Contributions to Discrete Mathematics

Volume 7, Number 1, Pages 21–34
ISSN 1715-0868

# ON CHARACTERIZING GAME-PERFECT GRAPHS BY FORBIDDEN INDUCED SUBGRAPHS

STEPHAN DOMINIQUE ANDRES

Abstract. A graph $G$ is called $g$-perfect if, for any induced subgraph $H$ of $G$, the game chromatic number of $H$ equals the clique number of $H$. A graph $G$ is called $g$-col-perfect if, for any induced subgraph $H$ of $G$, the game coloring number of $H$ equals the clique number of $H$. In this paper we characterize the classes of $g$-perfect resp. $g$-col-perfect graphs by a set of forbidden induced subgraphs. Moreover, we study similar notions for variants of the game chromatic number, namely $B$-perfect and $[A, B]$-perfect graphs, and for several variants of the game coloring number, and characterize the classes of these graphs.

## 1. Introduction

A well-known maker-breaker game is one of Bodlaender's graph coloring games [9]. We are given an initially uncolored graph $G$ and a color set $C$. Two players, Alice and Bob, move alternately with Alice beginning. A move consists in coloring an uncolored vertex with a color from $C$ in such a way that adjacent vertices receive distinct colors. The game ends if no move is possible any more. The maker Alice wins if the vertices of the graph are completely colored, otherwise, i.e. if there is an uncolored vertex surrounded by colored vertices of each color, the breaker Bob wins. For a graph $G$, the *game chromatic number* $\chi_g(G)$ of $G$ is the smallest cardinality of a color set $C$ such that Alice has a winning strategy in the game described above.

During the last 18 years, initiated by the paper of Faigle et al. [14], there have been a lot of attempts to obtain good upper bounds for the game chromatic number of the members of interesting classes of graphs, such as trees, forests, and interval graphs [14], outerplanar graphs [16], planar graphs [19, 12, 25, 18, 28], $(a, b)$-pseudo partial $k$-trees and graphs embeddable into some surface [26], special graphs embedabble into some surface [17, 22, 4], cactuses [21], Halin graphs [24], Cartesian product graphs [27], line graphs of $k$-degenerate graphs [10], line graphs of graphs of arboricity $k$ [7], line graphs of forests of maximum degree $\Delta \neq 4$ [10, 13, 1, 2], line graphs of

Received by the editors February 16, 2011, and in revised form September 30, 2011.
2000 *Mathematics Subject Classification.* 05C15, 05C17, 05C57, 91A43.
*Key words and phrases.* game chromatic number, game coloring number, game-perfect graph, perfect graph, graph coloring game.

wheels [5], certain Cartesian product graphs [6], etc. Obviously, for any graph $G$, we have

$$\omega(G) \leq \chi(G) \leq \chi_g(G), \tag{1}$$

where $\omega(G)$ denotes the clique number and $\chi(G)$ denotes the chromatic number of $G$. In most cases the above-mentioned upper bounds for $\chi_g(G)$ are much larger than $\omega(G)$, even larger than $\chi(G)$. For any $n \in \mathbb{N}$, there are bipartite graphs with $\chi_g(G) = n$ (see [18]), so that the gap between $\omega(G)$ (resp. $\chi(G)$) and $\chi_g(G)$ may be arbitrarily large. In this paper we are interested in the other extremal case: those graphs for which $\omega(G) = \chi_g(G)$. It turns out that the class of these graphs does not have a nice structure, so we modify our task as follows. A graph $G$ is called *$g$-perfect* if $\omega(H) = \chi_g(H)$ holds for any induced subgraph $H$ of $G$. This notion of game-perfectness was introduced in [3], following a suggestion of Maria Chudnovsky. It is a game-theoretic analog of perfectness of graphs, moreover by (1) $g$-perfect graphs are in particular perfect.

The characterization of perfect graphs by forbidden induced subgraphs was given by the Strong Perfect Graph Theorem of Chudnovsky et al. [11], which proves a conjecture of Berge [8].

**Theorem 1** (Chudnovsky, Robertson, Seymour, Thomas (2006))**.** *A graph is perfect if, and only if, it does neither contain an odd hole nor an odd antihole as induced subgraph.*

We will give a similar characterization of the class of $g$-perfect graphs, which is, of course, more easy since we will see that the class of $g$-perfect graphs is a very small subclass of the class of perfect graphs. However, in order to prove the characterization, we need an explicit characterization which makes use of another notion of game-perfectness, namely of $B$-perfect graphs. So we will also consider other variants of Bodlaender's game that will lead to other variants of game-perfectness.

Let "$A$" denote "Alice," and "$B$" denote "Bob," and "$-$" denote "none of the players." Let $X \in \{A, B\}$ and $Y \in \{A, B, -\}$. The game $[X, Y]$ is defined as follows. As in Bodlaender's game we are given an uncolored graph $G$ and a color set $C$. The players, $A$ and $B$, move alternately. Player $X$ has the first move. $Y$ may miss one or several turns, this possibly includes the right to pass in the first move. However the other player(s) (which are not $Y$) must always move. A move consists in coloring an uncolored vertex with a color from $C$ which is different from the colors of its neighbors, as in Bodlaender's game, until this is not possible any more. Alice wins if the graph is completely colored in the end, otherwise Bob wins. For a graph $G$, the *$[X, Y]$-game chromatic number* $\chi_{[X,Y]}(G)$ of $G$ is the smallest cardinality of a color set, so that Alice has a winning strategy for this game. A graph $G$ is *$[X, Y]$-perfect* if, for any induced subgraph $H$ of $G$, $\omega(H) = \chi_{[X,Y]}(H)$.

