Contributions to Discrete Mathematics

Volume 8, Number 2, Pages 1–18
ISSN 1715-0868

TRANSLATION PLANES OF ORDER 23^2

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Abstract. We give a complete classification of translation planes of order 23^2 whose translation complement contains a subgroup G such that the quotient group $\overline{G}$ modulo scalars is isomorphic to A_6. Up to isomorphism, there are exactly 23 such planes and six of them have a larger translation complement being modulo scalars isomorphic to S_6. We also survey the known results on this subject and compare our results with those due to Moorhouse [7].

1. Introduction

In the translation complement of a translation plane of order q^2 and kernel $GF(q)$ with $q = p^h$ and p prime, some irreducible subgroup may happen to have order relatively prime to p. This was pointed out by Ostrom in his pioneering work [9] dating back to 1977. For the quotient group $\overline{G}$ of such a group G modulo scalars, he was able to prove that if $\overline{G}$ coincides with its commutator group then the "largest" possibilities for $\overline{G}$ occur in two cases only, namely for $\overline{G} \cong A_6$ and for $\overline{G}/\overline{E} \cong A_5$ with an elementary abelian normal subgroup $\overline{E}$ of order 16. Following the terminology introduced by Moorhouse [7], the corresponding translation planes are called of type (A) and (B), respectively. Infinite families, as well as, several sporadic planes of translation planes of type (B) are known in the literature; see [2, 9, 10, 11].

In this paper, we consider translation planes of type (A). It seems that such planes do not exist for $h > 1$, but this has been proven so far only for $p = 5, 7$. From now on we limit ourselves to the case $h = 1$. For $p = 5$, there exists a unique translation plane of type (A), up to isomorphism, namely the Hering plane of order 25 (see [5]). In 1985, Mason [6] found two more planes of order 49 using ordinary character theory. Later on Nakagawa constructed translation planes of type (A) of order 121 (see [8]). In 1990 Biliotti and Korchmáros [1] used a different approach relying on the canonical representation of a translation plane on the Klein quadric and obtained all translation planes of type (A) for each $q = p^2$ in the range $5 \leq p \leq 19$. Their results are summarized in Table 1.

Received by the editors October 17, 2011, and in revised form March 3, 2013.
2010 *Mathematics Subject Classification.* 51A40.
Key words and phrases. Translation plane, Klein quadric, ovoid, alternating group.

p	5	7	11	13	17	19
N	1	2	8	3	4	14

TABLE 1. The number, N, of non-isomorphic examples for $q = p^2$, $5 \leq p \leq 19$

The existence problem of translation planes of type (A) of any order $q = p^2$ was solved affirmatively in 1998 by Moorhouse [7], who pointed out that slicing the Conway-Kleidman-Wilson binary ovoids of O_8^+ given in [3] provides A_6-invariant ovoids of the Klein quadric of $PG(5, p)$ for every $p > 3$ prime. Dempwolff and Guthmann [4] combined the slicing method with number theory to construct more examples. Nevertheless, as Moorhouse himself observed, the slicing method is not exhaustive. The question of deciding which of these A_6-invariant ovoids can be obtained by the slicing method has not been treated so far.

In the present paper we go on to investigate the case $q = 23^2$ using the approach due to Biliotti and Korchmáros. Our main result, which actually provides a complete classification, is stated in the following theorem.

Theorem 1.1. *Let π be a translation plane of order 23^2 whose translation complement contains a subgroup G such that the quotient group $\overline{G}$ modulo scalars is isomorphic to A_6. There is a complete enumeration of such planes π; in particular the number of pairwise non-isomorphic planes is equal to 23. Six of these planes have a larger translation complement being modulo scalars isomorphic to S_6.*

In Section 9 we review Moorhouse's A_6-invariant ovoid in $PG(5, 23)$ described in [7, Section 3, Case II], and verify that it is indeed isomorphic to one of the ovoids given in Theorem 1.1.

2. Translation planes, spreads and ovoids on the Klein quadric

Assume that π is a projective plane. It is well known that a line ℓ of π is called a translation line if π is (P, ℓ)-transitive for any point $P \in \ell$. In this situation π is called a translation plane.

A translation plane of dimension two over its kernel $GF(q)$ may be obtained in the following way. The points are the elements of a four dimensional vector space V over $GF(q)$. The lines through the zero vector (also called components) are two dimensional subspaces which partition the non zero vectors. The other lines are translates of the components. The linear translation complement is the group of linear transformations of V mapping components onto components. Let us regard V as the underlying vector space of $PG(3, q)$. Then the components correspond to mutually disjoint lines which form a spread, that is, a partition of the point-set of $PG(3, q)$. Thus an arbitrary translation plane of dimension two over its kernel $GF(q)$

arises from a spread of $PG(3,q)$, and the converse also holds. The translation complement modulo the scalar transformations is the group of all linear collineations of $PG(3,q)$ preserving the corresponding spread.

Since lines of $PG(3,q)$ can be interpreted as points on the Klein quadric Q in $PG(5,q)$, spreads can be viewed as subsets of points on Q. Let ρ be the one-to-one mapping from lines of $PG(3,q)$ to points of Q via Plücker coordinates. The main property of ρ is that two lines ℓ_1 and ℓ_2 of $PG(3,q)$ are incident if and only if $\rho(\ell_1)$ and $\rho(\ell_2)$ are orthogonal with respect to the polarity relative to Q. Therefore, any set of mutually skew lines of $PG(3,q)$ is mapped to a cap of Q. In particular, spreads of lines of $PG(3,q)$ and ovoids of Q correspond under ρ where an ovoid is defined to be a point-set of Q meeting each singular plane of Q in exactly one point. Every linear collineation of $PG(3,q)$ preserving a spread lifts to an element of the orthogonal group $PSO_+(6,q)$ of Q which preserves the corresponding ovoid. The converse is also true, thus the group K of all linear collineations of $PG(3,q)$ preserving a spread is isomorphic to the subgroup K of $PSO_+(6,q)$ which preserves the corresponding ovoid. Furthermore, K acts on the spread in the same manner as on the corresponding ovoid. Also, two ovoids O_1 and O_2 of Q define two isomorphic spreads (hence two isomorphic translation planes) if and only if there is an element $g \in PSO_+(6,q)$ which takes O_1 to O_2. If both O_1 and O_2 are K-invariant ovoids and their stabilizer in $PSO_+(6,q)$ is K, then the condition $g(O_1) = O_2$ yields that $g \in \mathcal{N} \setminus K$ where $\mathcal{N}$ stands for the normalizer of K in $PSO_+(6,q)$.

From now on, assume that $p \geq 5$. Then a canonical equation of Q which is suitable for a useful representation of a linear collineation group $M \cong S_6$ preserving Q is

$$\sum_{i=1}^{6} x_i^2 \; - \sum_{1\leq j<h\leq 6} x_j x_h = 0 \tag{2.1}$$

(see [1]). In fact, the action of G on the six vertices of the fundamental simplex is the natural representation of S_6 of degree 6, and the subgroup $G \cong A_6$ is identified with those elements in H which induce even permutations. Therefore, the problem of classifying all A_6-invariant spreads of $PG(3,q)$ is equivalent to the problem of classifying all G-invariant ovoids of the Klein quadric with equation (2.1). The collineations associated to the matrices

$$g_1 = \begin{bmatrix} 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{bmatrix}$$

preserve the quadric Q and they generate the group G isomorphic to A_6.

