Contributions to Discrete Mathematics

Volume 6, Number 2, Pages 36–69
ISSN 1715-0868

COMPUTATION OF ATOMIC FIBERS OF $\mathbb{Z}$-LINEAR MAPS

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ABSTRACT. For given matrix $A \in \mathbb{Z}^{d\times n}$, the set $P_b = \{ z : Az = b, z \in \mathbb{Z}^n_+ \}$ describes the preimage or fiber of $b \in \mathbb{Z}^d$ under the $\mathbb{Z}$-linear map $f_A : \mathbb{Z}^n_+ \to \mathbb{Z}^d$, $x \mapsto Ax$. The fiber P_b is called atomic if it has no nontrivial Minkowski decomposition, that is, $P_b = P_{b_1} + P_{b_2}$ implies $b = b_1$ or $b = b_2$. In this paper we present a novel algorithm to compute such atomic fibers. An algorithmic solution to subproblems, computational examples and applications in optimization and algebra are included as well.

1. Introduction

Following [13], we study the family of integer point sets

$$P_b := \{ z : Az = b,\ z \in \mathbb{Z}^n_+ \}$$

for a given matrix $A \in \mathbb{Z}^{d\times n}$ and varying right-hand sides $b \in \mathbb{Z}^d$. The set P_b thus is the preimage or *fiber* of $b \in \mathbb{Z}^d$ under the $\mathbb{Z}$-linear map $f_A : \mathbb{Z}^n_+ \to \mathbb{Z}^d$, $x \mapsto Ax$.

We study this infinite family under the operation of taking Minkowski sums. We call a fiber P_b *atomic* or *indecomposable* if $P_b = P_{b_1} + P_{b_2}$ implies $b = b_1$ or $b = b_2$. Note that $P_b = P_{b_1} + P_{b_2}$ means that *every lattice point* of P_b is the sum of a *lattice point* of P_{b_1} and a *lattice point* of P_{b_2} (and vice versa). This is indeed a very strong condition, but it was shown that there are only finitely many (nonempty) atomic fibers for a given matrix A [13]. Note that atomic fibers are not only minimal (with respect to decomposability) within the given family, but also generate every fiber P_b in this family as a Minkowski sum $P_b = \sum_{i=1}^k \alpha_i P_{b_i}$, $\alpha_i \in \mathbb{Z}_+$, where $\alpha_i P_{b_i}$ stands for iterated Minkowski-addition of P_{b_i} with itself.

Received by the editors November 3, 2009, and in revised form June 11, 2011.

2000 *Mathematics Subject Classification.* 13P10, 68W30, 90C10, 52B20.

Key words and phrases. Decomposition of polyhedra, Minkowski sums, generating sets, Hilbert bases of cones, strong SAGBI bases.

The second author wishes to thank Jesús De Loera for some useful suggestions on improving the presentation of an earlier version of this paper. The paper was completed while all authors were at the Otto-von-Guericke University of Magdeburg, Germany. The authors wish to thank all referees for their thorough work.

Atomic fibers appear in the computation of minimal vanishing sums of roots of unity [18]. Atomic fibers also appear in the capacitated design of telecommunication networks for a given communication demand under survivability constraints [5, 6]. A variant of the notion of atomic fibers appears in [3] to construct strong SAGBI bases for subalgebras of polynomial rings. We explain the detailed relation of this variant and other notions of decomposition and their applications in Section 2.

Next we introduce the notion of atomic *extended* fibers. We call the set

$$Q_b := \{\, z : Az = b,\ z \in \mathbb{Z}^n \,\}$$

an *extended fiber* of the linear map of A. We call it *atomic* if $(Q_b \cap \mathbb{O}_j) = (Q_{b_1} \cap \mathbb{O}_j) + (Q_{b_2} \cap \mathbb{O}_j)$ holds for all the 2^n orthants $\mathbb{O}_j$ of $\mathbb{R}^n$, then $b = b_1$ or $b = b_2$. Here, as well, it can be shown that there are only finitely many (nonempty) extended atomic fibers for a given matrix. Extended atomic fibers have an important application in the primal integer programming theory of stochastic integer optimization. The set $\mathcal{H}_\infty$ constructed in [10] for use in two-stage stochastic integer programming is in fact the set of extended atomic fibers of the family of extended fibers

$$\{\, (x, y) : x = b, Tx + Wy = 0,\ x \in \mathbb{Z}^m,\ y \in \mathbb{Z}^n \,\}$$

where T and W are kept fixed and where b varies.

Outline. In this paper, we are mainly concerned about designing fast algorithms for computing atomic and atomic extended fibers. The outline of the paper is as follows. In Section 3 we first define a hierarchy of *partially extended fibers* that interpolate between fibers and extended fibers. This hierarchy not only generalizes the notions of fibers and extended fibers, but also plays a significant role in our algorithms. Motivated by our application in survivable network design, we define decomposability with respect to a given finitely generated monoid of feasible right-hand side vectors. We prove that, in this more general situation as well, there are only finitely many atomic fibers. We also present an algorithmic way to decompose a fiber into a Minkowski sum of atomic fibers.

In Section 4 we present a first algorithm to compute the atomic extended fibers of a given matrix, following the pattern of a completion procedure. By restricting the atomic extended fibers to the positive orthants and performing a simple reduction step, the atomic fibers (or partially extended fibers) of a matrix can be easily obtained. However, this method is not a very efficient one for computing atomic fibers as we will illustrate in Section 7. Therefore, we present a more efficient way to compute atomic fibers via a project-and-lift approach in Sections 5 and 6.

