bar{x}_{r-1}^0)^{\sigma+2}+\bar{x}_{r-1}^0\bar{x}_{r-1}^1+(\bar{x}_{r-1}^1)^\sigma+\\ &m_2^1\bar{x}_2^0+(m_2^0+m_2^1)\bar{x}_2^1+\cdots+m_{r-1}^1\bar{x}_{r-1}^0+(m_{r-1}^0+m_1^1)\bar{x}_{r-1}^1+\\ &(m_1^1)^{\sigma+2}+(m_1^0+m_1^1)m_1^1+(m_1^0+m_1^1)^\sigma+d^1 \end{aligned}$$

or not. The induction hypothesis applied to $r-1$ yields that (3.7) has either
$$n_1=\frac{(q^{r-1}+(-1)^{r-2})(q^{r-2}-(-1)^{r-2})}{q^2-1}-\left|\mathcal{H}(r-3,q^2)\right|$$
or
$$n_2=\frac{(q^{r-1}+(-1)^{r-2})(q^{r-2}-(-1)^{r-2})}{q^2-1}+(-1)^{r-2}q^{r-2}-\left|\mathcal{H}(r-3,q^2)\right|$$

solutions. This implies that the number of solutions of (3.6) is either

$$a = n_1 + (q^{2(r-2)} - n_1)(q+1)$$

or

$$b = n_2 + (q^{2(r-2)} - n_2)(q+1).$$

A direct computation shows that $a = N_1$ and $b = N_2$ and our lemma follows □

Since the points at infinity of a hyperplane of $\mathrm{AG}(r, q^2)$ are also the points at infinity of the corresponding hyperplane in the projective closure of Π_ε, the assertion is proven. □

Theorem 3.3. *The quasi–Hermitian variety $\mathcal{V}$ defined in Theorem* (3.1) *is not projectively equivalent to the Hermitian variety $\mathcal{H}$ of* $\mathrm{PG}(r, q^2)$.

Proof. First assume $r = 2$. In this case $\mathcal{V}$ consists of the infinity point $(0,0,1)$ together with the points $(1, x, y)$ such that

$$y^q + y = [x + (x^q + x)\varepsilon]^{\sigma+2} + (x^q + x)^\sigma + (x^{2q} + x^2)\varepsilon + x^{q+1} + x^2.$$

Setting $x^q + x = t$, and $x + (x^q + x)\varepsilon = s$, we have

$$x = s + t\varepsilon,$$

and

$$y^q + y = s^{\sigma+2} + t^\sigma + ts,$$

that is,

$$y = (s^{\sigma+2} + t^\sigma + t\sigma)\varepsilon + r,$$

where $r \in GF(q)$. Therefore,

$$\mathcal{V} = \big\{(1, s + t\varepsilon, (s^{\sigma+2} + t^\sigma + t\sigma)\varepsilon + r) \,|\, r, s, t \in GF(q)\big\} \cup \{(0,0,1)\},$$

namely, $\mathcal{V}$ coincides with a Buekenhout-Tits unital which is not projectively equivalent to the hermitian curve of $\mathrm{PG}(2, q^2)$; see [2, 9].

In the case $r > 2$ let π be the plane of affine equations $x_2 = \cdots = x_{r-1} = 0$, and let $\mathcal{U}$ denote the intersection of $\mathcal{V}$ and π. We can choose homogeneous coordinates in π in such a way that $\mathcal{U}$ is the set of points

$$\big\{(1, s + t\varepsilon, (s^{\sigma+2} + t^\sigma + t\sigma)\varepsilon + r) \,|\, r, s, t \in GF(q)\big\} \cup \{(0,0,1)\}$$

that is, a Buekenhout-Tits unital of π, and thus the assertion is proven. □

Remark. For each $\gamma = (\gamma_1, \dots, \gamma_r) \in GF(q)^r$, let ψ_γ be the collineation of $\mathrm{PG}(r, q^2)$ induced by the non-singular matrix

$$\begin{pmatrix} 1 & \gamma_1\varepsilon & \gamma_2\varepsilon & \dots & \gamma_{r-1}\varepsilon & \gamma_r + (\gamma_1 + \dots + \gamma_{r-1})^\sigma\varepsilon) \\ 0 & 1 & 0 & \dots & 0 & \gamma_1 + \gamma_1\varepsilon \\ 0 & 0 & 1 & \dots & 0 & \gamma_2 + \gamma_2\varepsilon \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \gamma_{r-1} + \gamma_{r-1}\varepsilon \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Let G denote the following collineation group of order q^r,

$$G = \{\psi_\gamma \mid \gamma \in GF(q)^r\}.$$

Straightforward computations show that G is an abelian group which leaves $\mathcal{V}$ invariant; in particular, it fixes P_∞ and has q^{r-1} orbits of size q^r on $\mathcal{V} \setminus \mathcal{F}$. Furthermore, for $r = 2$, it coincides with the stabilizer in $PGL(3, q^2)$ of $\mathcal{V}$; see [9].

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