Every G-invariant ovoid is partitioned in caps of Q, each being a point-orbit under the action of G. Ostrom pointed out that one of these G-orbit caps has size 20. In our representation, this G-orbit cap K_1 consists of all points with three 0 and three 1 entries; later the cap K_1 will be called the "compulsory 20-cap". For the other G-orbit caps in an ovoid, we do not have any compulsory choice.

According to [1], all G-invariant ovoids of Q arise in the form $K_1 \cup K_2 \cup \cdots \cup K_s$ where

(i) $K_1, K_2, \ldots, K_s$ are G-orbits,
(ii) the sum of their lengths is equal to $1 + q^2$,
(iii) any two caps K_i, K_j are consistent, that is, $K_i \cup K_j$ is still a cap on Q.

The procedure for the construction of the G-orbit caps consists of four steps. For a non-trivial subgroup H of G, let $F(H)$ be the set of all points $P \in Q$ whose stabilizer G_P in G coincides with H.

Step 1. Take a representative H for each conjugacy class of subgroups of G and determine $F(H)$.

Step 2. Take an element g from each coset of H in G and select those points $P \in F(H)$ which are not orthogonal to any $g(P)$. This ensures that the G-orbit $\langle P \rangle$ of P is a cap on Q. Among the selected points, there might be distinct points P and P' such that $\langle P \rangle = \langle P' \rangle$. Obviously, such duplicates ought to be eliminated. So, a set $C(H) \subseteq F(H)$ arises such that the G-orbits $\{\langle P \rangle \mid P \in C(H)\}$ are pairwise distinct and are caps on Q. Let $C = \cup C(H)$. Then the sets $\{\langle P \rangle \mid P \in C\}$ are exactly the G-orbit caps on Q.

Step 3. Find all maximal sets of pairwise consistent G-orbit caps $\{\langle P \rangle \mid P \in C\}$ such that (ii) holds. They are all pairwise distinct G-invariant ovoids on Q.

Step 4. Determine the stabilizer of each G-invariant ovoid of Q in the subgroup $W \cong PGL(4, q)$ of index two of $PGO_+(6, q)$, leaving invariant both maximal families of singular planes. Two G-invariant ovoids O and O' of Q belong to the same isomorphism class if they have the same stabilizer S in W and there exists an element $h \notin S$ in the normalizer $N(S)$ of S in W that takes O to O'. Here $N(S)$ is the unique group isomorphic to S_6 containing G (see [1, Section 2]).

3. The conjugacy classes of A_6-isomorphic subgroups preserving the Klein quadric

3.1. **The general case.** To obtain all G-orbit caps on the Klein quadric Q, take a representative subgroup H for each conjugacy class of G and determine the set $F(H)$ consisting of all points $P \in Q$ whose stabilizer G_P is just H. Clearly, the G-orbit $\langle P \rangle$ of the point $P \in F(H)$ has length

$k = [G : H]$ and it consists of points S whose stabilizer G_S is a conjugate of H under G.

Here we consider all A_6-isomorphic subgroups. There are 22 conjugacy classes including the trivial one, therefore we also consider caps of length 360. The selected subgroups are isomorphic to A_5, S_4, A_4, S_3, $C_2 \times C_2$, C_3 (two classes for each type), and A_6, $(C_3 \times C_3) \rtimes C_4$, $(C_3 \times C_3) \rtimes C_2$, D_5, $C_3 \times C_3$, D_4, C_5, C_4, C_2, 1 (one class for each type).

According to Step 1, a representative H is listed in Table 2 for each isomorphism class of subgroups of G, and all points $P \in Q$ fixed by H. Here, H may happen to be a proper subgroup of the stabilizer G_P. We have to delete the fixed points P from Table 2 which do not generate a G-orbit cap.

We find only two points without parameters: namely $P_1(1, 1, 1, 0, 0, 0)$ and $P_2(0, 0, 0, 1, 1, 1)$; these points belong to the compulsory 20-cap K_1 mentioned in Section 2. The remaining points in Table 2 have coordinates depending on one, two, or three parameters satisfying the conditions given in Table 3.

3.2. **The case $q = 23$.** In $GF(23)$, some equations in Table 2 have no solution (e.g. $a^2 + 1 = 0$); therefore Table 2 can be refined. This is done in Table 4.

4. Step 1

From Table 2, some conjugacy classes (namely 22, 21, 19, 18 and 15) have no fixed point, while others (namely 20, 13 and 8) are inconsistent, since the related second degree equations have no solution in $GF(23)$.

Class 17 is represented by a group isomorphic to S_4, and has exactly two fixed points, $P_3(1, 1, 1, 1, 12, 12)$ and $P_4(1, 1, 1, 1, 19, 19)$. Both P_3 and P_4 have G-orbits of length 15, and these G-orbits are different.

Class 16 is represented by a group isomorphic to $(C_3 \times C_3) \rtimes C_2$, and has exactly two fixed points, $P_1(1, 1, 1, 0, 0, 0)$ and $P_2(0, 0, 0, 1, 1, 1)$. The G-orbit of the points P_1 and P_2 is the compulsory orbit K_1 of length 20.

Class 14 is represented by a group isomorphic to A_4, and has exactly twenty-four fixed points $P(1, 1, 1, 1, a, b)$ satisfying the condition $a^2 + b^2 - ab - 4a - 4b - 2 = 0$. In fact, the other possibility, namely $(0, 0, 0, 0, 1, a)$ with $a^2 - a + 1 = 0$, does not actually occur since $a^2 - a + 1 = 0$ has no solution in $GF(23)$. Two of these twenty-four points are P_3 and P_4, and therefore their stabilizer is S_4 in class 17. The remaining twenty-two points have G-orbits of length 30, and therefore A_4 is their stabilizer in A_6. Their G-orbits are pairwise equal, and we should consider eleven G-orbits of length 30.

Classes 12, 10 and 4 are represented by groups isomorphic, respectively, to $C_3 \times C_3$, S_3 and C_3. They have exactly two fixed points, P_1 and P_2, which are the points of Ostrom.

