

NOTES ON THE ILLUMINATION PARAMETERS OF  
CONVEX POLYTOPES

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ABSTRACT. Convex bodies with large illumination parameters are constructed in each dimension. The exact values of the illumination parameters of the centrally symmetric Platonic solids are calculated, and estimates on the illumination parameters of the centrally symmetric Archimedean solids are given.

## 1. INTRODUCTION

Let  $\mathbf{K}$  be a convex body of  $\mathbf{E}^d$ ,  $d \geq 1$  (i.e. a compact convex set of the  $d$ -dimensional Euclidean space  $\mathbf{E}^d$  with non-empty interior). We call a point  $l \in \mathbf{E}^d \setminus \mathbf{K}$  a light-source and say that it illuminates the boundary point  $p$  of  $\mathbf{K}$  if the half-line starting at  $l$  and passing through  $p$  intersects the interior of  $\mathbf{K}$  somewhere not between  $l$  and  $p$ . Furthermore, we say that the light-sources  $\{l_1, l_2, \dots, l_n\} \subset \mathbf{E}^d \setminus \mathbf{K}$  illuminate  $\mathbf{K}$  if each boundary point of  $\mathbf{K}$  is illuminated by at least one of the light-sources  $l_1, l_2, \dots, l_n$ . The smallest number of light-sources that can illuminate  $\mathbf{K}$  is called the *illumination number*  $I(\mathbf{K})$  of  $\mathbf{K}$ . The well-known illumination conjecture phrased independently by Boltyanski (1960) and Hadwiger (1960), says that any  $d$ -dimensional convex body can be illuminated by  $2^d$  light-sources in  $\mathbf{E}^d$ , that is the inequality  $I(\mathbf{K}) \leq 2^d$  holds for any convex body  $\mathbf{K} \in \mathbf{E}^d$ . This has been proved only for  $d \leq 2$ , but there are a number of partial results supporting the conjecture for  $d > 2$  (see [3] and [6]).

The following quantitative version of the illumination numbers of convex bodies was introduced by K. Bezdek in [2]. If  $\mathbf{K}_o$  is a convex body of  $\mathbf{E}^d$  symmetric about the origin  $o$  of  $\mathbf{E}^d$ , then  $\mathbf{K}_o$  defines a norm

$$\|x\|_{\mathbf{K}_o} = \inf\{0 < \lambda : \lambda^{-1}x \in \mathbf{K}_o\},$$

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which turns  $\mathbf{E}^d$  into a normed space. Then let the *illumination parameter* of  $\mathbf{K}_o$  be defined as

$$IP(\mathbf{K}_o) = \inf \left\{ \sum_i \|p_i\|_{\mathbf{K}_o} : \{p_i\} \text{ illuminates } \mathbf{K}_o \right\}.$$

This ensures that far-away light-sources are penalised. Let

$$IP(d) = \sup\{IP(\mathbf{K}_o) : \mathbf{K}_o \text{ is an } o\text{-symmetric convex body in } \mathbf{E}^d\}.$$

Obviously  $IP(1) = 2$ . It is known (see [2] and [4]) that  $IP(2) = 6$ , and if  $\mathbf{K}_o$  is a planar convex body, then  $IP(\mathbf{K}_o) = 6$  holds only for (affine) regular convex hexagons. Also, [2] has raised the fundamental problem of computing or estimating  $IP(d)$ . Recently Swanepoel [7] has shown that  $IP(d) \leq O(2^d d^2 \log d)$ . Perhaps  $IP(d) = O(2^d)$ . In Section 2 we construct some polytopes with large illumination parameters in each dimension. It follows from our construction, that  $IP(d) \geq 3 \cdot 2^{d-1}$  if  $d > 1$ .

The exact values of the illumination parameters have been calculated only for cubes and cross-polytopes in any dimension [4], and for spheres in dimensions 2 and 3 [5]. In Section 3 we investigate some three-dimensional polyhedra. We determine the exact values of the illumination parameters of the regular dodecahedron and the regular icosahedron, and give estimates on the illumination parameters of the ten centrally symmetric Archimedean solids.