Both our algorithms enable us to compute not only the atomic fibers P_b but also atomic fibers $\widetilde{P}_b$ according to the definition in [3] for the application in SAGBI bases. This will be shown at the end of Section 4.

Finally, in Section 7, we compare both algorithms with the help of first computational results.

2. Related Notions of Decomposition of Polyhedra

In this section, we discuss the relation of atomic fibers to other notions of Minkowski decompositions of polyhedra and their integer points.

A classic notion is that of linear decomposition of polyhedra. Two polyhedra $P, Q \subseteq \mathbb{R}^n$ are called *homothetic* if $P = \lambda Q + t$ for some $\lambda > 0$ and $t \in \mathbb{R}^n$. Here, a polyhedron P is called *indecomposable* if any decomposition $P = Q_1 + Q_2$ implies that both Q_1 and Q_2 are homothetic to P. It can be shown that there are only finitely many indecomposable rational polyhedra that are not homothetic to each other. For further details on this type of decomposition, we refer the reader to, for example, [7, 11, 12, 14, 15, 17].

Let us now come to a more restrictive decomposition. Here we consider only polyhedra of the form $\{\, x \in \mathbb{R}^n : Ax \leq b \,\}$ for a given matrix $A \in \mathbb{Z}^{d\times n}$ and varying $b \in \mathbb{Z}^d$. To emphasize that we only consider integer right-hand sides, we say that a polyhedron P is *integrally indecomposable* if any decomposition $P = Q_1 + Q_2$ (into polyhedra with integer right-hand sides) implies that both Q_1 and Q_2 are homothetic to P. This decomposition is more restrictive than the linear decomposition, since only such polyhedra Q_1 and Q_2 are allowed that have an *integer* right-hand side. Henk, Köppe, and Weismantel [11] showed finiteness of the system of integrally indecomposable polytopes. This result implies important applications: total dual integrality (TDI-ness) of each member of a family of systems $Ax \leq b$, $b \in \mathbb{Z}^d$, can be concluded from the TDI-ness of the integrally indecomposable systems. Furthermore the finiteness of the system of integrally indecomposable polytopes enables us to compute a finite representation of a test set for a mixed-integer linear optimization problem.

Another important application of integral decomposition of polyhedra is that of factorizing a multivariate polynomial, see for example [2] and the references therein. Here, one considers only polyhedra of the form $\{\, x \in \mathbb{R}^n : Ax \leq b \,\}$ for given matrix $A \in \mathbb{Z}^{d\times n}$ and varying $b \in \mathbb{Z}^d$, where each polyhedron is integral, that is, where each polyhedron has only integral vertices. Note that the notion of integral decomposability is restricted to integral polyhedra in this application whereas the definition of [11] is valid for arbitrary rational polyhedra with integral right-hand side. The reason for this restriction is the simple observation that the so-called Newton polytope $\operatorname{Newt}(f) := \operatorname{conv}\{\alpha_i \in \operatorname{supp}(f)\}$ associated to a polynomial $f = \sum_{i\in I} a_i x^{\alpha_i}$ with $\operatorname{supp}(f) = \{\alpha_i \colon a_i \neq 0\}$ is integral by definition. Moreover, the relation $f = gh$ among the three polynomials f, g, and h implies $\operatorname{Newt}(f) = \operatorname{Newt}(g) + \operatorname{Newt}(h)$, a theorem due to Ostrowski [16].

A direct generalization of the above notion of integral decomposition of integral polyhedra was introduced by [3]. The authors considered polytopes

$$\widetilde{P}_b := \operatorname{conv}\{\, z : Az = b, z \in \mathbb{Z}^n_+ \,\}$$

called the *fibers* of b under the linear map $f_A \colon \mathbb{Z}^n_+ \to \mathbb{Z}^d$, $x \mapsto Ax$. A fiber $\widetilde{P}_b$ is called *atomic* if $\widetilde{P}_b = \widetilde{P}_{b_1} + \widetilde{P}_{b_2}$ implies $b = b_1$ or $b = b_2$. Note that

$\widetilde{P}_b = \widetilde{P}_{b_1} + \widetilde{P}_{b_2}$ means that *every vertex* of $\widetilde{P}_b$ is the sum of a *vertex* of $\widetilde{P}_{b_1}$ and a *vertex* of $\widetilde{P}_{b_2}$ (and vice versa). Atomic fibers were used by the authors of [3] to construct strong SAGBI bases for subalgebras of polynomial rings. They proved that the family of atomic fibers is finite and also gave an algorithm to compute atomic fibers via certain standard pairs. Via this algorithm, the authors of [3] computed the atomic fibers of the twisted cubic, see Example 3.7.

In the present paper, we consider the variant of the notion of atomic fibers that was introduced by [13]. Instead of considering convex hulls $\widetilde{P}_b$ of the preimages

$$P_b := \{ z : Az = b, z \in \mathbb{Z}_+^n \}$$

of the map $f_A\colon \mathbb{Z}_+^n \to \mathbb{Z}^d$, $x \mapsto Ax$, we consider the preimages P_b themselves. In the more general terminology of [13], the sets P_b are called $((0), A)$-fibers.

3. (Partially Extended) Atomic Fibers

Let us now start our treatment with a formal definition of partially extended fibers.