num	**ord**	**type**	**generators**	**fixed points**	**cond**
22	360	A_6	(12345), (16)(23)	no fixed point	
21	60	A_5	(12346), (14)(56)	no fixed point	
20	60	A_5	(12345), (123), (12)(34)	$(1,1,1,1,1,a)$	(20)
19	36	$(C_3 \times C_3) \rtimes C_4$	(456), (123), (23)(56), (14)(2536)	no fixed point	
18	24	S_4	(135)(246), (12)(56), (34)(56), (35)(46)	no fixed point	
17	24	S_4	(123), (1324)(56)	$(1,1,1,1,a,a)$	(17)
16	18	$(C_3 \times C_3) \rtimes C_2$	(123), (456), (12)(56)	$P_1(1,1,1,0,0,0)$ $P_2(0,0,0,1,1,1)$	
15	12	A_4	(135)(246), (12)(34), (34)(56)	no fixed points	
14	12	A_4	(123), (12)(34), (14)(23)	$(1,1,1,1,a,b)$ $(0,0,0,0,1,a)$	(14.1) (14.2)
13	10	D_5	(23456), (36)(45)	$(a,1,1,1,1,1)$	(13)
12	9	$C_3 \times C_3$	(123), (456)	$(1,a,a^2,0,0,0)$ $(0,0,0,1,a,a^2)$ P_1, P_2	(12) (12)
11	8	D_4	(12)(56), (35)(46), (34)(56)	$(1,1,a,a,a,a)$ $(1,1,a,a,-a,-a)$	(11.1) (11.2)
10	6	S_3	(123)(456), (12)(45)	P_1, P_2	
9	6	S_3	(123), (12)(45)	(1,1,1,a,a,b)	(9)
8	5	C_5	(12345)	$(1,a,a^2,a^3,a^4,0)$ $(1,1,1,1,1,a)$	(8.1) (8.2)
7	4	C_4	(1234)(56)	$(1,-1,1,-1,a,-a)$ $(1,1,1,1,a,a)$	(7.1) (7.2)
6	4	$C_2 \times C_2$	(12)(34), (12)(56)	$(1,1,a,a,b,b)$ $(0,0,1,1,a,a)$	(6.1) (6.2)
5	4	$C_2 \times C_2$	(12)(34), (13)(24)	$(1,1,1,1,a,b)$ $(0,0,0,0,1,a)$	(5.1) (5.2)
4	3	C_3	(123)(456)	P_1, P_2	
3	3	C_3	(123)	$(1,1,1,a,b,c)$ $(0,0,0,1,a,b)$ $(0,0,0,0,1,a)$	(3.1) (3.2) (3.3)
2	2	C_2	(12)(34)	$(1,1,a,a,b,c)$ $(1,-1,a,-a,0,0)$ $(0,0,1,1,a,b)$ $(0,0,0,0,1,a)$	(2.1) (2.2) (2.3) (2.4)
1	1		()	each point is fixed	

TABLE 2. General case

(20)	$a^2-5a-5=0$	(7.2)	$a^2-8a-2=0$
(17)	$a^2-8a-2=0$	(6.1)	$a^2+b^2-4ab-4a-4b+1=0$
(14.1)	$a^2+b^2-ab-4a-4b-2=0$	(6.2)	$a^2-4a+1=0$
(14.2)	$a^2-a+1=0$	(5.1)	$a^2+b^2-ab-4a-4b-2=0$
(13)	$a^2-5a-5=0$	(5.2)	$a^2-a+1=0$
(12)	$a^3-1=0$	(3.1)	$a^2+b^2+c^2-ab-ac-bc-3a-3b-3c=0$
(11.1)	$2a^2+8a-1=0$	(3.2)	$a^2+b^2-ab-a-b+1=0$
(11.2)	$8a^2+1=0$	(3.3)	$a^2-a+1=0$
(9)	$a^2+b^2-2ab-6a-3b=0$	(2.1)	$a^2+b^2+c^2-2ab-2ac-bc-4a-2b-2c+1=0$
(8.1)	$a^4+a^3+a^2+a+1=0$	(2.2)	$a^2+1=0$
(8.2)	$a^2-5a-5=0$	(2.3)	$a^2+b^2-ab-2a-2b+1=0$
(7.1)	$a^2+2=0$	(2.4)	$a^2-a+1=0$

TABLE 3. Conditions on parameters in Table 2

Class 11 is represented by a group isomorphic to the dihedral group D_4, and has exactly two fixed points, $P_7(1,1,2,2,2,2)$ and $P_8(1,1,17,17,17,17)$. Both these points belong to the point orbits of P_3 and P_4 of class 17.

Class 9 is represented by a group isomorphic to S_3, and has exactly twenty-three fixed points, $P(1,1,1,a,a,b)$, satisfying the condition $a^2+b^2-2ab-6a-3b=0$. Besides the compulsory cap K_1, there are two G-orbits of length 15, and twenty G-orbits of length 60.

Class 7 is represented by a group isomorphic to C_4, and its fixed points are P_3 and P_4 which were already considered in class 17. In fact, the other possibility, namely $(1,1,1,1,a,a)$ with $a^2+2=0$, does not actually occur since $a^2+2=0$ has no solution in $GF(23)$.

Class 6 is represented by a group isomorphic to $C_2\times C_2$. Its fixed points are $P_5(0,0,1,1,9,9)$ and $P_6(0,0,1,1,18,18)$, together with twenty-two further points, $P(1,1,a,a,b,b)$, satisfying the condition $a^2+b^2-4ab-4a-4b+1=0$. The points P_5 and P_6 give rise to distinct G-orbits of length 90. The remaining twenty-two fixed points determine six G-orbits of length 15 (the examined group is not the stabilizer), and sixteen G-orbits of length 90. These latter sixteen G-orbits are pairwise equal. Therefore, class 6 gives ten G-orbits of length 90, but they reduce to three after the normalization of the coordinates.

Class 5 is represented by another group isomorphic to $C_2\times C_2$, and the results are those already obtained from class 14.

Class 3 is represented by a group isomorphic to C_3, and besides the Ostrom point P_2 (the condition (3.2) gives a reducible conic consisting of the single point P_2), has five hundred twenty-nine fixed points, $P(1,1,1,a,b,c)$, satisfying the condition $a^2+b^2+c^2-ab-ac-bc-3a-3b-3c=0$. These points give one hundred different G-orbits; aside from the compulsory cap, there are two G-orbits of length 15, eleven G-orbits of length 30, twenty G-orbits of length 60, and sixty-six G-orbits of length 120.

num	**ord**	**type**	**generators**	**fixed points**
22	360	A_6	(12345), (16)(23)	no fixed points
21	60	A_5	(12346), (14)(56)	no fixed points
20	60	A_5	(12345), (123), (12)(34)	no fixed points
19	36	$(C_3 \times C_3) \rtimes C_4$	(456), (123), (23)(56), (14)(2536)	no fixed points
18	24	S_4	(135)(246), (12)(56), (34)(56), (35)(46)	no fixed points
17	24	S_4	(123),(1324)(56)	$P_3(1,1,1,1,12,12)$ $P_4(1,1,1,1,19,19)$
16	18	$(C_3 \times C_3) \rtimes C_2$	(123),(456),(12)(56)	$P_1(1,1,1,0,0,0)$ $P_2(0,0,0,1,1,1)$
15	12	A_4	(135)(246), (12)(34), (34)(56)	no fixed points
14	12	A_4	(123), (12)(34), (14)(23)	24 points (14.1)
13	10	D_5	(23456), (36)(45)	no fixed points
12	9	$C_3 \times C_3$	(123), (456)	P_1, P_2
11	8	D_4	(12)(56), (35)(46), (34)(56)	$P_7(1,1,2,2,2,2)$ $P_8(1,1,17,17,17,17)$
10	6	S_3	(123)(456), (12)(45)	P_1, P_2
9	6	S_3	(123), (12)(45)	23 points (9)
8	5	C_5	(12345)	no fixed points
7	4	C_4	(1234)(56)	P_3, P_4
6	4	$C_2 \times C_2$	(12)(34), (12)(56)	22 points (6.1) $P_5(0,0,1,1,9,9)$ $P_6(0,0,1,1,18,18)$
5	4	$C_2 \times C_2$	(12)(34), (13)(24)	24 points (14.1)
4	3	C_3	(123)(456)	P_1, P_2
3	3	C_3	(123)	529 points (3.1) $P_2(0,0,0,1,1,1)$ (3.2)
2	2	C_2	(12)(34)	506 points (2.1) 24 points (2.3)
1	1		()	each point is fixed

Table 4. The case $q = 23$

Class 2 is represented by a group isomorphic to C_2, and has twenty-four fixed points, $P(0,0,1,1,a,b)$, satisfying the condition $a^2+b^2-ab-2a-2b+1=0$, and five hundred and six fixed points, $R(1,1,a,a,b,c)$, satisfying the condition $a^2+b^2+c^2-2ab-2ac-bc-4a-2b-2c+1=0$. The former case gives thirteen G-orbits; aside from the compulsory cap, there is one G-orbit of length 60, two G-orbits of length 90, and nine G-orbits of length 180. The latter case gives two hundred fifty-four G-orbits; aside from the compulsory cap, there are four G-orbits of length 15, eleven G-orbits of length 30, thirty-nine G-orbits of length 60, eight G-orbits of length 90, and one hundred ninety-one G-orbits of length 180.