So Bodlaender's game is the game $[A, -]$ and $g$-perfectness is the same as $[A, -]$-perfectness. With a slight abuse of notation we define the following abbreviations

$$A := [A, A], \quad B := [B, B], \quad g := g_A := [A, -], \quad g_B := [B, -].$$

Then our notation matches with the notations of all previous references.

The following observation was shown in [2].

**Observation 2.** *For any graph $H$,*

$$\omega(H) \le \chi(H) \le \chi_A(H) \le \left\{ \begin{array}{ccc} \chi_g(H) & \le & \chi_{[A,B]}(H) \\ \chi_{[B,A]}(H) & \le & \chi_{g_B}(H) \end{array} \right\} \le \chi_B(H).$$

In particular, $B$-perfect graphs are $[A, B]$-perfect, $[A, B]$-perfect graphs are $g$-perfect, $g$-perfect graphs are $A$-perfect, and $A$-perfect graphs are perfect.

In this paper we will characterize $B$-perfect graphs (in Section 2), and $[A, B]$-perfect and $g$-perfect graphs (in Section 3) by forbidden induced subgraphs and explicitly. The characterization of $A$-perfect, $g_B$-perfect and $[B, A]$-perfect graphs is still open, some partial results are given in Section 5. In the Section 4 we examine a similar notion to game-perfectness, namely game-col-perfectness, which is based on the game coloring number instead of the game chromatic number. For all variants of the underlying marking game we characterize the class of game-col-perfect graphs.

## 2. Characterizing $B$-perfect graphs

We start with the smallest class of game-perfect graphs: the class of $B$-perfect graphs. Its characterization is given in the next theorem. For its formulation we need the following notation. Let $G_1$ and $G_2$ be two graphs. By $G_1 \cup G_2$ we denote the *disjoint union* of $G_1$ and $G_2$, i.e. the graph that consists of an isomorphic copy of $G_1$ and an isomorphic copy of $G_2$ which is disjoint from the copy of $G_1$, and there are no edges between the copies of $G_1$ and $G_2$.. By $G_1 \vee G_2$ we denote the *join* of $G_1$ and $G_2$, i.e. the graph constructed from $G_1 \cup G_2$ by connecting every vertex of the copy of $G_1$ with every vertex of the copy of $G_2$ by an edge.

**Theorem 3.** *Let $G$ be a (nonempty) graph. Then the following conditions are equivalent:*

(i) *$G$ is $B$-perfect.*
(ii) *$G$ does neither contain a $C_4$, nor a $P_4$, nor a split 3-star, nor a double fan as an induced subgraph (see Figure 1).*
(iii) *For every connected component $H$ of $G$, there is $k \ge 0$, so that*

$$H = K_1 \vee (H_0 \cup H_1 \cup \cdots \cup H_k),$$

*where the $H_i$'s are complete graphs for $i \ge 1$, and $H_0$ is either empty or there are $p, q, r \in \mathbb{N}$, so that $H_0 = K_r \vee (K_p \cup K_q)$ (see Figure 2).*

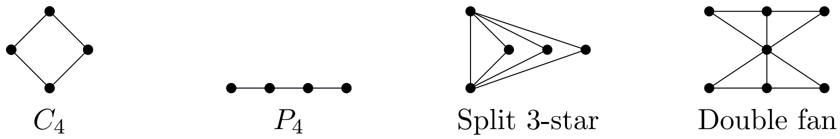


FIGURE 1. Four forbidden induced subgraphs for $B$-perfect graphs

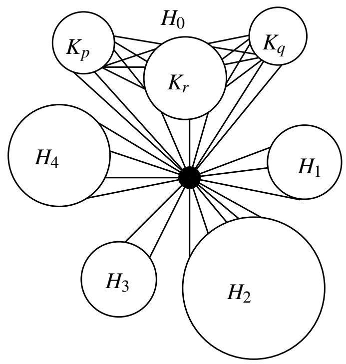


FIGURE 2. Structure of a connected component according to (iii)

*Proof.* (i) $\Longrightarrow$ (ii): Winning strategies for Bob in game $B$ with 2 colors on $C_4$ resp. $P_4$ are obvious. (He has to make use of his right to miss the first turn.) On the split 3-star, Bob wins with 3 colors, since after his second move he can achieve a situation in which two vertices of degree 2 are differently colored. We now prove that Bob wins on the double fan with 3 colors. Note that, in any proper coloring of the double fan, the two upper resp. the two lower vertices of degree 2 must be colored with the same color. Bob has the following winning strategy: He colors an upper vertex of degree 2 with color 1. Then Alice is forced to color the other upper vertex of degree 2 with the same color. Now Bob colors a lower vertex of degree 2 with color 2. Again, Alice is forced to color the other lower vertex of degree 2 with color 2. Now Bob colors a vertex with degree 3 with color 3, hence he wins.