## 2. BODIES WITH LARGE ILLUMINATION PARAMETERS

It is easy to construct bodies with small illumination parameters. If we illuminate the  $d$ -dimensional ball  $\mathbf{B}_d$  by the set of vertices of a slightly enlarged circumscribed cross-polytope, then we get  $IP(\mathbf{B}_d) \leq 2d\sqrt{d}$ . In this section we construct polytopes with large illumination parameters in each dimension.

**Proposition 2.1.** *Let  $\mathbf{K}_o$  be a convex body of  $\mathbf{E}^d$  symmetric about the origin  $o$  of  $\mathbf{E}^d$  with  $IP(\mathbf{K}_o) = k$ . If  $\mathbf{C}_o$  is a right cylinder of  $\mathbf{E}^{d+1}$  symmetric about the origin of  $\mathbf{E}^{d+1}$ , whose base is congruent with  $\mathbf{K}_o$ , then  $IP(\mathbf{C}_o) \geq 2k$ .*

*Proof.* Let  $\mathbf{T}$  and  $\mathbf{B}$  be the top and the bottom of  $\mathbf{C}_o$  respectively. Without loss of generality, we may assume that  $\mathbf{T}$  and  $\mathbf{B}$  are in the hyperplanes given by  $X_{d+1} = 1$  and  $X_{d+1} = -1$ , and we can regard the intersection of  $\mathbf{C}_o$  and the hyperplane  $X_{d+1} = 0$  as  $\mathbf{K}_o$ . If  $l = (x_1, x_2, \dots, x_{d+1})$  illuminates a boundary point  $p = (p_1, p_2, \dots, p_{d+1})$  of  $\mathbf{T} \subset \mathbf{C}_o$  in  $\mathbf{E}^{d+1}$  then  $l' = (x_1, x_2, \dots, x_d, 0)$  illuminates  $p' = (p_1, p_2, \dots, p_d, 0) \in \mathbf{K}_o$  in the hyperplane  $X_{d+1} = 0$ . Furthermore we have that

$$\|l'\|_{\mathbf{K}_o} \leq \|l\|_{\mathbf{C}_o}.$$

If a set of light-sources  $\{l_1, l_2, \dots, l_n\}$  illuminates the boundary of  $\mathbf{T} \subset \mathbf{C}_o$  in  $\mathbf{E}^{d+1}$ , then the set  $\{l'_1, l'_2, \dots, l'_n\}$  illuminates  $\mathbf{K}_o$  in  $\mathbf{E}^d$ .

The proposition follows from a similar argument for the boundary points of  $\mathbf{B}$ , and the fact that no light-source can illuminate a point on  $\mathbf{T}$  and a point on  $\mathbf{B}$  simultaneously.  $\square$

**Theorem 2.2.** *If  $d \geq 2$  then  $IP(d) \geq 3 \cdot 2^{d-1}$ .*

*Proof.* We use induction on  $d$  and show that there is an  $o$ -symmetric convex body  $\mathbf{K}_o$  for each dimension such that  $IP(\mathbf{K}_o) \geq 3 \cdot 2^{d-1}$ . For  $d = 2$ , this follows from the result of Bezdek [2]. Now suppose that there exists an  $o$ -symmetric convex body  $\mathbf{K}_o$  in  $\mathbf{E}^d$  such that  $IP(\mathbf{K}_o) \geq 3 \cdot 2^{d-1}$ , then according to Proposition 2.1, there exists an  $o$ -symmetric convex body  $\mathbf{C}_o$  in  $\mathbf{E}^{d+1}$  such that  $IP(\mathbf{C}_o) \geq 3 \cdot 2^d$ .  $\square$

### 3. THE ILLUMINATION PARAMETERS OF SOME THREE DIMENSIONAL POLYHEDRA

Throughout this section we will use the following notations:

- if  $\pi$  is a supporting plane of a convex body  $\mathbf{K}$  in  $\mathbf{E}^3$ , then  $\pi^+$  is the closed half-space containing  $\mathbf{K} \cup \pi$ , and  $\pi^- = \mathbf{E}^3 \setminus \pi^+$ ;
- if  $u, v$  and  $w$  are three non-collinear points in  $\mathbf{E}^3$ , then  $\pi_{uvw}$  is the unique plane containing them;
- if the light-source  $l$  illuminates exactly  $k$  vertices of a convex body  $\mathbf{K}$ , then the efficiency of  $l$  is  $e(l) = k/\|l\|_{\mathbf{K}}$ ;
- if  $v$  and  $w$  are two vertices of a graph  $\Gamma$ , then  $d(v, w)$  is their distance in  $\Gamma$ , that is the length of the shortest path joining them;
- $\tau$  is the positive root of the equation  $x^2 = x + 1$ , that is  $\tau = (\sqrt{5} + 1)/2$ .