Class 1 is represented by the trivial group, and obviously in this case, each point of the Klein quadric in $PG(5, 23)$ is a fixed point. The number of points in the Klein quadric of $PG(5, 23)$ is 293 090; they split into four G-orbits of length 15, one G-orbit of length 20 (the compulsory orbit), thirty-one G-orbits of length 30, fifty-eight G-orbits of length 60, eight G-orbits of length 90, two hundred fifty-six G-orbits of length 120, three hundred eighty-three G-orbits of length 180, and three thousand nine hundred two G-orbits of length 360. Actually, these G-orbits partition the Klein quadric and we look for caps among them.

As a conclusion of this section, we obtain exactly eight sporadic points, four families of points whose coordinates satisfy a condition of degree 2 in two parameters (these can be read as conics in an affine plane), and two families of points whose coordinates satisfy a condition of degree 2 in three parameters (these can be read as quadrics in an affine space). Besides class 1, we obtain exactly one thousand one hundred thirty-six fixed points with respect to the 21 conjugacy classes. Each of these points gives a G-orbit, but some of these orbits coincide; there are precisely three hundred twenty pairwise distinct orbits.

5. Step 2

The computation in Step 1 provides three hundred twenty G-orbits, and only eighty-three of them are caps. We will denote them by $K_1, \dots, K_{83}$. Precisely, we obtain

- two caps of length 15,
- one cap of length 20 - the compulsory cap,
- eleven caps of length 30,
- sixteen caps of length 60,
- three caps of length 90,
- thirty-seven caps of length 120,
- thirteen caps of length 180.

Actually, some more computation shows that class 1 consists of these eighty-three caps together with another one of length 360. We enumerate these caps and give their length and one generator in the Table 5. We recall that these caps are also G-orbits.

name	length	point	name	length	point	name	length	point
K_1	20	(0,0,0,1,1,1)	K_{29}	120	(1,1,1,5,13,7)	K_{57}	120	(1,1,1,17,16,9)
K_2	60	(0,0,1,1,1,3)	K_{30}	120	(1,1,1,5,13,14)	K_{58}	60	(1,1,1,17,19,19)
K_3	180	(0,0,1,1,20,17)	K_{31}	60	(1,1,1,2,2,8)	K_{59}	120	(1,1,1,16,11,9)
K_4	90	(0,0,1,1,9,9)	K_{32}	60	(1,1,1,2,2,22)	K_{60}	60	(1,1,1,11,18,18)
K_5	30	(0,1,1,1,1,13)	K_{33}	120	(1,1,1,2,4,19)	K_{61}	120	(1,1,1,11,19,12)
K_6	30	(0,1,1,1,1,14)	K_{34}	120	(1,1,1,2,8,11)	K_{62}	120	(1,1,1,11,7,12)
K_7	120	(0,1,1,1,5,16)	K_{35}	120	(1,1,1,2,13,14)	K_{63}	120	(1,1,1,18,13,7)
K_8	120	(0,1,1,1,10,17)	K_{36}	60	(1,1,1,2,3,3)	K_{64}	120	(1,1,1,18,15,6)
K_9	120	(0,1,1,1,16,14)	K_{37}	120	(1,1,1,2,6,14)	K_{65}	120	(1,1,1,18,6,12)
K_{10}	120	(0,1,1,1,13,15)	K_{38}	60	(1,1,1,2,7,7)	K_{66}	60	(1,1,1,3,3,7)
K_{11}	120	(0,1,1,1,19,12)	K_{39}	60	(1,1,1,10,10,11)	K_{67}	120	(1,1,1,3,6,12)
K_{12}	120	(0,1,1,1,3,6)	K_{40}	120	(1,1,1,10,17,7)	K_{68}	60	(1,1,1,3,12,12)
K_{13}	60	(0,1,1,1,6,6)	K_{41}	120	(1,1,1,10,16,15)	K_{69}	60	(1,1,1,15,6,6)
K_{14}	30	(1,1,1,1,5,16)	K_{42}	120	(1,1,1,10,16,14)	K_{70}	90	(1,1,5,5,12,12)
K_{15}	30	(1,1,1,1,4,19)	K_{43}	120	(1,1,1,10,11,14)	K_{71}	180	(1,1,5,20,20,22)
K_{16}	30	(1,1,1,1,4,12)	K_{44}	120	(1,1,1,10,3,7)	K_{72}	180	(1,1,5,8,8,3)
K_{17}	30	(1,1,1,1,8,17)	K_{45}	120	(1,1,1,4,20,6)	K_{73}	180	(1,1,5,22,15,15)
K_{18}	30	(1,1,1,1,8,18)	K_{46}	120	(1,1,1,4,8,17)	K_{74}	180	(1,1,5,3,12,12)
K_{19}	30	(1,1,1,1,17,13)	K_{47}	120	(1,1,1,4,17,6)	K_{75}	180	(1,1,2,17,7,7)
K_{20}	30	(1,1,1,1,16,15)	K_{48}	60	(1,1,1,20,20,8)	K_{76}	360	(1,1,2,22,19,19)
K_{21}	30	(1,1,1,1,18,14)	K_{49}	60	(1,1,1,20,20,12)	K_{77}	180	(1,1,10,10,19,19)
K_{22}	15	(1,1,1,1,19,19)	K_{50}	120	(1,1,1,20,18,11)	K_{78}	90	(1,1,10,10,15,6)
K_{23}	30	(1,1,1,1,3,15)	K_{51}	120	(1,1,1,20,17,19)	K_{79}	180	(1,1,10,20,20,13)
K_{24}	15	(1,1,1,1,12,12)	K_{52}	120	(1,1,1,20,11,3)	K_{80}	180	(1,1,4,4,11,13)
K_{25}	120	(1,1,1,5,2,3)	K_{53}	120	(1,1,1,20,3,15)	K_{81}	180	(1,1,8,17,17,15)
K_{26}	120	(1,1,1,5,9,3)	K_{54}	120	(1,1,1,20,15,12)	K_{82}	180	(1,1,17,17,19,7)
K_{27}	120	(1,1,1,5,9,14)	K_{55}	120	(1,1,1,8,9,7)	K_{83}	180	(1,1,16,11,22,22)
K_{28}	60	(1,1,1,5,18,18)	K_{56}	60	(1,1,1,17,17,7)	K_{84}	180	(1,1,2,9,19,14)

TABLE 5. The case $q = 23$

6. STEP 3 - CONSISTENT CAPS

We are looking for ovoids on the Klein quadric in $PG(5,23)$ which are left invariant by the group generated by the collineations associated to the matrices g_1 and g_2 mentioned in Section 2.