Definition 3.1. *Let $A \in \mathbb{Z}^{d\times n}$ be a matrix, $b \in \mathbb{Z}^d$ and $0 \le k \le n$.*

(i) *The set*

$$Q_b^{(k)} := \{ z : Az = b,\ z \in \mathbb{Z}_+^k \times \mathbb{Z}^{n-k} \}$$

is called a partially extended fiber of order k *of the matrix A. The set $Q_b := Q_b^{(0)}$ is called an* extended fiber*, and $P_b := Q_b^{(n)}$ is called a* fiber *of the matrix A.*

(ii) *Let $0 \le l \le n$. For $u, v \in \mathbb{R}^n$, we say that $u \sqsubseteq_l v$ if $u^{(i)}v^{(i)} \ge 0$ and $|u^{(i)}| \le |v^{(i)}|$ for all components $i = 1, \ldots, l$. We will abbreviate $\sqsubseteq_n$ by $\sqsubseteq$. For $U, V, W \subseteq \mathbb{R}^n$, we say that*

$$U = V \overset{(l)}{\oplus} W$$

and call U the l-restricted Minkowski sum *of V and W if for all $u \in U$ there exist $v \in V$, $w \in W$ with $v, w \sqsubseteq_l u$ and $u = v + w$. Note that $V \oplus^{(0)} W$ is just the ordinary Minkowski sum $V + W$. We will abbreviate $\oplus^{(n)}$ by $\oplus$.*

(iii) *For $0 \le m \le n$, we will denote by $\pi_m\colon \mathbb{R}^n \to \mathbb{R}^m$ with $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ the projection onto the first m components.*

Now we will go on defining *atomic* partially extended fibers with respect to a certain monoid $M \subseteq \mathbb{Z}^d$. To accompany the hierarchy of partially extended fibers, we define a hierarchy of notions of decomposition that interpolates between ordinary Minkowski sums and orthant-wise Minkowski sums.

Definition 3.2. *Let $A \in \mathbb{Z}^{d\times n}$ be a matrix, $b \in \mathbb{Z}^d$ and $0 \le k, l \le n$. Additionally, let $M \subseteq \mathbb{Z}^d$ be a monoid.*

(i) *We call* $Q_b^{(k)}$ atomic with respect to $\oplus^{(l)}$ and M *if there is no decomposition*

$$Q_b^{(k)} = Q_{b_1}^{(k)} \overset{(l)}{\oplus} Q_{b_2}^{(k)}$$

with $b_1, b_2 \in M$ *and* $\pi_l(Q_{b_1}^{(k)}), \pi_l(Q_{b_2}^{(k)}) \neq \pi_l(Q_0^{(k)})$. *By* $E_l^{(k)}(A, M)$ *we denote the set of partially extended fibers of order* k *which are atomic with respect to* $\oplus^{(l)}$ *and* M.

(ii) *We denote by* $E^{(k)}(A, M)$ *the set* $E_n^{(k)}(A, M)$ *and call it the set of* partially extended atomic fibers with respect to the monoid M. *We denote by* $F(A, M)$ *the set* $E^{(n)}(A, M)$ *and call it the set of* atomic fibers with respect to M.

Note that Definition 3.2 also applies to the special case where the monoid M is a lattice. In this paper we will develop two algorithms to compute the atomic fibers for this special case. However, in [5] the second algorithm is generalized in such a way that the atomic fibers of a matrix with respect to a monoid may be computed using it.

As a first step, we prove a generalization of the finiteness result for the family of atomic fibers.

Lemma 3.3. *Let* $0 \leq k \leq n$ *be fixed. There are only finitely many partially extended fibers* $Q_b^{(k)}$ *which are atomic with respect to a finitely generated monoid* M.

The proof of this lemma is based on the following nice theorem.

Theorem 3.4 ([13])**.** *Let* k *be a field. Let* $\mathcal{I}$ *be an infinite family of monomial ideals in a polynomial ring* $\mathsf{k}[x_1, \ldots, x_n]$. *Then there must exist ideals* $I, J \in \mathcal{I}$ *with* $I \subseteq J$.

To apply this theorem in our situation of partially extended fibers which are atomic with respect to a certain finitely generated monoid M, we introduce the following definition.

Definition 3.5. *Let* $A \in \mathbb{Z}^{d \times n}$ *and* $M = \langle m_1, \ldots, m_t \rangle \subseteq \mathbb{Z}^d$ *a finitely generated monoid. Let* $0 \leq k \leq n$ *be fixed.*

(i) *Let* $\alpha, \overline{\alpha} \in \mathbb{Z}_+^t$ *with*

$$b := \sum_{i=1}^{t} \alpha_i m_i \quad \text{and} \quad \overline{b} := \sum_{i=1}^{t} \overline{\alpha}_i m_i.$$

We say that $\left(\overline{\alpha}, Q_{\overline{b}}^{(k)}\right)$ reduces $\left(\alpha, Q_b^{(k)}\right)$ *and denote*

$$\left(\overline{\alpha}, Q_{\overline{b}}^{(k)}\right) \trianglelefteq \left(\alpha, Q_b^{(k)}\right)$$

if $\overline{\alpha} \sqsubseteq \alpha$ *and* $Q_b^{(k)} = Q_{\overline{b}}^{(k)} \oplus Q_{b-\overline{b}}^{(k)}$. *In particular,* $b - \overline{b} \in M$.