The most famous polyhedra are the five Platonic solids. One of them, the regular tetrahedron, is not centrally symmetric. The illumination parameters of the cube and the regular octahedron are known [4]. In this section we calculate the illumination parameters of the remaining two centrally symmetric Platonic solids.

If we would like to illuminate a polytope  $\mathbf{P}$ , then it is enough to illuminate the vertices of  $\mathbf{P}$ . This follows because, if a light-source  $l$  illuminates a vertex  $v$  which belongs to a  $k$ -face  $F$  of  $\mathbf{P}$ , then  $l$  obviously illuminates each point in the relative interior of  $F$ . So when we compute the illumination parameter of  $\mathbf{P}$ , we always compute the sum of norms of a set of light-sources illuminating the vertices of  $\mathbf{P}$ .

**Theorem 3.1.** *The illumination parameter of the regular dodecahedron is  $4\sqrt{5} + 2$ .*

*Proof.* Let  $\mathbf{D}$  be a regular dodecahedron, and let  $\Gamma_{\mathbf{D}}$  be the edge-graph of  $\mathbf{D}$  (see Figure 1). It is easy to see that for any two vertices  $v$  and  $w$  of  $\Gamma_{\mathbf{D}}$  if  $d(v, w) \geq 3$  holds, then there are two parallel faces of  $\mathbf{D}$  such that one of them contains  $v$  and the other contains  $w$ . Hence no light-source can illuminate  $v$  and  $w$  simultaneously. If a subset of at least four vertices,

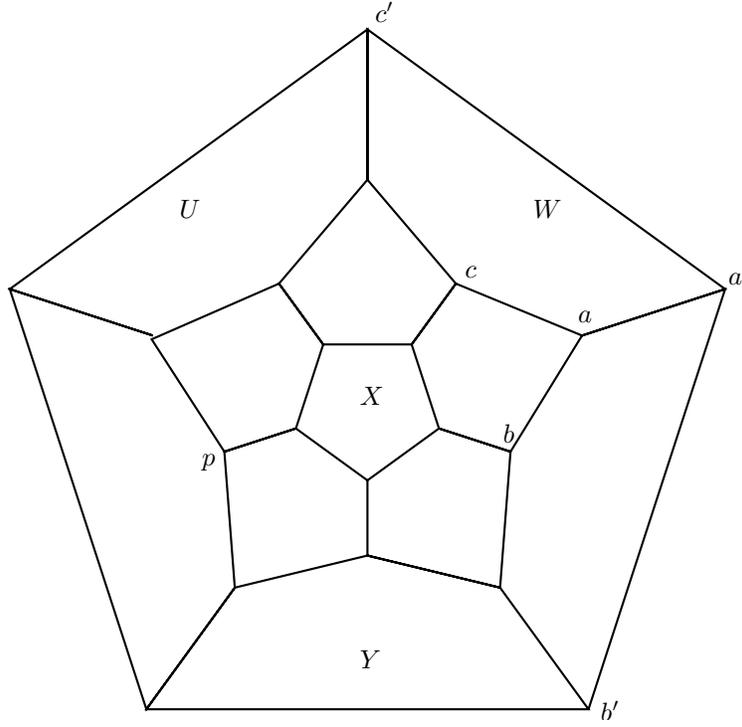


FIGURE 1. Dodecahedral graph

$V = \{v_1, v_2, \dots, v_k\}$  has the property, that  $d(v_i, v_j) \leq 2$  for all  $i, j$ , then there are two possibilities: either  $k = 4$  and there exists a vertex  $v \in V$  such that  $d(v, v_i) = 1$  for all  $v \neq v_i \in V$  (we call this a *star with centre*  $v$ ), or there is a face of  $\mathbf{D}$  which contains each vertex of  $V$ .