(iii) $\Longrightarrow$ (i): We describe a winning strategy for Alice with $\omega(G)$ colors on a graph $G$ as in (iii). This is sufficient since every induced subgraph of $G$ is of the same type as described in (iii). For $H_0 = K_r \vee (K_p \cup K_q)$ let the $K_p$ and the $K_q$ be the *ears*. We call the the vertices of the $K_r$ and the universal vertex of a connected component *dangerous vertices*, since these are the only vertices that might have more than $\omega(G) - 1$ neighbors, all other vertices can always be colored with $\omega(G)$ colors.

Alice always responds to Bob's moves in the same connected component $H$ (if Bob passes, in an arbitrary component). As long as Bob does not play in an ear, Alice does not play in an ear; she first colors the universal vertex of $H$. If Bob plays in an ear $K_p$, Alice colors a vertex in the corresponding ear $K_q$ with the same color (in case there is no uncolored vertex she uses the strategy described before). If Alice is forced to start coloring an ear, then all dangerous vertices of its connected component are colored, so she will win in any case.

(ii) $\Longrightarrow$ (iii): We examine the structure of a graph $G$ without induced $P_4$, $C_4$, split 3-star, double fan. Let $H$ be a connected component of $G$. We use the following lemma of Wolk [23].

**Lemma 4** (Wolk (1965))**.** *A connected graph without induced $C_4$ and $P_4$ (a so-called* trivially perfect graph [15]*) has a universal vertex.*

So, $H$ has a universal vertex $v$. Let $H_0, \dots, H_n$ be the connected components of $H \setminus v$.

**Claim 5.** *At most one of the $H_i$'s is not complete.*

*Proof.* Assume $H_1$, $H_2$ are not complete. Then both contain a $P_3$. So $H$ contains a double fan, which contradicts (ii). □

Let $H_0$ be the (only) connected component of $H \setminus v$ which is not complete. Let $K$ be the largest clique of $H_0$. We are done if we show the following:

**Claim 6.**

(a) *$H_0 \setminus K$ induces a clique.*
(b) *$H_0 \setminus K$ induces a module of $H_0$ (i.e. if $x \in K$, either $x$ is adjacent to all $y \in H_0 \setminus K$ or to none.)*

*Proof of Claim 6.*
(a) Assume there are non-adjacent vertices $x, y \in H_0 \setminus K$. Since $K$ is a maximal clique, there are $z, z' \in K$ such that neither $x, z$ nor $y, z'$ are adjacent. We note that, again by Lemma 4, $H_0$ has a universal vertex $w \in K$. If $y, z$ are not adjacent, $x, y, z, w, v$ induce a split 3-star, contradicting (ii). So we may assume that $y, z$ are adjacent and, by symmetry, $x, z'$ are adjacent. This implies that $z \neq z'$ and $x, y, z, z'$ induce a $P_4$, contradicting (ii).
(b) Assume that there are $x \in K$, $s, t \in H_0 \setminus K$, so that $s, x$ are not adjacent, but $t, x$ are adjacent. By Claim 6 (a), $s, t$ are adjacent. Since $K$ is a maximal clique, there is $y \in K$, so that $t, y$ are not adjacent. This implies that $s, t, x, y$ induces either a $P_4$ or a $C_4$, which contradicts (ii). □

By Claim 6, $H_0 \setminus K$ corresponds to the $K_p$, its neighbors correspond to the $K_r$, and the rest of $H_0$ corresponds to the $K_q$. Thus $G$ has the structure as described in (iii). This completes the proof of Theorem 3. □

## 3. $[A, B]$- AND $g$-PERFECT GRAPHS

Now we are ready to prove our main result, the characterization of $g$-perfect graphs. It turns out that this class of graphs is the same as the class of $[A, B]$-perfect graphs.

**Theorem 7.** *Let $G$ be a (nonempty) graph. Then the following conditions are equivalent:*

(i) *$G$ is $[A, B]$-perfect*
(ii) *$G$ is $g$-perfect.*
(iii) *$G$ does neither contain a $C_4$, nor a $P_4$, nor a triangle star, nor a $\Xi$-graph nor a graph with two connected components which are a split 3-star or/and a double fan as an induced subgraph (see Figure 3).*
(iv) *Let $G_1, G_2, \ldots, G_m$ be the connected components of $G$. $G_2, \ldots, G_m$ are $B$-perfect and $G_1$ contains a universal vertex $v$, so that every connected component of $G_1 \setminus v$ is $B$-perfect.*

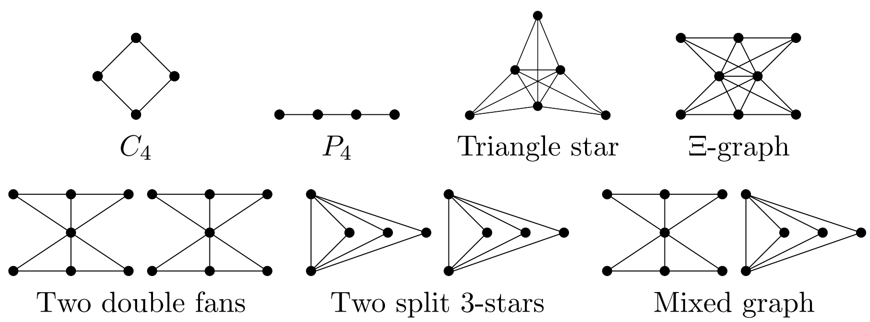


FIGURE 3. Seven forbidden configurations for $g$-perfectness

*Proof.* (i) $\Longrightarrow$ (ii): This follows directly from Observation 2.