(ii) *We call a pair* $\big(\alpha, Q_b^{(k)}\big)$ irreducible with respect to $\trianglelefteq$ *if there is no pair* $\big(\overline{\alpha}, Q_{\overline{b}}^{(k)}\big)$ *different from* $\big(\alpha, Q_b^{(k)}\big)$ *and* $\big(0, Q_0^{(k)}\big)$ *with*

$$\big(\overline{\alpha}, Q_{\overline{b}}^{(k)}\big) \trianglelefteq \big(\alpha, Q_b^{(k)}\big).$$

Lemma 3.6. *Let* $0 \leq k \leq n$ *be fixed. Let* $\mathcal{A} = \big\{\big(\alpha^1, Q_{b_1}^{(k)}\big), \big(\alpha^2, Q_{b_2}^{(k)}\big), \dots\big\}$ *be a set of pairs.*

(i) *Let* $\big(\alpha^i, Q_{b_i}^{(k)}\big) \ntrianglelefteq \big(\alpha^j, Q_{b_j}^{(k)}\big)$ *for all* $\big(\alpha^i, Q_{b_i}^{(k)}\big), \big(\alpha^j, Q_{b_j}^{(k)}\big) \in \mathcal{A}$ *with* $i < j$. *Then* $\mathcal{A}$ *is finite.*

(ii) *There are only finitely many pairs* $\big(\alpha, Q_b^{(k)}\big)$ *which are irreducible with respect to* $\trianglelefteq$.

Proof. (i): We associate with a pair $\big(\alpha^j, Q_{b_j}^{(k)}\big)$ the monomial ideal

$$I_\alpha = \Big\langle x^{(z_1,\dots,z_k,z_{k+1}^+,z_{k+1}^-,\dots,z_n^+,z_n^-,\alpha_1^j,\dots,\alpha_t^j)} : Az = \sum_{i=1}^t \alpha_i^j m_i (= b_j),\ z \in \mathbb{Z}_+^k \times \mathbb{Z}^{n-k} \Big\rangle \subseteq \mathbb{Q}[x_1, \dots, x_{2n-k+t}],$$

where $z_i^+ = \max\{0, z_i\}$ and $z_i^- = \max\{0, -z_i\}$. Then $\big(\alpha^j, Q_{b_j}^{(k)}\big) \ntrianglelefteq \big(\alpha^l, Q_{b_l}^{(k)}\big)$ if I_{α^j} is not contained in I_{α^l}. Consider the set $\mathcal{I} = \{I_{\alpha^1}, I_{\alpha^2}, \dots\}$ of ideals associated to the elements in the set $\mathcal{A}$. The set $\mathcal{I}$ then is an antichain of ideals and is thus finite according to Theorem 3.4 (see [13]). The finiteness of $\mathcal{A}$ follows from the finiteness of $\mathcal{I}$.

(ii): A pair $\big(\alpha, Q_b^{(k)}\big)$ is irreducible with respect to $\trianglelefteq$ if and only if $\big(\alpha, Q_b^{(k)}\big) \ntrianglelefteq \big(\overline{\alpha}, Q_{\overline{b}}^{(k)}\big)$ for any $\overline{\alpha} \neq \alpha$. Let $\mathcal{A} = \big\{\big(\alpha^1, Q_{b_1}^{(k)}\big), \big(\alpha^2, Q_{b_2}^{(k)}\big), \dots\big\}$ be the set of pairs which are irreducible with respect to $\trianglelefteq$. Part (i) then yields that $\mathcal{A}$ is finite. □

We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3. It is sufficient to show that for every $Q_b^{(k)}$ atomic with respect to M, there exists $\alpha \in \mathbb{Z}_+^t$ with $b = \sum_{i=1}^t \alpha_i m_i$ such that $\big(\alpha, Q_b^{(k)}\big)$ is irreducible with respect to $\trianglelefteq$. Then there is an injective mapping from the set of atomic extended fibers $Q_b^{(k)}$ into the set of irreducible pairs $\big(\alpha, Q_b^{(k)}\big)$ and, thus, there are only finitely many extended atomic fibers with respect to M.

Let b be fixed with $Q_b^{(k)}$ an extended atomic fiber with respect to M. Let $\alpha \in \mathbb{Z}_+^t$ with $b = \sum_{i=1}^t \alpha_i m_i$ be minimal with respect to $\sqsubseteq$, i.e., there is no $\mathbb{Z}_+^t \ni \overline{\alpha} \neq \alpha$ with $\overline{\alpha} \sqsubseteq \alpha$ and $b = \sum_{i=1}^t \overline{\alpha}_i m_i$. We claim that the pair $\big(\alpha, Q_b^{(k)}\big)$ is irreducible with respect to $\trianglelefteq$. Suppose that it is not. Then there is $\big(\overline{\alpha}, Q_{\overline{b}}^{(k)}\big) \trianglelefteq \big(\alpha, Q_b^{(k)}\big)$, i.e., $\overline{\alpha} \sqsubseteq \alpha$ and $Q_b^{(k)} = Q_{\overline{b}}^{(k)} \oplus Q_{b-\overline{b}}^{(k)}$ implying $b - \overline{b} \in M$. As $Q_b^{(k)}$ is an extended atomic fiber we may assume w.l.o.g. that

$\overline{b} = b$ and $b - \overline{b} = 0$. Therefore $b = \sum_{i=1}^{t} \overline{\alpha}_i m_i$ and as α is minimally chosen with respect to $\sqsubseteq$ we have $\overline{\alpha} = \alpha$. This proves our claim. $\square$

Example 3.7. *In [3], it was shown how atomic fibers could be used to construct strong SAGBI bases for monomial subalgebra over principal ideal domains. As an example, they computed the atomic fibers of the matrix*

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

by hand via an approach different from the one we present below.