Let  $l$  be a light-source illuminating some vertices of  $\mathbf{D}$ . We estimate  $\|l\|_{\mathbf{D}}$ . For the computation we use Cartesian coordinates. It is well-known [1] that there exists a coordinate system such that the vertices of  $\mathbf{D}$  are the eight vertices of a cube  $(\pm 1, \pm 1, \pm 1)$  and twelve other vertices  $(0, \pm\tau, \pm\tau^{-1})$ ,  $(\pm\tau^{-1}, 0, \pm\tau)$  and  $(\pm\tau, \pm\tau^{-1}, 0)$ . Then the twelve faces of  $\mathbf{D}$  are contained in the planes with equations  $\pm\tau X \pm Z = \tau^2$ ,  $\pm\tau Z \pm Y = \tau^2$  and  $\pm\tau Y \pm X = \tau^2$ .

Let  $a = (\tau^{-1}, 0, \tau)$ ,  $b = (1, -1, 1)$ ,  $c = (1, 1, 1)$ ,  $a' = (-\tau^{-1}, 0, \tau)$ ,  $b' = (-1, -1, 1)$  and  $c' = (-1, 1, 1)$  be six vertices of  $\mathbf{D}$ .

If  $l$  illuminates only one vertex of  $\mathbf{D}$ , then  $l$  can be arbitrarily close to that vertex, hence  $\|l\|_{\mathbf{D}}$  tends to 1. In this case  $e(l) < 1$ , but  $e(l)$  can be arbitrarily close to 1.

If  $l$  illuminates two endpoints of an edge of  $\mathbf{D}$ , then we prove that  $\|l\|_{\mathbf{D}} \geq \tau$ . Without loss of generality we may assume that  $l$  illuminates the edge  $aa'$ . This means that  $l \in \mathcal{L} = \pi_{abc}^- \cap \pi_{a'ab}^- \cap \pi_{a'ac}^- \cap \pi_{a'b'c'}$ . (The set  $\mathcal{L}$  looks like an infinite sloping roof whose ridge has endpoints  $(0, \pm\tau, \tau^2)$ .)

The points of  $\pi_{abc}^-$  satisfy the inequality

$$(1) \quad \tau X + Z - \tau^2 > 0,$$

while the points of  $\pi_{a'b'c'}^-$  satisfy the inequality

$$(2) \quad -\tau X + Z - \tau^2 > 0.$$

Hence  $\mathcal{L}$  is contained in the open half-space  $Z > \tau^2$ . But  $\mathbf{D}$  is contained in the half-space  $Z \leq \tau$ , so  $\|l\|_{\mathbf{D}} \geq \tau^2/\tau = \tau$ . On the other hand,  $l$  could be arbitrarily close to the point  $(0, 0, \tau^2)$ , and in this case  $\|l\|_{\mathbf{D}}$  tends to  $\tau$ . Hence  $e(l) < 2/\tau = \sqrt{5} - 1$ , but  $e(l)$  can be arbitrarily close to  $2/\tau$ .

If  $l$  illuminates two vertices,  $v$  and  $w$  of  $\mathbf{D}$  with  $d(v, w) = 2$ , then without loss of generality we may assume that  $l$  illuminates the edges  $aa'$  and  $ab$ . Thus  $l$  satisfies the inequalities (1) and (2), and also satisfies

$$(3) \quad X - \tau Y - \tau^2 > 0,$$

because the plane  $X - \tau Y = \tau^2$  contains the vertex  $b$  and a face of  $\mathbf{D}$ . Hence  $l$  satisfies

$$\frac{1}{\tau}(X - \tau Y - \tau^2) + \frac{1}{\tau}(\tau X + Z - \tau^2) + (-\tau X + Z - \tau^2) > 0,$$

and so  $l$  is contained in the open half-space

$$-Y + \tau Z > \tau^2 + 2\tau.$$

But  $\mathbf{D}$  is contained in the half-space  $-Y + \tau Z \leq \tau^2$ , so