Any ovoid of $PG(5,23)$ consists of five hundred thirty points and each ovoid must contain the compulsory 20-cap; therefore we are looking for the remaining five hundred ten points. With this aim in mind we should check that the union of two caps is still a cap, i.e. the caps are "consistent". We remind the reader that if A, B, C are three pairwise consistent caps, then the union $A \cup B \cup C$ is still a cap.

In the present case, all available caps are consistent with the compulsory cap K_1. No ovoid can contain a single cap of length 15. The cap K_{75} is consistent with no cap apart from K_1. Tables 6 and 7 give the adjancency list, with each cap being indicated by its label.

K_2	4, 5, 6, 10, 11, 12, 13, 14, 15, 17, 18, 20, 21, 22, 23, 24, 25, 26, 28, 30, 32, 34, 36, 42, 43, 48, 49, 50, 52, 53, 56, 60, 62, 66, 67, 68, 69, 70, 73, 79, 82
K_3	5, 6, 7, 13, 14, 15, 18, 21, 22, 39, 46, 68, 69
K_4	5, 6, 7, 10, 13, 16, 17, 18, 19, 22, 23, 24, 27, 31, 32, 34, 36, 39, 41, 45, 48, 53, 55, 57, 58, 60, 68, 70, 71, 78, 84
K_5	6, 7, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 41, 42, 44, 45, 48, 49, 50, 51, 52, 56, 58, 59, 61, 63, 64, 65, 66, 67, 68, 71, 72, 80, 82
K_6	7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 52, 54, 55, 57, 58, 59, 60, 61, 65, 66, 67, 70, 71, 74, 75, 78, 83
K_7	15, 16, 17, 18, 19, 20, 21, 24, 31, 32, 47, 49, 56, 60, 69, 74
K_8	12, 13, 14, 16, 20, 22, 23, 24, 25, 28, 30, 32, 36, 37, 41, 42, 58, 59, 60, 66, 78
K_9	10, 14, 15, 18, 19, 20, 22, 28, 32, 36, 38, 41, 47, 66, 68, 69, 70
K_{10}	14, 18, 19, 20, 21, 23, 24, 32, 35, 45, 50, 58, 63, 68
K_{11}	15, 21, 22, 24, 26, 31, 43, 46, 47, 49, 51, 54, 55, 56, 58, 60, 69, 78
K_{12}	16, 18, 19, 23, 33, 40, 48, 49, 54, 66, 68, 69, 70
K_{13}	14, 15, 16, 17, 18, 19, 20, 21, 24, 32, 33, 35, 36, 38, 40, 43, 45, 48, 50, 51, 53, 54, 56, 57, 59, 60, 62, 64, 66, 77, 78, 84
K_{14}	15, 16, 17, 18, 19, 21, 22, 23, 24, 28, 29, 31, 32, 33, 34, 35, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 51, 53, 54, 55, 56, 57, 60, 62, 64, 65, 66, 67, 68, 69, 70, 71, 72, 79, 83
K_{15}	16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 39, 40, 41, 44, 45, 46, 48, 49, 50, 51, 52, 53, 56, 58, 59, 60, 61, 62, 63, 64, 66, 68, 69, 71, 73, 74, 75, 78, 80, 83
K_{16}	17, 18, 19, 20, 21, 22, 23, 24, 26, 28, 29, 30, 31, 32, 33, 35, 36, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 56, 57, 60, 61, 62, 63, 64, 67, 68, 69, 72, 73, 75, 76, 82
K_{17}	18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 35, 36, 37, 39, 40, 41, 42, 43, 45, 46, 47, 49, 50, 51, 54, 56, 57, 59, 62, 63, 65, 66, 67, 68, 69, 70, 73, 74, 77, 78, 79, 80, 81, 82, 83
K_{18}	19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 31, 34, 35, 36, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 52, 54, 55, 56, 58, 59, 60, 61, 62, 64, 65, 66, 68, 70, 73, 74, 78, 79, 80, 83
K_{19}	20, 21, 22, 24, 25, 26, 28, 29, 30, 32, 34, 3537, 38, 39, 40, 41, 43, 44, 46, 48, 49, 51, 52, 53, 55, 56, 58, 59, 60, 62, 64, 65, 66, 67, 68, 69, 70, 72, 77, 78, 80, 81, 82, 83
K_{20}	21, 22, 23, 24, 26, 27, 28, 29, 31, 32, 33, 34, 36, 38, 39, 40, 42, 43, 44, 46, 49, 50, 53, 54, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 77, 78, 80, 81, 83, 84
K_{21}	22, 23, 24, 25, 28, 29, 30, 31, 33, 36, 37, 38, 39, 41, 42, 43, 44, 45, 48, 50, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 65, 66, 67, 68, 69, 70, 76, 79, 81, 84
K_{22}	23, 24, 26, 27, 29, 30, 31, 32, 33, 35, 36, 37, 38, 39, 40, 41, 43, 44, 45, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 73, 74, 77, 78, 79, 80, 81, 82, 83, 84
K_{23}	24, 26, 27, 28, 30, 31, 32, 33, 34, 35, 37, 38, 40, 41, 43, 44, 45, 46, 47, 48, 49, 51, 52, 53, 54, 55, 56, 58, 59, 60, 62, 63, 64, 66, 67, 68, 69, 70, 71, 72, 77, 78, 80, 84
K_{24}	25, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 46, 47, 48, 49, 5053, 54, 57, 58, 60, 61, 63, 64, 65, 66, 67, 68, 69, 70, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83
K_{25}	26, 31, 32, 36, 44, 48, 5051, 57, 58, 64, 77
K_{26}	28, 29, 32, 33, 38, 42, 44, 60, 63, 69, 74
K_{27}	28, 29, 38, 40, 44, 55, 56, 68, 69
K_{28}	29, 31, 33, 38, 41, 45, 48, 49, 51, 58, 61, 62, 65, 68, 69, 70, 74, 78
K_{29}	37, 46, 48, 56, 66, 78
K_{30}	35, 38, 39, 45, 54, 55, 56, 60, 66, 69, 79

TABLE 6. The adjacency list (part 1)