In the following table, we list the right-hand sides and all (finitely many) elements in these 18 *atomic fibers.*

Fiber	*Element*
(0,3)	$\{(0,0,0,1)\}$
(1,2)	$\{(0,0,1,0)\}$
(2,1)	$\{(0,1,0,0)\}$
(2,4)	$\{(0,1,0,1),(0,0,2,0)\}$
(3,0)	$\{(1,0,0,0)\}$
(3,3)	$\{(1,0,0,1),(0,1,1,0)\}$
(3,6)	$\{(1,0,0,2),(0,1,1,1),(0,0,3,0)\}$
(4,2)	$\{(0,2,0,0),(1,0,1,0)\}$
(4,5)	$\{(0,2,0,1),(0,1,2,0),(1,0,1,1)\}$
(4,8)	$\{(0,2,0,2),(1,0,1,2),(0,1,2,1),(0,0,4,0)\}$
(5,4)	$\{(1,1,0,1),(0,2,1,0),(1,0,2,0)\}$
(6,3)	$\{(2,0,0,1),(1,1,1,0),(0,3,0,0)\}$
(6,6)	$\{(2,0,0,2),(0,3,0,1),(1,1,1,1),(1,0,3,0),$ $(0,2,2,0)\}$
(6,9)	$\{(2,0,0,3),(0,3,0,2),(1,1,1,2),(1,0,3,1),$ $(0,2,2,1),(0,1,4,0)\}$
(6,12)	$\{(2,0,0,4),(0,3,0,3),(1,1,1,3),(1,0,3,2),$ $(0,2,2,2),(0,1,4,1),(0,0,6,0)\}$
(8,4)	$\{(2,1,0,1),(0,4,0,0),(1,2,1,0),(2,0,2,0)\}$
(9,6)	$\{(3,0,0,2),(1,3,0,1),(2,1,1,1),(2,0,3,0),$ $(1,2,2,0),(0,4,1,0)\}$
(12,6)	$\{(4,0,0,2),(2,3,0,1),(3,1,1,1),(3,0,3,0),$ $(2,2,2,0),(0,6,0,0),(1,4,1,0)\}$

Thus, for example, the fiber given by the right-hand side $(8,7)$ *is not atomic, since it can be decomposed into atomic fibers as*

$$P_{\binom{8}{7}} = P_{\binom{2}{4}} \oplus P_{\binom{6}{3}}.$$

This can be quickly verified by looking at the elements in these fibers:

$$\begin{aligned} &\{(2,1,0,2),(2,0,2,1),(1,1,3,0),(1,2,1,1),(0,4,0,1),(0,3,2,0)\} \\ &\qquad = \{(0,1,0,1),(0,0,2,0)\} \oplus \{(2,0,0,1),(1,1,1,0),(0,3,0,0)\}. \end{aligned}$$

Indeed, we have

$$\begin{aligned}(2,1,0,2) &= (0,1,0,1) + (2,0,0,1),\\ (2,0,2,1) &= (0,0,2,0) + (2,0,0,1),\\ (1,1,3,0) &= (0,0,2,0) + (1,1,1,0),\\ (1,2,1,1) &= (0,1,0,1) + (1,1,1,0),\\ (0,4,0,1) &= (0,1,0,1) + (0,3,0,0),\end{aligned}$$

and

$$(0,3,2,0) = (0,0,2,0) + (0,3,0,0).$$

In Example 3.7 above, it was easy to verify whether a given fiber is a summand in the decomposition of another fiber by simply checking the finitely many elements in the fiber for a decomposition. If the fibers are not bounded, however, this would not give a finite procedure. The following lemma tells us how to solve this problem via the (finitely many!) $\sqsubseteq$-minimal elements in the given fibers.

Definition 3.8. *Let $A \in \mathbb{Z}^{d\times n}$ and $b \in \mathbb{Z}^d$. Let $0 \le k \le l \le n$.*

(i) *An element $v \in Q_b^{(k)}$ is called* minimal with respect to $\sqsubseteq_l$ *if there is no $w \in Q_b^{(k)}$ with $v \neq w$ and $w \sqsubseteq_l v$.*

(ii) *We define $z, \widetilde{z} \in Q_b^{(k)}$ to be equivalent if and only if $\pi_l(z) = \pi_l(\widetilde{z})$.*

For $l < n$, there are infinitely many $\sqsubseteq_l$-minimal elements in general. Therefore we have to restrict ourselves to representatives of equivalence classes *of $\sqsubseteq_l$-minimal elements. Let $R_{b,l}^{(k)}$ denote a set of representatives of the equivalence classes of the $\sqsubseteq_l$-minimal elements in $Q_b^{(k)}$. Let these representatives be chosen arbitrarily but fixed.*

Remark: Let $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$ and $0 \le k, l \le n$. Then the set of representatives of $\sqsubseteq_l$-minimal elements in $Q_b^{(k)}$, $R_{b,l}^{(k)}$, is finite by the Lemma of Gordan–Dickson (see for example [4]).

Lemma 3.10. *Let $0 \le k \le l \le n$ and let $Q_{b_1}^{(k)} \neq \varnothing$, $Q_{b_2}^{(k)} \neq \varnothing$. Then $Q_{b_1+b_2}^{(k)} = Q_{b_1}^{(k)} \oplus^{(l)} Q_{b_2}^{(k)}$ if and only if for every $\sqsubseteq_l$-minimal vector $v \in R_{b_1+b_2,l}^{(k)}$ there is a vector $w \in Q_{b_1}^{(k)}$ with $w \sqsubseteq_l v$.*

Proof. Let $v \in Q_{b_1+b_2}^{(k)}$. Then there is $\overline{v} \in R_{b_1+b_2,l}^{(k)}$ with $\overline{v} \sqsubseteq_l v$. Thus, by the assumption in the lemma, there is some $\overline{w} \in Q_{b_1}^{(k)}$ such that $\overline{w} \sqsubseteq_l \overline{v} \sqsubseteq_l v$. As $k \le l$ we have $\overline{v} - \overline{w} \in \mathbb{Z}_+^k \times \mathbb{Z}^{n-k}$ and thus $\overline{v} - \overline{w} \in Q_{b_2}^{(k)}$ with $\overline{v} - \overline{w} \sqsubseteq_l \overline{v} \sqsubseteq_l v$.