(ii) $\Longrightarrow$ (iii): Consider the game $g$, Bodlaender's original game. Obviously, Bob wins on the $C_4$ and $P_4$ with 2 colors. Bob wins on two double fans, two split 3-stars resp. the mixed graph with 3 colors since, after Alice has played in a certain connected component, Bob can follow his strategy as in the proof of Theorem 3 in the other component. Bob also wins on the triangle star with 4 colors: Bob's goal is to have two vertices of degree 3 colored in two different colors and at least one universal vertex uncolored. Obviously, he can achieve this goal within his first two moves.

Now we prove that Bob wins on the $\Xi$-graph with 4 colors. If Alice colors a vertex of degree 3, Bob colors the uncolored vertex of degree 3 in the same row with a different color and wins.

If Alice, in her first move, colors a vertex of degree 4 with color 1, Bob colors the second vertex of degree 4 with color 2. If Alice now plays a universal vertex (necessarily with color 3), Bob colors a vertex of degree 3 with color 4 and wins. If, on the other hand, Alice colors a vertex of degree 3, Bob colors another vertex of degree 3 with color 3 and wins, since for the two universal vertices two new colors would be needed.

Finally consider the last case, i.e., in her first move Alice colors a universal vertex. Then the remaining game reduces to a game $[B, -]$ with 3 colors on a double fan, which Bob will win using the strategy described in the proof of Theorem 3. Therefore Bob always wins on the $\Xi$-graph.

(iii) $\Longrightarrow$ (iv): First we prove that every connected component $H$ of $G$ has a universal vertex $v$ with the property that, if $v$ is deleted, the connected components of $H \setminus v$ are $B$-perfect. This is seen as follows: Since $G$ does not contain a $P_4$ or a $C_4$, by Lemma 4 every connected component $H$ has a universal vertex $v_H$. Assume $H \setminus v_H$ is not $B$-perfect. Then, by Theorem 3, it has either an induced $C_4$ or $P_4$ (which is not possible since $G$ does neither contain a $C_4$ nor a $P_4$) or an induced double fan or split 3-star. Assume $H \setminus v_H$ contains an induced double fan $D$. Then the graph induced by the vertices of $D$ and $v_H$ is a $\Xi$-graph in $G$, a contradiction. Assume $H \setminus v_H$ contains an induced split 3-star $S$. Then the graph induced by the vertices of $S$ and $v_H$ is a triangle star in $G$, a contradiction.

Now assume two connected components $H_1, H_2$ of $G$ are not $B$-perfect. This means that $H_1$ and $H_2$ each contain either a split 3-star or a double fan. But then $G$ contains either two split 3-stars, two double fans or a mixed graph, a contradiction.

(iv) $\Longrightarrow$ (i): We describe a winning strategy for Alice in the game $[A, B]$ on a graph as defined in (iv). This is sufficient since every induced subgraph of such a graph is of the same type. Alice's strategy is very simple: in her first move she colors the universal vertex $v$ of the special connected component $G_1$. Now the game is reduced to the game $[B, B]$ on $G \setminus v$, which is a $B$-perfect graph. So Alice has a winning strategy with $\omega(G \setminus v) \in \{\omega(G), \omega(G) - 1\}$ colors. This proves the theorem. $\square$

**Remark.** Note that, in spite of the fact that the classes of $[A, B]$-perfect and $g$-perfect graphs are the same this does not mean that, for any graph $G$, $\chi_g(G) = \chi_{[A,B]}(G)$. Consider the graph $G = C_6 \cup K_1$. Here $\chi_g(G) = 2$ (Alice first colors the isolated vertex, then a vertex at distance 3 from the vertex Bob has colored), but $\chi_{[A,B]}(G) = 3$.

## 4. Game-col-perfect graphs

In the theory of graph coloring games, a main idea for gaining upper bounds is the so-called *game coloring number* of a graph which was introduced by Zhu [25]. The game underlying this concept is a marking game. In our context we may define some marking games as follows. Let "$A$" denote "Alice," and "$B$" denote "Bob," and "$-$" denote "none of the players." Let $X \in \{A, B\}$ and $Y \in \{A, B, -\}$. The game $[X, Y]$-col is defined as follows. We are given a graph $G = (V, E)$ whose vertices are unmarked at the beginning. Alice and Bob alternately mark unmarked vertices. Player $X$ begins. Player $Y$ may miss one or several turns (possibly also the first), but the other player(s) always have to move. Whenever a vertex is marked it is assigned a score. The score $s(v)$ of a vertex $v$ is 1 plus the number of neighbors of $v$ which have been marked before $v$. The score of the game is $S = \max_{v \in V} s(v)$. Alice's goal is to minimize the score, Bob's goal to maximize the score. The $[X,Y]$-*game coloring number* $\mathrm{col}_{[X,Y]}(G)$ is the smallest integer $S$ such that Alice has a strategy to achieve the score $S$ in game $[X,Y]$-col. Obviously, for a graph $G$,