K_{31}	32, 34, 35, 36, 38, 39, 40, 42, 43, 48, 50, 53, 60, 64, 66, 67, 77, 78, 79
K_{32}	33, 36, 38, 39, 41, 45, 49, 51, 52, 54, 55, 56, 58, 61, 66, 69, 70, 79, 81, 82
K_{33}	48, 50, 52, 56, 58, 66
K_{34}	36, 37, 39, 43, 48, 49, 52, 69
K_{35}	36, 39, 42, 44, 57, 58, 63, 65, 69
K_{36}	38, 40, 45, 46, 47, 48, 54, 55, 62, 63, 65, 66, 67, 68, 69, 71, 72, 82
K_{37}	61, 62, 66, 68, 69
K_{38}	39, 49, 50, 55, 56, 57, 58, 59, 60, 63, 64, 66, 69, 70, 73, 74, 81, 83, 84
K_{39}	40, 41, 42, 46, 47, 49, 52, 55, 58, 59, 66, 68, 69, 70, 72, 74, 75, 78
K_{40}	44, 48, 64, 70
K_{41}	43, 56, 60
K_{42}	45, 53, 56, 58, 60, 84
K_{43}	58, 68, 70
K_{44}	65, 78
K_{45}	49, 58, 60, 69, 73
K_{46}	48, 56, 57, 58, 59, 68, 81
K_{47}	49, 53, 64
K_{48}	49, 54, 56, 57, 58, 60, 63, 66, 68, 69, 73, 75, 77, 80
K_{49}	50, 53, 54, 55, 56, 58, 60, 63, 64, 66, 68, 69, 71, 74, 78, 79, 83, 84
K_{50}	55, 56, 60
K_{51}	58
K_{52}	57, 60, 65, 67, 70
K_{53}	58
K_{54}	58, 65, 67, 68, 82
K_{55}	66
K_{56}	58, 61, 63, 65, 67, 69, 70, 72, 74, 75, 78
K_{57}	58, 65, 66, 69, 78, 82
K_{58}	62, 66, 67, 70, 79
K_{59}	60, 63, 64, 68, 69, 72, 73
K_{60}	61, 63, 65, 66, 67, 68, 70, 82
K_{61}	66, 70
K_{62}	66, 70, 72, 77
K_{63}	65, 70, 78, 79
K_{64}	78
K_{65}	68, 69
K_{66}	67, 68, 71, 75, 77, 78, 80, 82
K_{67}	69, 70, 81
K_{68}	70, 72, 77, 78, 84
K_{69}	70, 71, 79
K_{70}	71, 75, 78
K_{71}	79, 82
K_{73}	84
K_{75}	81
K_{77}	79, 82

Table 7. The adjacency list (part 2)

According to Tables 6 and 7, we obtain forty ovoids $\mathcal{O}_j$, as union of caps K_i, which are listed in Table 8.

7. Step 4

Isomorphic translation planes arise from isomorphic ovoids, and the converse also holds. Therefore, it is of interest to find a complete list of pairwise non-isomorphic ovoids. From an exhaustive computer aided investigation,

(1) the two caps of length 15 are isomorphic,
(2) the ten caps of length 30, apart from one of them, are pairwise isomorphic,
(3) the sixteen caps of length 60 are pairwise isomorphic,
(4) two of the three caps of length 90 are isomorphic,
(5) the thirty-six caps of length 120, apart from one of them, are pairwise isomorphic,
(6) the twelve caps of length 180, apart one of them, are pairwise isomorphic.

At this stage we have found twenty-three pairwise non-isomorphic ovoids, namely those listed in Table 9.

8. Groups preserving the ovoids

All the ovoids obtained in this paper are G-invariant with $G \cong A_6$. A computer aided investigation shows that a few of them, namely

$$\mathcal{O}_{25}, \mathcal{O}_{27}, \mathcal{O}_{28}, \mathcal{O}_{30}, \mathcal{O}_{32}, \mathcal{O}_{40},$$

have a larger collineation group. So each of these six ovoids, and the corresponding spreads, are S_6-invariant.

9. The binary ovoids of Moorhouse

Our terminology and background on root systems, their Weyl groups, and lattices come from [7, Section 2]. Consider the root lattice defined by:

$$E = \left\{ \tfrac{1}{2}(a_1, \ldots, a_8) \mid a_i \in \mathbb{Z},\, a_1 \equiv \cdots \equiv a_8 \,(\mathrm{mod}\, 2),\, \sum \mathrm{a_i} \equiv 0 \,(\mathrm{mod}\, 4) \right\}.$$

The vectors in E form a $\mathbb{Z}$-module. The root vectors of E are the 240 vectors $\mathbf{e} \in E$ satisfying the condition $||\mathbf{e}||^2/2 = 1$, and we can choose among them a system of fundamental roots