We now claim that $v = (\overline{w} + v - \overline{v}) + (\overline{v} - \overline{w})$ with $\overline{w} + v - \overline{v} \in Q_{b_1}^{(k)}$, $\overline{v} - \overline{w} \in Q_{b_2}^{(k)}$, $\overline{w} + v - \overline{v} \sqsubseteq_l v$, and $\overline{v} - \overline{w} \sqsubseteq_l v$, is a desired representation of v. The first two relations are trivial, if we keep in mind that $Av = A\overline{v} = b$, $A\overline{w} = b_1$, $b = b_1 + b_2$ and $k \le l$. We get the other two relations as follows:

(a) $\overline{w}+v-\overline{v} \sqsubseteq_l \overline{v}+v-\overline{v}=v$, since by construction $\pi_l(\overline{w})$ and $\pi_l(v-\overline{v})$ lie in the same orthant, and
(b) $\overline{v}-\overline{w} \sqsubseteq_l \overline{v} \sqsubseteq_l v$, since $\overline{w} \sqsubseteq_l \overline{v}$.

Thus, we have constructed for arbitrary $v \in Q^{(k)}_{b_1+b_2}$ a valid representation of v as a sum of two elements from $Q^{(k)}_{b_1}$ and $Q^{(k)}_{b_2}$ whose projection onto the first l components lie in the same orthant as the projection of v onto its first l components. This concludes the proof. $\square$

Using this lemma repeatedly, we are now able to find, for a given right-hand side $b \in M$, a decomposition $Q^{(k)}_b = \bigoplus_{i=1}^s \alpha_i Q^{(k)}_{b_i}$, $\alpha_i \in \mathbb{Z}_+$, that is, we can find a decomposition of a partially extended fiber into a sum of partially extended fibers which are atomic with respect to the monoid M.

Algorithm 3.1 Algorithm to decompose extended fibers into sums of extended atomic fibers

Input: A, right-hand sides $\{b_1,\dots,b_s\}$ of the set of extended atomic fibers $E^{(k)}(A,M)$

Output: $\alpha_1,\dots,\alpha_s$ such that $Q^{(k)}_b = \bigoplus\limits_{i=1}^{s} \alpha_i Q^{(k)}_{b_i}$

```
1: α_1 := ··· := α_s := 0
2: for i = 1 to s do
3:    while Q_b^(k) = Q_{b_i}^(k) ⊕ Q_{b−b_i}^(k) and b − b_i ∈ M do
4:       b := b − b_i
5:       α_i := α_i + 1
6:    end while
7: end for
8: return α_1, ..., α_s.
```

It remains to state an algorithm that computes the finitely many $\sqsubseteq$-minimal elements in $Q^{(k)}_b$ for fixed k. We will do this in the following paragraphs.

We have to find for some $l \in \{1,\dots,n\}$ and some $k \in \{1,\dots,l\}$ all $\sqsubseteq_l$-minimal elements in (projections of) fibers of the form

$$\pi_l(Q^{(k)}_b) = \left\{ (x,y) \in \mathbb{Z}^k_+ \times \mathbb{Z}^{(l-k)} : \exists\, z \in \mathbb{Z}^{(n-l)} \text{ with } A(x,y,z)=b \right\}.$$

If $b=0$, then 0 is the only $\sqsubseteq_l$-minimal element. If not, we reduce this problem to the problem of finding a Hilbert basis of a cone. It is not hard to show that all $\sqsubseteq_l$-minimal elements (x,y,z) correspond to the elements $(x,y^+,y^-,z,1)$ in a Hilbert basis of the cone

$$\left\{(x,y^+,y^-,z,u) \in \mathbb{Z}^{n+(l-k)+1} : A(x,y^+-y^-,z)-bu=0; x,y^+,y^-,u \geq 0\right\}.$$

In general, this is not a pointed rational polyhedral cone (and thus need not have a unique inclusion-minimal Hilbert basis), since there can be linear relations among the (free) variables z. However, projected onto the space of the variables x, y^+, y^-, u, the nonnegativity constraints lead to a pointed rational polyhedral cone that possesses a unique inclusion-minimal Hilbert basis. Such a minimal Hilbert basis can be computed for example with `4ti2` (see [1]).

Note that the splitting of y into y^+ and y^- is only used for exposition here. In practice, one can directly use y when computing the $\sqsubseteq_l$-minimal elements, see [9, Section 2.6] for more details.

4. Computation of (Extended) Atomic Fibers

In the following we show how to compute the finitely many (extended) atomic fibers of a matrix $A \in \mathbb{Z}^{m\times n}$ In this section we will present a simple algorithm; we will give a more complex and much more efficient algorithm in the following sections. Both algorithms use the algorithmic pattern of a completion procedure.

We will denote the columns of matrix A by $A_1, \ldots, A_n \in \mathbb{Z}^m$. Note that the function normal-form(s, G) in Algorithm 4.2 is listed in Algorithm 4.3.