$$\|l\|_{\mathbf{D}} \geq \frac{\tau^2 + 2\tau}{\tau^2} = \sqrt{5}.$$

On the other hand,  $l$  could be arbitrarily close to the point  $(0, -\tau, \tau^2)$  in the open infinite cone formed by those five planes, which meet  $\mathbf{D}$  in the neighbouring faces of the face  $\mathbf{D} \cap \pi_{aa'b}$ . In this case  $\|l\|_{\mathbf{D}}$  tends to  $\sqrt{5}$ , and  $l$  illuminates each of the five vertices of the face  $\mathbf{D} \cap \pi_{aa'b}$ . Hence  $e(l) < 5/\sqrt{5} = \sqrt{5}$ , but  $e(l)$  can be arbitrarily close to  $\sqrt{5}$ .

If  $l$  illuminates a star, then we prove that  $\|l\|_{\mathbf{D}} \geq \sqrt{5} + 2 > 4$ . Without loss of generality we may assume that  $l$  illuminates the star with centre  $a$ . Thus  $l$  satisfies the inequalities (2) and (3), and also satisfies

$$(4) \quad X + \tau Y - \tau^2 > 0,$$

because the plane  $X + \tau Y = \tau^2$  contains the vertex  $c$  and a face of  $\mathbf{D}$ . Hence  $l$  satisfies

$$X > \tau^2 \text{ and } Z > \tau X + \tau^2 > \tau^3 + \tau^2 = \tau^4,$$

so  $l$  is contained in the open half-space

$$X + \tau^2 Z > \tau^2 + \tau^6.$$

But  $\mathbf{D}$  is contained in the half-space  $X + \tau^2 Z \leq \tau^{-1} + \tau^3$ , so

$$\|l\|_{\mathbf{D}} \geq \frac{\tau^6 + \tau^2}{\tau^{-1} + \tau^3} = \tau^3 = \sqrt{5} + 2.$$

On the other hand,  $l$  could be arbitrarily close to the point  $(\tau^2, 0, \tau^4)$  in the open infinite cone formed by the three planes  $-\tau X + Z = \tau^2$ ,  $X - \tau Y = \tau^2$  and  $X + \tau Y = \tau^2$ . In this case  $\|l\|_{\mathbf{D}}$  tends to  $\sqrt{5} + 2$ , and  $l$  illuminates each vertex of the star with centre  $a$ . Hence  $e(l) < 4/(\sqrt{5} + 2) < 1$ .

We get the illumination parameter of  $\mathbf{D}$  if we cover the vertices of  $\Gamma_{\mathbf{D}}$  in the most effective way. The faces  $U, W, X$  and  $Y$  cover 18 vertices (see Figure 1). If we illuminate each of these faces by its own light-source and the remaining two vertices by two light-sources, then we get a set of six light-sources such that the sum of their norms tends to  $4\sqrt{5} + 2$ . We prove that this is the most effective way.

It follows from the previous computations that the most effective light-sources illuminate five vertices of a face of  $\mathbf{D}$ . If the covering contains at least five faces, then the sum of the norms of the corresponding light-sources is at least  $5\sqrt{5} > 4\sqrt{5} + 2$ . On the other hand, if the covering contains  $0 \leq f \leq 3$  faces, then at least  $20 - 5f$  vertices are not covered by them. Each optimal covering of these vertices contains at most  $\lfloor (20 - 5f)/2 \rfloor$  edges, hence the sum of the norms of the light-sources is at least

$$f \cdot \sqrt{5} + \left\lfloor \frac{20 - 5f}{2} \right\rfloor \cdot \frac{\sqrt{5} + 1}{2} + 2 \left( \frac{20 - 5f}{2} - \left\lfloor \frac{20 - 5f}{2} \right\rfloor \right) \cdot 1 > 4\sqrt{5} + 2.$$

Now consider those coverings that contain exactly four faces. First we prove that the maximum number of pairwise disjoint faces in the covering is three. If two opposite faces are chosen, then any other face is a neighbour of exactly one of them. Suppose that the covering does not contain opposite faces. Then without loss of generality we may assume, that the faces  $X$  and  $Y$  are chosen (see Figure 1). Then they have eight neighbours in all, hence there are only two possible choices for the remaining faces,  $U$  and  $W$ , but these two are neighbours.