$$\omega(G) \leq \chi(G) \leq \chi_{[X,Y]}(G) \leq \mathrm{col}_{[X,Y]}(G), \tag{2}$$

and we have, in line with Observation 2,

**Observation 8.** *For any graph $H$,*

$$\mathrm{col}_{[A,A]}(H) \leq \left\{ \begin{array}{ccc} \mathrm{col}_{[A,-]}(H) & \leq & \mathrm{col}_{[A,B]}(H) \\ \mathrm{col}_{[B,A]}(H) & \leq & \mathrm{col}_{[B,-]}(H) \end{array} \right\} \leq \mathrm{col}_{[B,B]}(H).$$

The game $[A,-]$-col is Zhu's marking game. A graph $G$ is $[X,Y]$-*col-perfect* if, for any induced subgraph $H$ of $G$, $\mathrm{col}_{[X,Y]}(G) = \omega(G)$. By (2) a $[X,Y]$-col-perfect graph is in particular $[X,Y]$-perfect.

In this section we characterize $[X,Y]$-col-perfect graphs for any $X \in \{A,B\}$ and $Y \in \{A,B,-\}$. As we shall see, the only significant difference lies in the player who is allowed to have the first move. Our proof uses an idea of Zhu ([27], Lemma 3) who already proved the first equation of (a).

**Theorem 9.** *For any graph $G$,*

(a) $\mathrm{col}_{[B,B]}(G) = \mathrm{col}_{[B,-]}(G) = \mathrm{col}_{[B,A]}(G)$, *and*
(b) $\mathrm{col}_{[A,B]}(G) = \mathrm{col}_{[A,-]}(G) = \mathrm{col}_{[A,A]}(G)$.

*Proof.* We first prove that missing a turn is not an advantage for Bob, then that missing a turn is not an advantage for Alice.

Let $X \in \{A,B\}$. First, assume that Alice has a strategy in game $[X,-]$-col to achieve a score $S_{[X,-]} \leq \alpha$. We prove that she has a strategy in game $[X,B]$-col to achieve a score $S_{[X,B]} \leq \alpha$. Alice uses basically her strategy in game $[X,-]$-col and further sometimes keeps in mind a special unmarked vertex, the *memory vertex*. In case Bob misses his turn and there is a memory vertex, Alice marks the memory vertex, which thereby looses his role as memory vertex. In case Bob misses his turn and there is no memory vertex or if Bob marks the memory vertex, Alice imagines that Bob has marked the unmarked vertex of smallest degree, which now is the new memory vertex, and chooses a vertex for marking by her strategy for game $[X,-]$-col. If Bob marks any other vertex, Alice replies by using her strategy for the game $[X,-]$-col. In case the memory vertex is the only unmarked vertex, Alice marks the memory vertex, of course.

Using this strategy the only vertices which might obtain a higher score in game $[X,B]$-col than in game $[X,-]$-col are the memory vertices. However, at the time they are chosen, they have smallest degree $d$ among all unmarked vertices, therefore $\alpha \geq d+1$. Since a memory vertex can obtain at most a score of $d+1$, the score of the game will be $\leq \alpha$.

Secondly, assume that Bob has a strategy in game $[X,-]$-col to obtain a score $S_{[X,-]} \geq \alpha$. We prove that he has a strategy in game $[X,A]$-col to achieve a score $S_{[X,A]} \geq \alpha$. Basically, Bob here uses his strategy in game $[X, -]$ and keeps in mind a memory vertex and proceeds as in Alice's strategy above with the only difference that, if Alice passes and there is no memory vertex or if Alice marks the memory vertex, Bob imagines that Alice has marked the unmarked vertex of largest degree, which is now the new memory vertex.

In this strategy, at any time there has been constructed already a vertex with score $\alpha$, or there is a memory vertex with degree $d \geq \alpha - 1$, or there is no memory vertex. If the last unmarked vertex is the memory vertex with degree $d \geq \alpha - 1$, then the score is $\geq d + 1 \geq \alpha$. Otherwise the score of all vertices marked after a memory vertex has been marked (before there is a new memory vertex) is equal to their scores for the game $[X, -]$-col. Therefore, in any case Bob can achieve a score $\geq \alpha$ in game $[X, A]$-col. This proves the theorem. □

The previous theorem justifies to use the short notations $A$-col resp. $B$-col instead of the games $[A, -]$-col resp. $[B, -]$-col. Moreover, in view of Observation 8, beginning is always an advantage in the marking game:

**Corollary 10.** *For any graph $G$, $\mathrm{col}_A(G) \leq \mathrm{col}_B(G)$.*

Before formulating the theorems on the characterization of game-col-perfect graphs we give a definition. A *diamond* $D$ is the graph $K_4$ minus one edge. By a $2D$ we mean $D \cup D$, the disjoint union of two diamonds.