Algorithm 4.2 Algorithm to compute extended atomic fibers

Input: $F := \{\pm A_1, \ldots, \pm A_n\}$
Output: A set G, such that $\{\, Q_b : b \in G \,\}$ contains all extended atomic fibers of A

$G := F$
$C := \bigcup_{f,g\in G} \{f + g\}$ /* *Forming S-vectors* */
while $C \neq \varnothing$ **do**
  $s :=$ an element in C
  $C := C \setminus \{s\}$
  $f :=$ normal-form (s, G)
  if $f \neq 0$ **then**
    $G := G \cup \{f\}$
    $C := C \cup \bigcup_{g\in G}\{f + g\}$ /* *Adding S-vectors* */
  end if
end while
$G := G \cup \{0\}$
return G.

Algorithm 4.3 Normal form algorithm

Input: s, G
Output: A normal form of s with respect to G

while there is some $g \in G$ such that $Q_s = Q_g \oplus Q_{s-g}$ **do**
  $s := s - g$
end while
return s

Lemma 4.1. *Algorithm 4.2 terminates and computes a set G such that $\{ Q_b : b \in G \}$ contains all atomic fibers of A.*

Proof. Associate with $b \in A\mathbb{Z}^n$ the monomial ideal

$$I_{A,b} := \left\langle x^{(z^+,z^-)} : Az = b,\ z \in \mathbb{Z}^n \right\rangle \subseteq \mathbb{Q}[x_1, \ldots, x_{2n}],$$

where $(z^+)^j = \max(0, z^j)$ and $(z^-)^j = \max(0, -z^j)$ for all components $j = 1, \ldots, n$. Algorithm 4.2 generates a sequence $\{f_1, f_2, \ldots\}$ in $G \setminus F$ such that $Q_{f_j} \neq Q_{f_i} \oplus Q_{f_j - f_i}$ whenever $i < j$. Thus, the corresponding sequence $\{I_{A,f_1}, I_{A,f_2}, \ldots\}$ of monomial ideals satisfies $I_{A,f_j} \not\subseteq I_{A,f_i}$ whenever $i < j$. We conclude, by Theorem 3.4 [13], that this sequence of monomial ideals must be finite and, thus, Algorithm 4.2 must terminate.

It remains to prove correctness. For this, let G denote the set that is returned by Algorithm 4.2. Moreover, let $Q_{\overline{b}}$ be an extended atomic fiber of A with $\overline{b} \neq 0$. We will show that $\overline{b} \in G$.

Since $F \setminus \{0\} \subseteq G \setminus \{0\}$, we know that $Q_{\overline{b}} = \sum Q_{b_j}$ for finitely many (not necessarily distinct) $b_j \in G \setminus \{0\}$. This implies in particular, that every $z \in Q_{\overline{b}}$ can be written as a sum $z = \sum v_j$ with $v_j \in Q_{b_j}$. We will show that we can find vectors $b_j \in G$ such that every $z \in Q_{\overline{b}}$ can be written as a sum $z = \sum v_j$ with $v_j \in Q_{b_j}$ and $v_j \sqsubseteq z$. This implies $Q_{\overline{b}} = \bigoplus Q_{b_j}$. Since $Q_{\overline{b}}$ is atomic, this representation must be trivial, that is, it has to be $Q_{\overline{b}} = Q_{\overline{b}}$, and therefore we conclude $\overline{b} \in G$.

With Lemma 3.10 it is sufficient to consider the $\sqsubseteq$-minimal elements in $Q_{\overline{b}}$, $R^{(0)}_{\overline{b},n} = \{z_1, \ldots, z_k\}$, to decide if it decomposes with respect to $\oplus$. From all representations $Q_{\overline{b}} = \sum_{j \in J} Q_{b_j}$ with $b_j \in G \setminus \{0\}$, choose a representation and elements $v_{i,j} \in Q_{b_j}$ with $z_i = \sum_{j \in J} v_{i,j}$, $i = 1, \ldots, k$, such that the sum

$$\sum_{i=1}^{k} \sum_{j \in J} \|v_{i,j}\|_1 \tag{4.1}$$

is minimal. By the triangle inequality, we have that

$$\sum_{i=1}^{k} \sum_{j \in J} \|v_{i,j}\|_1 \geq \sum_{i=1}^{k} \|z_i\|_1. \tag{4.2}$$

Herein, equality holds if and only if all $v_{i,j}$ have the same sign pattern as z_i, $i = 1, \ldots, k$, that is, if and only if we have $v_{i,j} \sqsubseteq z_i$ for all i and all j. Thus, if we have equality in (4.2) for such a minimal representation $Q_{\overline{b}} = \sum_{j \in J} Q_{b_j}$, then $v_{i,j} \in Q_{b_j}$ and $v_{i,j} \sqsubseteq z_i$ for all occurring $v_{i,j}$, and we are done.

(It should be noted that we have required $b_j \in G \setminus \{0\}$ for all appearing b_j, that is, in particular, $b_j \neq 0$. Those b_j will be sufficient to generate all $\sqsubseteq$-minimal elements in the extended fiber $Q_{\overline{b}}$. We get the remaining elements in $Q_{\overline{b}}$ by adding elements from Q_0.)

Therefore, let us assume that

$$\sum_{i=1}^{k}\sum_{j\in J}\|v_{i,j}\|_1 > \sum_{i=1}^{k}\|z_i\|_1. \tag{4.3}$$

In the following we construct a new representation $Q_{\overline{b}} = \sum_{j'\in J'} Q_{b'_j}$ and elements $v'_{i,j}$ whose corresponding sum (4.1) is smaller than the minimally chosen sum. This contradiction proves that we have indeed equality in (4.2) and our claim is proved.