If the covering contains less than three pairwise disjoint faces, then the four faces cover at most 16 vertices. Thus the sum of the norms of the light-sources is at least

$$4 \cdot \sqrt{5} + 2 \cdot \frac{\sqrt{5} + 1}{2} > 4\sqrt{5} + 2.$$

If the covering contains three pairwise disjoint faces, then because of the symmetry we may assume that these are  $X, Y$  and  $U$ . They cover 15 vertices, the remaining five vertices form a star with centre  $a$  and a singleton  $p$  (see Figure 1). The fourth face covers either  $p$  or at most three vertices of the star with centre  $a$ . In both cases there are two non-covered vertices whose distance is at least 2. Hence the sum of the norms of the light-sources is at least  $4\sqrt{5} + 2$ .

This finishes the proof of the theorem. □

**Theorem 3.2.** *The illumination parameter of the regular icosahedron is 12.*

*Proof.* Let  $\mathbf{I}$  be a regular icosahedron, and let  $\Gamma_{\mathbf{I}}$  be the edge-graph of  $\mathbf{I}$  (see Figure 2). It is easy to see that for any two vertices  $v$  and  $w$  of  $\Gamma_{\mathbf{I}}$   $d(v, w) \leq 3$ , and equality holds if and only if  $v$  and  $w$  are opposite vertices.

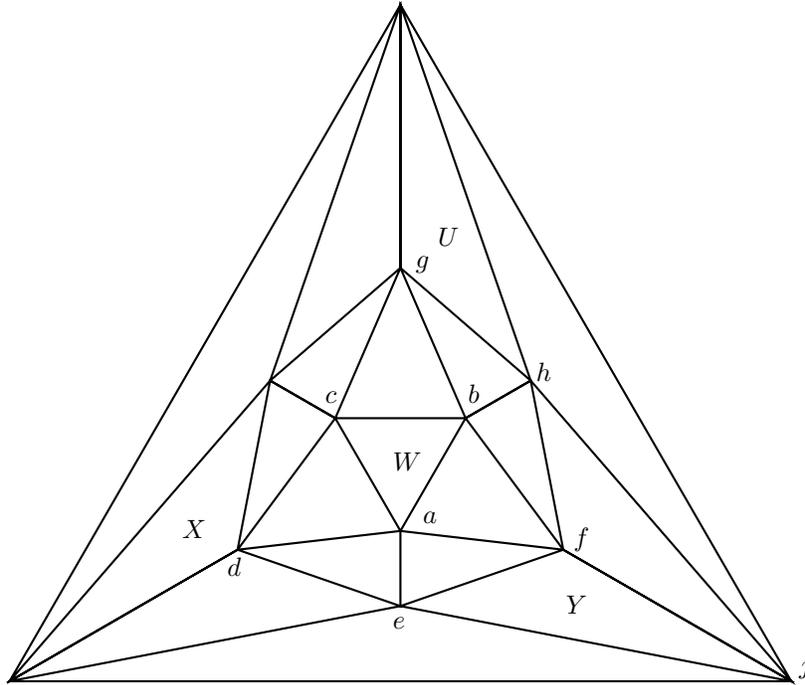


FIGURE 2. Icosahedral graph

If  $d(v, w) \geq 2$ , then there are two parallel faces of  $\mathbf{I}$  such that one of them contains  $v$  and the other contains  $w$ . Hence no light-source can illuminate  $v$  and  $w$  simultaneously. This implies that no light-source can illuminate more than three vertices of  $\mathbf{I}$ , because any subset of more than three vertices contains a pair of vertices having distance at least 2.

Let  $l$  be a light-source illuminating some vertices of  $\mathbf{I}$ . We estimate  $\|l\|_{\mathbf{I}}$ . It is well-known [1] that there exists a coordinate system such that the vertices of  $\mathbf{I}$  are the points  $(0, \pm\tau, \pm 1)$ ,  $(\pm 1, 0, \pm\tau)$  and  $(\pm\tau, \pm 1, 0)$ .