**Theorem 11.** *Let $G$ be a (nonempty) graph. Then the following conditions are equivalent:*

(i) *$G$ is $B$-col-perfect.*

(ii) *$G$ does neither contain a $P_4$ nor a $C_4$ nor a $D$ as an induced subgraph.*

(iii) *Every connected component of $G$ is of the form*

$$K_1 \vee (H_1 \cup H_2 \cup \cdots \cup H_m)$$

*for some $m \geq 0$, where $H_i$ are complete graphs.*

*Proof.* (i) $\Longrightarrow$ (ii): Obviously there is a strategy for Bob to obtain a score of 3 resp. 4 in the game $B$-col played on the $P_4$ and $C_4$ resp. the diamond.

(ii) $\Longrightarrow$ (iii): Let (ii) be true. Since $G$ does neither contain an induced $C_4$ nor an $P_4$, $G$ is trivially perfect, hence by Lemma 4 every connected component $H$ of $G$ has a universal vertex $v_H$. Let $H$ be a connected component of $G$. Let $H'$ be a connected component of $H \setminus v_H$. Assume that $H'$ is not complete. Then, since it is connected it contains an induced $P_3$. This $P_3$ together with $v_H$ is an induced diamond in $G$, a contradiction to (ii). Thus (iii) holds.

(iii) $\Longrightarrow$ (i): We show that on $G$ as in (iii) Alice has a strategy to obtain a score of $\omega(G)$. The only *dangerous vertices*, i.e. vertices with more than $\omega(G)-1$ neighbors, are the universal vertices of the connected components of $G$ in case they are the only universal vertex of such a component (otherwise the component is a clique). Alice's strategy is simple. In case Bob marks a vertex in connected component $H$, Alice marks a vertex in the same component, preferably the universal vertex of the component. If this is not possible or Bob misses his turn, Alice marks some vertex, preferably a universal vertex of some connected component. In this way the universal vertices $v$ of the connected components have score $s(v) \leq 2$, and the score 2 is only achieved if the connected component is not trivial. Thus the score of the game is $\omega(G)$. □

**Theorem 12.** *Let $G$ be a (nonempty) graph. Then the following conditions are equivalent:*

(i) *$G$ is $A$-col-perfect.*
(ii) *$G$ does neither contain a $P_4$ nor a $C_4$ nor a $2D$ nor a $K_1 \vee D$ as an induced subgraph.*
(iii) *Let $G_1, G_2, \ldots, G_n$ be the connected components of $G$. Then the components $G_2, \ldots, G_n$ are $B$-col-perfect and $G_1$ contains a universal vertex $v$, so that every connected component of $G_1 \setminus v$ is $B$-col-perfect.*

*Proof.* (i) $\Longrightarrow$ (ii): Obviously, there is a strategy for Bob to obtain a score of 3 in the game $A$-col played on the $P_4$ and $C_4$. In the $2D$, Bob can force a score of 4 if in his first move he marks a vertex of degree 2 in a connected component in which Alice has not marked any vertex before. Now consider the $K_1 \vee D$. This graph has 3 vertices of degree 4 and 2 of degree 3. The score will be 4 if a vertex of degree 4 leaves as the last marked vertex. Bob can achieve this goal if in his first two moves he marks vertices of degree 3 preferably.

(ii) $\Longrightarrow$ (iii): Let $G$ be as in (ii). Assume there are two connected components of $G$ that are not $B$-col-perfect. Since $G$ does neither contain an induced $P_4$ nor an induced $C_4$ each of these components contains an induced diamond. So $G$ contains a $2D$, a contradiction.

Now let $H$ be the single connected component (if any) of $G$ that is not $B$-col-perfect. Since $H$ is trivially perfect and connected, by Lemma 4 $H$ has a universal vertex $v_H$. Let $H'$ be a connected component of $H \setminus v_H$. Assume $H'$ is not $B$-col-perfect. Then $H'$ contains a diamond $D$. So, together with $v_H$, $G$ contains a $K_1 \vee D$, a contradiction. Thus (iii) holds.

(iii) $\Longrightarrow$ (i): We show that on $G$ as in (iii) in game $[A, B]$-col Alice has a strategy to obtain a score of $\omega(G)$. In her first move she marks the universal vertex $v$ of $G_1$. Then the game is reduced to a game $[B, B]$-col on $G_2, \ldots, G_n$ and the connected components of $G_1 \setminus v$, which is a $[B, B]$-col-perfect graph. We have seen in the previous theorem that Alice now has a winning strategy to obtain score $\omega(G)$. □

The game coloring number is a game-theoretic analog of the well-known coloring number col($G$) of a graph $G$. We might also define col-perfectness and ask: what are the col-perfect graphs? The answer is very easy. Since every cycle of length $\geq 4$ is forbidden for col-perfectness, and chordal graphs have a perfect elimination scheme, the class of col-perfect graphs is the class of chordal graphs. This fact was already remarked by Kierstead and Yang [20].

## 5. Open problems

There is not much known about $A$-perfect graphs. The graphs depicted in Figure 4 are known to be minimal forbidden induced subgraphs, but there are probably many more. In [3], the author proved the following two theorems.