$\mathcal{O}_1$	K_1	K_5	K_{36}	K_{66}	K_{71}	K_{81}							
$\mathcal{O}_2$	K_1	K_{14}	K_{49}	K_{69}	K_{71}	K_{78}							
$\mathcal{O}_3$	K_1	K_2	K_6	K_{23}	K_{52}	K_{60}	K_{67}	K_{70}					
$\mathcal{O}_4$	K_1	K_3	K_{14}	K_{15}	K_{18}	K_{39}	K_{46}	K_{68}					
$\mathcal{O}_5$	K_1	K_4	K_6	K_{10}	K_{23}	K_{32}	K_{45}	K_{58}					
$\mathcal{O}_6$	K_1	K_5	K_{19}	K_{20}	K_{39}	K_{59}	K_{68}	K_{72}					
$\mathcal{O}_7$	K_1	K_7	K_{15}	K_{17}	K_{18}	K_{49}	K_{56}	K_{74}					
$\mathcal{O}_8$	K_1	K_{13}	K_{17}	K_{19}	K_{20}	K_{62}	K_{66}	K_{76}					
$\mathcal{O}_9$	K_1	K_{17}	K_{22}	K_{24}	K_{49}	K_{63}	K_{77}	K_{78}					
$\mathcal{O}_{10}$	K_1	K_{17}	K_{22}	K_{24}	K_{57}	K_{66}	K_{77}	K_{81}					
$\mathcal{O}_{11}$	K_1	K_2	K_{17}	K_{21}	K_{22}	K_{24}	K_{30}	K_{69}	K_{78}				
$\mathcal{O}_{12}$	K_1	K_5	K_{17}	K_{22}	K_{24}	K_{35}	K_{36}	K_{63}	K_{65}				
$\mathcal{O}_{13}$	K_1	K_6	K_{19}	K_{21}	K_{30}	K_{38}	K_{39}	K_{55}	K_{66}				
$\mathcal{O}_{14}$	K_1	K_{12}	K_{16}	K_{18}	K_{23}	K_{48}	K_{49}	K_{54}	K_{68}				
$\mathcal{O}_{15}$	K_1	K_{14}	K_{17}	K_{22}	K_{24}	K_{35}	K_{57}	K_{65}	K_{69}				
$\mathcal{O}_{16}$	K_1	K_{16}	K_{17}	K_{22}	K_{24}	K_{32}	K_{36}	K_{54}	K_{81}				
$\mathcal{O}_{17}$	K_1	K_2	K_5	K_{12}	K_{18}	K_{23}	K_{48}	K_{49}	K_{66}	K_{68}			
$\mathcal{O}_{18}$	K_1	K_2	K_5	K_{17}	K_{22}	K_{24}	K_{32}	K_{36}	K_{66}	K_{81}			
$\mathcal{O}_{19}$	K_1	K_2	K_{14}	K_{17}	K_{21}	K_{23}	K_{56}	K_{67}	K_{69}	K_{70}			
$\mathcal{O}_{20}$	K_1	K_2	K_{14}	K_{17}	K_{22}	K_{24}	K_{32}	K_{49}	K_{69}	K_{78}			
$\mathcal{O}_{21}$	K_1	K_4	K_5	K_6	K_{13}	K_{16}	K_{17}	K_{32}	K_{36}	K_{45}			
$\mathcal{O}_{22}$	K_1	K_5	K_{13}	K_{18}	K_{19}	K_{20}	K_{21}	K_{38}	K_{59}	K_{64}			
$\mathcal{O}_{23}$	K_1	K_5	K_{14}	K_{16}	K_{17}	K_{18}	K_{31}	K_{35}	K_{39}	K_{42}			
$\mathcal{O}_{24}$	K_1	K_5	K_{14}	K_{17}	K_{19}	K_{21}	K_{28}	K_{51}	K_{65}	K_{68}			
$\mathcal{O}_{25}$	K_1	K_5	K_{14}	K_{18}	K_{19}	K_{22}	K_{24}	K_{35}	K_{44}	K_{65}			
$\mathcal{O}_{26}$	K_1	K_6	K_{14}	K_{19}	K_{32}	K_{38}	K_{39}	K_{49}	K_{55}	K_{66}			
$\mathcal{O}_{27}$	K_1	K_6	K_{15}	K_{20}	K_{22}	K_{23}	K_{24}	K_{27}	K_{40}	K_{44}			
$\mathcal{O}_{28}$	K_1	K_{13}	K_{15}	K_{16}	K_{17}	K_{20}	K_{21}	K_{33}	K_{50}	K_{56}			
$\mathcal{O}_{29}$	K_1	K_{14}	K_{15}	K_{16}	K_{18}	K_{19}	K_{29}	K_{46}	K_{48}	K_{56}			
$\mathcal{O}_{30}$	K_1	K_{15}	K_{17}	K_{18}	K_{19}	K_{20}	K_{39}	K_{46}	K_{59}	K_{68}			
$\mathcal{O}_{31}$	K_1	K_2	K_{13}	K_{15}	K_{17}	K_{18}	K_{20}	K_{21}	K_{36}	K_{62}	K_{66}		
$\mathcal{O}_{32}$	K_1	K_4	K_{17}	K_{18}	K_{19}	K_{22}	K_{24}	K_{39}	K_{68}	K_{70}	K_{77}		
$\mathcal{O}_{33}$	K_1	K_7	K_{15}	K_{16}	K_{17}	K_{19}	K_{20}	K_{32}	K_{49}	K_{56}	K_{69}		
$\mathcal{O}_{34}$	K_1	K_{15}	K_{16}	K_{17}	K_{19}	K_{22}	K_{24}	K_{30}	K_{35}	K_{39}	K_{69}		
$\mathcal{O}_{35}$	K_1	K_{17}	K_{18}	K_{20}	K_{21}	K_{22}	K_{24}	K_{36}	K_{54}	K_{65}	K_{68}		
$\mathcal{O}_{36}$	K_1	K_2	K_{14}	K_{17}	K_{18}	K_{21}	K_{22}	K_{23}	K_{24}	K_{43}	K_{68}	K_{70}	
$\mathcal{O}_{37}$	K_1	K_4	K_5	K_6	K_{16}	K_{17}	K_{19}	K_{22}	K_{24}	K_{32}	K_{39}	K_{41}	
$\mathcal{O}_{38}$	K_1	K_2	K_{14}	K_{15}	K_{18}	K_{22}	K_{23}	K_{24}	K_{48}	K_{49}	K_{60}	K_{66}	K_{68}
$\mathcal{O}_{39}$	K_1	K_5	K_6	K_{19}	K_{20}	K_{22}	K_{24}	K_{32}	K_{38}	K_{39}	K_{49}	K_{58}	K_{66}
$\mathcal{O}_{40}$	K_1	K_{15}	K_{17}	K_{18}	K_{19}	K_{20}	K_{22}	K_{24}	K_{39}	K_{49}	K_{66}	K_{68}	K_{77}

TABLE 8. The case $q = 23$

$\mathcal{O}_1$	K_1	K_5	K_{36}	K_{66}	K_{71}	K_{81}							
$\mathcal{O}_3$	K_1	K_2	K_6	K_{23}	K_{52}	K_{60}	K_{67}	K_{70}					
$\mathcal{O}_4$	K_1	K_3	K_{14}	K_{15}	K_{18}	K_{39}	K_{46}	K_{68}					
$\mathcal{O}_7$	K_1	K_7	K_{15}	K_{17}	K_{18}	K_{49}	K_{56}	K_{74}					
$\mathcal{O}_9$	K_1	K_{17}	K_{22}	K_{24}	K_{49}	K_{63}	K_{77}	K_{78}					
$\mathcal{O}_{11}$	K_1	K_2	K_{17}	K_{21}	K_{22}	K_{24}	K_{30}	K_{69}	K_{78}				
$\mathcal{O}_{12}$	K_1	K_5	K_{17}	K_{22}	K_{24}	K_{35}	K_{36}	K_{63}	K_{65}				
$\mathcal{O}_{13}$	K_1	K_6	K_{19}	K_{21}	K_{30}	K_{38}	K_{39}	K_{55}	K_{66}				
$\mathcal{O}_{17}$	K_1	K_2	K_5	K_{12}	K_{18}	K_{23}	K_{48}	K_{49}	K_{66}	K_{68}			
$\mathcal{O}_{18}$	K_1	K_2	K_5	K_{17}	K_{22}	K_{24}	K_{32}	K_{36}	K_{66}	K_{81}			
$\mathcal{O}_{19}$	K_1	K_2	K_{14}	K_{17}	K_{21}	K_{23}	K_{56}	K_{67}	K_{69}	K_{70}			
$\mathcal{O}_{22}$	K_1	K_5	K_{13}	K_{18}	K_{19}	K_{20}	K_{21}	K_{38}	K_{59}	K_{64}			
$\mathcal{O}_{23}$	K_1	K_5	K_{14}	K_{16}	K_{17}	K_{18}	K_{31}	K_{35}	K_{39}	K_{42}			
$\mathcal{O}_{25}$	K_1	K_5	K_{14}	K_{18}	K_{19}	K_{22}	K_{24}	K_{35}	K_{44}	K_{65}			
$\mathcal{O}_{27}$	K_1	K_6	K_{15}	K_{20}	K_{22}	K_{23}	K_{24}	K_{27}	K_{40}	K_{44}			
$\mathcal{O}_{28}$	K_1	K_{13}	K_{15}	K_{16}	K_{17}	K_{20}	K_{21}	K_{33}	K_{50}	K_{56}			
$\mathcal{O}_{30}$	K_1	K_{15}	K_{17}	K_{18}	K_{19}	K_{20}	K_{39}	K_{46}	K_{59}	K_{68}			
$\mathcal{O}_{31}$	K_1	K_2	K_{13}	K_{15}	K_{17}	K_{18}	K_{20}	K_{21}	K_{36}	K_{62}	K_{66}		
$\mathcal{O}_{32}$	K_1	K_4	K_{17}	K_{18}	K_{19}	K_{22}	K_{24}	K_{39}	K_{68}	K_{70}	K_{77}		
$\mathcal{O}_{34}$	K_1	K_{15}	K_{16}	K_{17}	K_{19}	K_{22}	K_{24}	K_{30}	K_{35}	K_{39}	K_{69}		
$\mathcal{O}_{36}$	K_1	K_2	K_{14}	K_{17}	K_{18}	K_{21}	K_{22}	K_{23}	K_{24}	K_{43}	K_{68}	K_{70}	
$\mathcal{O}_{38}$	K_1	K_2	K_{14}	K_{15}	K_{18}	K_{22}	K_{23}	K_{24}	K_{48}	K_{49}	K_{60}	K_{66}	K_{68}
$\mathcal{O}_{40}$	K_1	K_{15}	K_{17}	K_{18}	K_{19}	K_{20}	K_{22}	K_{24}	K_{39}	K_{49}	K_{66}	K_{68}	K_{77}