Let  $a = (\tau, 1, 0)$ . Then the five neighbours of  $a$  are  $b = (0, \tau, 1)$ ,  $c = (0, \tau, -1)$ ,  $d = (1, 0, -\tau)$ ,  $e = (\tau, -1, 0)$  and  $f = (1, 0, \tau)$ . Let  $g = (-\tau, 1, 0)$ ,  $h = (-1, 0, \tau)$  and  $j = (0, -\tau, 1)$  be three other vertices. Then the triangles  $bgh$ ,  $bhf$  and  $fhj$  are faces of  $\mathbf{I}$ .

If  $l$  illuminates only one vertex of  $\mathbf{I}$ , then  $l$  can be arbitrarily close to that vertex, hence  $\|l\|_{\mathbf{I}}$  tends to 1. In this case  $e(l) < 1$ , but  $e(l)$  can be arbitrarily close to 1.

If  $l$  illuminates two endpoints of an edge of  $\mathbf{I}$ , then we prove that  $\|l\|_{\mathbf{I}} \geq \tau^2 = (\sqrt{5} + 3)/2$ . Without loss of generality we may assume, that  $l$  illuminates the edge  $ab$ . This means that  $l \in \mathcal{M} = \pi_{acd}^- \cap \pi_{ade}^- \cap \pi_{aef}^- \cap \pi_{bcg}^- \cap \pi_{bgh}^- \cap \pi_{bhf}^-$ .

The points of  $\pi_{ade}^-$  satisfy the inequality

$$(5) \quad \tau^2 X - Z - \tau^3 > 0,$$

while the points of  $\pi_{bgh}^-$  satisfy the inequality

$$(6) \quad -X + Y + Z - \tau^2 > 0.$$

Hence  $\mathcal{M}$  is contained in the open half-space

$$\tau(\tau^2 X - Z - \tau^3) + \tau^2(-X + Y + Z - \tau^2) > 0.$$

Rearranging the coefficients, we get

$$\tau X + \tau^2 Y + Z > 2\tau^4.$$

But  $\mathbf{I}$  is contained in the half-space  $\tau X + \tau^2 Y + Z \leq 2\tau^2$ , so

$$\|l\|_{\mathbf{I}} \geq \frac{2\tau^4}{2\tau^2} = \tau^2.$$

On the other hand,  $l$  could be arbitrarily close to the point  $(\tau^3/2, \tau^4/4, \tau^2/2)$ , and in this case  $\|l\|_{\mathbf{D}}$  tends to  $\tau^2$ . Hence  $e(l) < 2/\tau^2 < 1$ .

If  $l$  illuminates three vertices of a face of  $\mathbf{I}$ , then without loss of generality we may assume that  $l$  illuminates  $a$ ,  $b$  and  $f$ . Thus  $l$  satisfies the inequalities (5) and (6), and also satisfies

$$(7) \quad -Y + \tau^2 Z - \tau^3 > 0,$$

because the plane  $\pi_{fhj}$  has equation  $-Y + \tau^2 Z = \tau^3$ . Hence  $l$  satisfies

$$(\tau + 2)(\tau^2 X - Z - \tau^3) + 2\tau^2(-X + Y + Z - \tau^2) + (-Y + \tau^2 Z - \tau^3) > 0,$$

and so  $l$  is contained in the open half-space

$$X + Y + Z > 3\tau^2.$$

But  $\mathbf{I}$  is contained in the half-space  $X + Y + Z \leq \tau^2$ , so

$$\|l\|_{\mathbf{I}} \geq \frac{3\tau^2}{\tau^2} = 3.$$

On the other hand,  $l$  could be arbitrarily close to the point  $(\tau^2, \tau^2, \tau^2)$  in the open infinite cone  $\pi_{ade}^- \cap \pi_{bgh}^- \cap \pi_{fhj}^-$ . In this case  $l \in \mathcal{M}$  also holds, hence because of the rotational symmetry,  $l$  illuminates each vertex of the face  $\mathbf{I} \cap \pi_{abf}$ , and  $\|l\|_{\mathbf{I}}$  tends to 3. Thus  $e(l) < 3/3 = 1$ , but  $e(l)$  can be arbitrarily close to 1.