**Theorem 13.** *A triangle-free graph $G$ is $A$-perfect if, and only if, every connected component of $G$ is either $K_1$ or $K_{m,n}$ or $K_{m,n} - e$, where $e$ is an edge.*

**Theorem 14.** *Complements of bipartite graphs are $A$-perfect.*

In view of Observation 2 and the preceding results, the following open problem seems to be more difficult than Theorem 3.

**Problem 15.** *Characterize $A$-perfect graphs explicitly and/or by forbidden induced subgraphs.*

Even the following is an open question.

**Problem 16.** *Is the number of minimal forbidden induced subgraphs for $A$-perfectness that are different from odd antiholes finite?*

For the other variants of the game, the question of Problem 15 might be interesting, too.

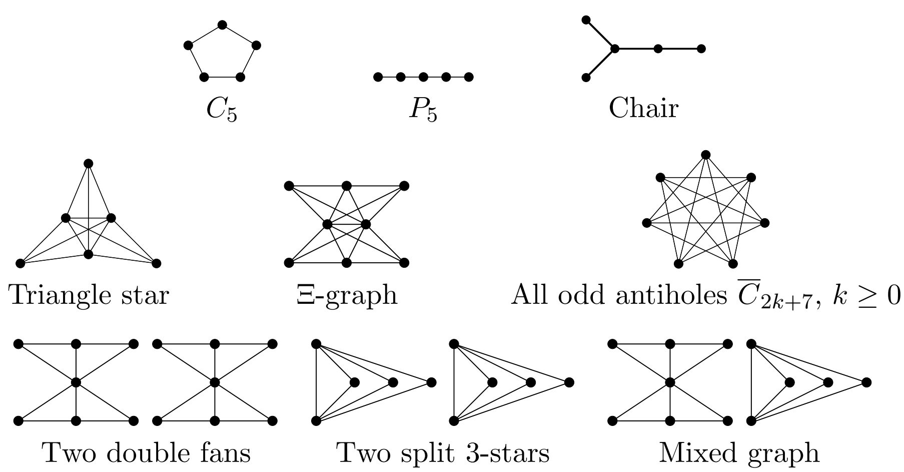


Figure 4. Some forbidden induced subgraphs for $A$-perfect graphs

**Problem 17.** *Characterize $g_B$-perfect graphs explicitly and/or by forbidden induced subgraphs.*

**Problem 18.** *Characterize $[B, A]$-perfect graphs explicitly and/or by forbidden induced subgraphs.*

In Figure 5 resp. Figure 6 sets of minimal forbidden induced subgraphs for $[B, A]$- resp. $g_B$-perfectness are displayed. It is not known whether these sets (together with the graphs of Figure 4 resp. the graphs of Figure 4 and Figure 5) are complete to characterize game-perfectness for the respective games. Of course the occurrence of the split 3-star and the double fan yields that the triangle star, the $\Xi$-graph, the two double fans, the two split stars, and the mixed graph are not minimal (for $[B, A]$- and $g_B$-perfectness). And the occurrence of the wheel $C_4 \vee K_1$ yields that the odd antiholes $\overline{C}_{2k+9}$, $k \geq 0$, are not minimal (for $g_B$-perfectness).

The following theorem is obvious.

**Theorem 19.** *Either every connected component of a $g_B$-perfect graph $G$ is $B$-perfect or $G$ has only one connected component.*

*Proof.* Assume $G$ is $g_B$-perfect, but not $B$-perfect. Then a connected component of $G$ must contain a $P_4$ or a $C_4$. If there is a second component, then $G$ contains a $P_4 \cup K_1$ or a $C_4 \cup K_1$, which are forbidden in $g_B$-perfect graphs. Thus $G$ is connected. □

Connected $g_B$-perfect graphs may have a richer structure which resisted a characterization until now.

We remark that the classes of $A$-, $g$-, $[B, A]$-, $g_B$- and $B$-perfect graphs are 5 distinct classes of graphs. This is easily seen as follows. The path $P_4$ is $[B, A]$-perfect, $g_B$-perfect, and $A$-perfect, but neither $g$-perfect nor $B$-perfect. The double fan is $A$-perfect and $g$-perfect, but neither $[B, A]$-perfect nor $g_B$-perfect. The $P_4 \vee K_1$ is $[B, A]$-perfect, but not $g_B$-perfect.

Split 3-star Double fan

FIGURE 5. Some additional forbidden induced subgraphs for $[B, A]$-perfect graphs

$C_4 \cup K_1$ $P_4 \cup K_1$ Wheel $C_4 \vee K_1$ Fan $P_4 \vee K_1$

FIGURE 6. Some additional forbidden induced subgraphs for $g_B$-perfect graphs

There are some more interesting questions which are related to the classifications of this paper. For example, one might consider the notion of $[X, Y]$-chi-col-perfectness. A graph $G$ is $[X, Y]$*-chi-col-perfect* if, for any induced subgraph $H$ of $G$, $\chi_{[X,Y]}(H) = \mathrm{col}_{[X,Y]}(H)$. Obviously, the class of $[X, Y]$-col-perfect graphs is the intersection of the classes of $[X, Y]$-perfect graphs and $[X, Y]$-chi-col-perfect graphs.