TABLE 9. The case $q = 23$

$$
\begin{aligned}
\mathbf{e}_1 &= \tfrac{1}{2}(1,-1,-1,1,1,1,1,1),\\
\mathbf{e}_2 &= \quad (-1,-1,0,0,0,0,0,0),\\
\mathbf{e}_3 &= \quad (0,1,-1,0,0,0,0,0),\\
\mathbf{e}_4 &= \quad (0,0,1,-1,0,0,0,0),\\
\mathbf{e}_5 &= \quad (0,0,0,1,-1,0,0,0),\\
\mathbf{e}_6 &= \quad (0,0,0,0,1,-1,0,0),\\
\mathbf{e}_7 &= \quad (0,0,0,0,0,1,-1,0),\\
\mathbf{e}_8 &= \quad (0,0,0,0,0,0,1,-1).
\end{aligned}
$$

For $i = 1, \ldots, 8$, let r_i be the reflection with equation $\mathbf{x}' = \mathbf{x} - (\mathbf{x}\mathbf{e}_i)\mathbf{e}_i$. Then r_i preserves E. Moreover, the group generated by r_i is the Weyl group of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ which is isomorphic to the orthogonal group $O_8^+(2)$.

Now fix an odd prime p, and look at the factor module $\overline{E} = E/pE$, which is an eight-dimensional vector space over $GF(p) = \mathbb{Z}/p\mathbb{Z}$, such that $\mathbf{e}_1, \ldots, \mathbf{e}_8$, regarded as vectors of $\overline{E}$, form a base. This vector space is equipped with a hyperbolic quadratic form defined by $Q(\mathbf{v}) = ||\mathbf{v}||^2/2$. It is known that for every root vector $\mathbf{e}$, the coset $\mathbf{e} + 2E$ consists of $2(p^3+1)$ vectors satisfying the condition $||\mathbf{v}||/2 = p$. Moreover, the vectors in $\mathbf{e} + 2E$ always occur in pairs $\pm\mathbf{v}$. Let Γ be the subgroup of the Weyl group which leaves the coset $\mathbf{e} + 2E$ invariant. In terms of the seven-dimensional projective space $PG(7,p)$ arising from $\overline{E}$, the above $2(p^3+1)$ vectors define p^3+1 distinct points lying on a hyperbolic quadric $O_8^+(p)$ associated to $Q(\mathbf{v})$. These points are pairwise non-orthogonal. Therefore, they form an ovoid Ω of $O_8^+(p)$ left invariant by the factor group $\overline{\Gamma} = \Gamma/\langle -I\rangle$.

From now on suppose $p = 23$. According to [7, Lemma 1], $p = 23$ can be written as $a^2 - ab + b^2 + c^2$ with $(a,b,c) \in \mathbb{Z}^3$. More precisely, this occurs in eight cases (up to sign) namely for

$$(1,3,4),(2,3,4),(2,5,2),(3,1,4),(3,2,4),(3,5,2),(5,2,2),(5,3,2). \tag{9.1}$$

Following Moorhouse's construction in [7, Section 3, Case II], we need to replace the first and third vectors in our previous base $\{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$ by the vectors $\mathbf{f}_1 = (1,-1,0,0,0,0,0,0)$ and $\mathbf{f}_2 = \frac{1}{2}(1,1,1,1,1,1,1,1)$, so that we obtain a new base. Define the vector $\mathbf{w}$ to be $a\mathbf{e}_2 + b\mathbf{f}_2 + c\mathbf{f}_1$. In the first case, $(a,b,c) = (1,3,4)$, and a straightforward computation shows that then $\mathbf{w} = \frac{1}{2}(9,-7,3,3,3,3,3,3)$. Apart from the factor $1/2$, all components in $\mathbf{w}$ are odd integers, and summing the components gives 20, which is a multiple of 4, and $||\mathbf{w}||^2/2 = 184/4 = 46 = 2\cdot 23$. Therefore, the point $P(\mathbf{w})$ arising from the vector $\mathbf{w}$ lies on the hyperbolic quadric $O_8^+(23)$, but it is off the ovoid Ω.

The first step in the slicing method due to Moorhouse [7] consists in taking from Ω those points which lie on the polar hyperplane $\pi(\mathbf{w})$ of $P(\mathbf{w})$ with respect to the hyperbolic quadric $O_8^+(23)$. In our case, $\pi(\mathbf{w})$ has equation

$$9x_1 - 7x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6 + 3x_7 + 3x_8 = 0,$$

and there are 23^2+1 points taken in this way. Let Δ denote the set of these 23^2+1 points. The hyperplane $\pi(\mathbf{w})$ cuts out of $O_8^+(23)$ a cone $\mathcal{K}(\mathbf{w})$ containing Δ. Let G be the subgroup of $\overline{\Gamma}$ which preserves $P(\mathbf{w})$ and Δ. A computer aided computation shows that $G = \langle r_4, r_5, r_6, r_7, r_8\rangle \cong S_6$, and that G has six orbits on Δ, namely the orbits of the following points:

$$\begin{array}{ll}
(0,0,1,-5,-1,-1,3,3)\ , & (1,-3,-4,-4,-2,0,0,0)\ , \\
(1,-3,-4,-2,-2,-2,-2,2)\ , & (3,3,-4,-2,0,0,2,2)\ , \\
(3,3,-2,-2,-2,0,0,4)\ , & (4,0,-3,-3,-3,-1,-1,-1)\ .
\end{array}$$

Moreover, these orbits have lengths $180, 60, 30, 180, 60, 20$, respectively. In particular, the identity is the only collineation in G which fixes every line joining $P(\mathbf{w})$ with a point in Δ. Let M be the subgroup of G isomorphic to A_6.

The lines of $\pi(\mathbf{w})$ through $P(\mathbf{w})$ can be regarded as the points of a projective space $PG(5,23)$ such that the generators of $\mathcal{K}(\mathbf{w})$ are the points of a Klein quadric $Q_5^+(23)$. Those generators which join $P(\mathbf{w})$ with a point in Δ are the points of an ovoid $\mathcal{O}$ left invariant by G. The subgroup of $PGO_+(5,23)$ preserving both families of planes lying on $Q_5^+(23)$ contains a subgroup isomorphic to A_6, but does not contain a subgroup isomorphic to S_6 (see [1, Section 2]). In terms of the translation plane Π arising from $\mathcal{O}$ with translation complement H, a subgroup $H \cong A_6$ of G is a subgroup of H modulo scalars.

For each of the other triples in (9.1), similar computations can be carried out showing that the numbers and the lengths of the orbits are the same as on $\mathcal{O}$.

From the above discussion, the translation plane Π arising from the S_6-invariant Moorhouse ovoid of the Klein quadric is equivalent to our ovoid $\mathcal{O}_1$ in Table 9. Also, Π is self-polar, since every projectivity in $G \setminus M$ provides a polarity. This agrees with Moorhouse's assertion in [7, Remark 4].

10. Acknowledgments

The authors would like to thank both anonymous referees for their thorough work and useful comments that we believe improved the paper considerably.

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