We get the illumination parameter of  $\mathbf{I}$  if we cover the vertices of  $\Gamma_{\mathbf{I}}$  in the most effective way. It follows from the previous computations that the most effective light-sources illuminate either a single vertex or three vertices of a face of  $\mathbf{I}$ . In both cases  $e(l)$  tends to 1 for any light-source. Hence  $IP(\mathbf{I}) \geq 12$ . On the other hand, if each vertex  $v$  of  $\mathbf{I}$  has its own light-source  $l_v$ , then  $l_v$  could be arbitrarily close to  $v$ , hence  $IP(\mathbf{I}) = 12$ .  $\square$

It is easy to see that the vertices of  $\Gamma_{\mathbf{I}}$  can be covered by four faces  $U$ ,  $W$ ,  $X$  and  $Y$  (see Figure 2). Thus the most effective illumination of  $\mathbf{I}$  is not unique; we can mix the two types of light-sources having efficiency 1. Hence the number of light-sources in a set of optimal configuration could be 4, 6, 8, 10 or 12.

There are 13 semiregular convex polyhedra, the so-called Archimedean solids. Ten of them are centrally symmetric. Most of these can be constructed from the Platonic solids by truncation. For a more detailed description of the Archimedean solids we refer to [8].

The following simple observation is very useful when we estimate the illumination parameters of these polyhedra.

**Proposition 3.3.** *Let  $\mathbf{P}$  be a convex polytope in  $\mathbf{E}^d$  with vertex set  $V = \{v_1, v_2, \dots, v_k\}$ . Let  $N_{v_1} = \{v_i : v_1v_i \text{ is an edge of } \mathbf{P}\}$  be the set of the neighbouring vertices of  $v_1$ . Suppose that there exists a hyperplane  $H$  such that  $H \cap v_1v_i = w_i \in \text{relint}(v_1v_i)$  holds for each  $v_i \in N_{v_1}$ , and let  $\mathbf{P}'$  be the convex hull of  $V \setminus \{v_1\} \cup \{w_i : v_i \in N_{v_1}\}$ . (This means that  $H$  cuts off the vertex  $v_1$  of  $\mathbf{P}$ .) If a light-source  $l$  illuminates the vertex  $v_1$  of  $\mathbf{P}$ , then  $l$  illuminates the vertices  $w_i$  of the polytope  $\mathbf{P}'$ , as well.*

We do not determine the exact values of the illumination parameters, but give upper estimates. These estimates come from constructions of the set of light-sources. Most of these sets are highly symmetric configurations, and we conjecture that they give the optimal solutions.

**Theorem 3.4.** *There are sets of light-sources which give upper estimates on the illumination parameters of the 10 centrally symmetric Archimedean solids as follows.*

Name of the polyhedron	Upper estimate on the illumination parameter	Number of light-sources
Truncated cube	$24(\sqrt{2} - 1) \approx 9.941$	8
Truncated octahedron	9	6
Truncated dodecahedron	$(68\sqrt{5} + 10)/15 \approx 10.804$	6
Truncated icosahedron	12	4
Cuboctahedron	12	4 or 12
Rhombicuboctahedron	12	4
Truncated cuboctahedron	12	4
Icosidodecahedron	$3(\sqrt{5} + 1) \approx 9.708$	6
Truncated icosidodecahedron	$15(3\sqrt{5} + 1)/11 \approx 10.511$	6
Rhombicosidodecahedron	12	4

*Proof.* We omit the elementary computation. The starting positions of the light-sources are as follows:

- For the truncated cube and octahedron, the light-sources are at the vertices of the corresponding cube and octahedron.
- For the truncated dodecahedron and icosahedron, the light-sources are at the same positions as in Theorems 3.1 and 3.2.
- For the cuboctahedron, rhombicuboctahedron, truncated cuboctahedron and rhombicosidodecahedron, the light-sources are at the vertices of the circumscribed regular tetrahedron.
- For the icosidodecahedron and the truncated icosidodecahedron, the light-sources are at the midpoints of six corresponding edges of the circumscribed regular dodecahedron.

Finally, in each case we have to push the light-sources away from the corresponding polytope slightly.  $\square$

These results support our concluding conjecture.

**Conjecture 3.5.**

$$IP(d) = 3 \cdot 2^{d-1}.$$

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