**Problem 20.** *For any $X \in \{A, B\}$ and $Y \in \{A, B, -\}$, characterize the class of $[X, Y]$-chi-col-perfect graphs explicitly and/or by forbidden induced subgraphs.*

The difference between the chromatic number and the game chromatic number is an important parameter in the theory of graph coloring games. So we may define a graph $G$ as $[X, Y]$*-exact* if, for any induced subgraph $H$ of $G$, $\chi(H) = \chi_{[X,Y]}(H)$. Then the class of $[X, Y]$-perfect graphs is the intersection of the classes of perfect graphs and $[X, Y]$-exact graphs.

**Problem 21.** *For any $X \in \{A, B\}$ and $Y \in \{A, B, -\}$, characterize the class of $[X, Y]$-exact graphs explicitly and/or by forbidden induced subgraphs.*

## Acknowledgements

Thanks to Winfried Hochstättler for useful comments, in particular for some hints which helped me to formulate Claim 6 in a compact way, and for a hint concerning reference [20]. I also acknowledge a discussion with Stefan Felsner concerning the minimality of the odd antiholes for $A$-perfectness.

## References

1. S. D. Andres, *Spieltheoretische Kantenfärbungsprobleme auf Wäldern und verwandte Strukturen*, Diploma thesis, University of Cologne, 2003.
2. ———, *The game chromatic index of forests of maximum degree* $\Delta \geq 5$, Discrete Applied Math. **154** (2006), 1317–1323.
3. ———, *Game-perfect graphs*, Math. Methods Oper. Res. **69** (2009), 235–250.
4. ———, *Lightness of digraphs in surfaces and directed game chromatic number*, Discrete Math. **309** (2009), 3564–3579.
5. S. D. Andres, W. Hochstättler, and C. Schallück, *The game chromatic index of wheels*, Discrete Applied Math. **159** (2011), 1660–1665.
6. T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, and I. Peterin, *Game chromatic number of Cartesian product graphs*, Electronic J. Comb. **15** (2008), R72.
7. T. Bartnicki and J. Grytczuk, *A note on the game chromatic index of graphs*, Graphs and Combinatorics **24** (2008), 67–70.
8. C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1976.
9. H. L. Bodlaender, *On the complexity of some coloring games*, Int. J. Found. Comput. Sci. **2** (1991), 133–147.
10. L. Cai and X. Zhu, *Game chromatic index of $k$-degenerate graphs*, J. Graph Theory **36** (2001), 144–155.
11. M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The strong perfect graph theorem*, Ann. Math. **164** (2006), 51–229.
12. T. Dinski and X. Zhu, *A bound for the game chromatic number of graphs*, Discrete Math. **196** (1999), 109–115.

13. P. Erdös, U. Faigle, W. Hochstättler, and W. Kern, *Note on the game chromatic index of trees*, Theoretical Comp. Sci. **313** (2004), 371–376.
14. U. Faigle, W. Kern, H. Kierstead, and W. T. Trotter, *On the game chromatic number of some classes of graphs*, Ars Combin. **35** (1993), 143–150.
15. M. C. Golumbic, *Trivially perfect graphs*, Discrete Math. **24** (1978), 105–107.
16. D. J. Guan and X. Zhu, *Game chromatic number of outerplanar graphs*, J. Graph Theory **30** (1999), 67–70.
17. W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, and X. Zhu, *Edge-partitions of planar graphs and their game coloring numbers*, J. Graph Theory **41** (2002), 307–317.
18. H. A. Kierstead, *A simple competitive graph coloring algorithm*, J. Comb. Theory **B** **78** (2000), 57–68.
19. H. A. Kierstead and W. T. Trotter, *Planar graph coloring with an uncooperative partner*, J. Graph Theory **18** (1994), 569–584.
20. H. A. Kierstead and D. Yang, *Orderings on graphs and game coloring number*, Order **20** (2003), 255–264.
21. E. Sidorowicz, *The game chromatic number and the game colouring number of cactuses*, Information Processing Letters **102** (2007), 147–151.
22. W.-F. Wang, *Edge-partitions of graphs of nonnegative characteristic and their game coloring numbers*, Discrete Math. **306** (2006), 262–270.
23. E. S. Wolk, *A note on "the comparability graph of a tree"*, Proc. Am. Math. Soc. **16** (1965), 17–20.
24. J.-J. Wu, *Game chromatic number of Halin graphs*, M.Sc. thesis, National Sun Yat-sen University, 2001.
25. X. Zhu, *The game coloring number of planar graphs*, J. Combin. Theory **B** **75** (1999), 245–258.
26. ———, *The game coloring number of pseudo partial $k$-trees*, Discrete Math. **215** (2000), 245–262.
27. ———, *Game coloring the Cartesian product of graphs*, J. Graph Theory **59** (2008), 261–278.
28. ———, *Refined activation strategy for the marking game*, J. Comb. Theory **B** **98** (2008), 1–18.

Dept. of Mathematics and Computer Science, FernUniversität in Hagen, Lützowstr. 125, 58084 Hagen, Germany
*E-mail address*: dominique.andres@fernuni-hagen.de