From (4.3) we conclude that there are indices i_0, j_1, j_2 and a component $m \in \{1,\ldots,n\}$ such that $v^{(m)}_{i_0,j_1}\cdot v^{(m)}_{i_0,j_2} < 0$. As $b_{j_1}, b_{j_2} \in G$, the sum $b_{j_1}+b_{j_2}$ has been built and the extended fiber $Q_{b_{j_1}+b_{j_2}}$ has either been reduced to Q_0 by sets $Q_{b_{j''}}$, $j''\in J''$, during the Algorithm 4.3 or $b_{j_1}+b_{j_2}$ has been added to G. In the latter case we set $J'' := \{j''\}$ with $b_{j''} := b_{j_1}+b_{j_2}$. This gives representations

$$v_{i,j_1}+v_{i,j_2} = \sum_{j''\in J''} w_{i,j''} \quad\text{with}\quad w_{i,j''}\in Q_{b_{j''}} \quad\text{and}\quad w_{i,j''}\sqsubseteq v_{i,j_1}+v_{i,j_2}$$

for $i = 1,\ldots,k$. As all $w_{i,j''}$ lie in the same orthant of $\mathbb{R}^n$ as $v_{i,j_1}+v_{i,j_2}$, we get

$$\Big\|\sum_{j''\in J''} w_{i,j''}\Big\|_1 = \|v_{i,j_1}+v_{i,j_2}\|_1 \le \|v_{i,j_1}\|_1+\|v_{i,j_2}\|_1,$$

with strict inequality for $i = i_0$.

Thus, replacing in $Q_{\overline{b}} = \sum_{j\in J} Q_{b_j}$ the term $Q_{b_{j_1}}+Q_{b_{j_2}}$ by $\sum_{j''\in J''} Q_{b_{j''}}$, we arrive at a new representation $Q_{\overline{b}} = \sum_{j'\in J'} Q_{b_{j'}}$ whose corresponding sum (4.1) is at most

$$\sum_{i=1}^{k}\sum_{j'\in J'}\|v_{i,j'}\|_1 < \sum_{i=1}^{k}\sum_{j\in J}\|v_{i,j}\|_1,$$

contradicting the minimality of the representation $Q_{\overline{b}} = \sum_{j\in J} Q_{b_j}$. This concludes the proof. □

Remark: One may use Algorithm 4.2 for the problem of finding all extended fibers that are atomic with respect to a certain lattice $\Lambda \subseteq \mathbb{Z}^m$. Let $b_1,\ldots,b_s$ be a lattice basis of the lattice $\Lambda \cap A\mathbb{Z}^n$. Then Algorithm 4.2 with input set $F = \{\pm b_1,\ldots,\pm b_s\}$ computes the right-hand sides of all extended fibers that are atomic with respect to the lattice Λ.

Having an algorithm available that computes all extended atomic fibers, we can use it to compute partially extended atomic fibers with respect to $\oplus$: if $Q^{(k)}_b$ is atomic then so is Q_b, as any decomposition of Q_b, restricted to $\mathbb{Z}^k_+ \times \mathbb{Z}^{n-k}$, would give a decomposition of $Q^{(k)}_b$. This way of computing partially extended atomic fibers of a given matrix $A \in \mathbb{Z}^{m\times n}$ is illustrated in Figure 1 and formalized in Algorithm 4.4.

Algorithm 4.4 Computing partially extended atomic fibers

Input: $F := \{\pm A_1, \ldots, \pm A_n\}$, $k \in \mathbb{Z}_+$

Output: A set G^* such that $\{\, Q_b^{(k)} : b \in G^* \,\}$ contains all partially extended fibers of order k which are atomic with respect to $\oplus$

Apply Algorithm 4.2 to the set F. Let G denote the output.
$G^* := \varnothing$.
for $b \in G$ with $Q_b^{(k)} \neq \varnothing$ **do**
  if $Q_b^{(k)} \neq Q_g^{(k)} \oplus Q_{b-g}^{(k)}$ for all $g \neq b \in G$ **then**
    $G^* := G^* \cup \{b\}$
  end if
end for
return G^*

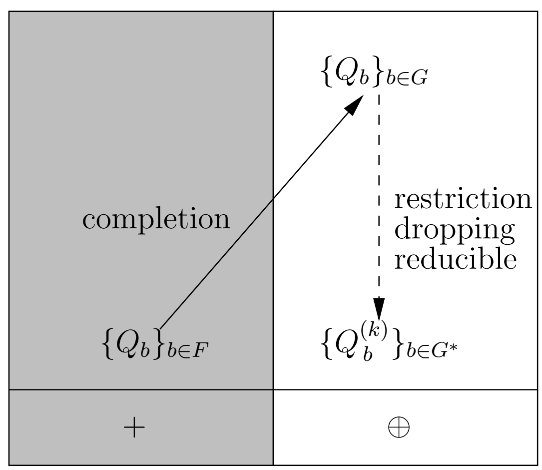

FIGURE 1. Computing (partially extended) atomic fibers via extended atomic fibers

The solid arrow from the bottom up in Figure 1 stands for the completion procedure which is given by Algorithm 4.2. The dashed arrow from the top to the bottom illustrates the procedure of intersecting the extended atomic fibers with $\mathbb{Z}_+^k \times \mathbb{Z}^{n-k}$ and dropping the reducible (or empty) fibers afterwards.

Being given the atomic fibers P_b of a matrix A it is easy to compute the atomic fibers $\widetilde{P}_b$ which have been defined in [3]. Recall that $\widetilde{P}_b := \operatorname{conv}\{\, z : Az = b,\ z \in \mathbb{Z}_+^n \,\}$ and that $\widetilde{P}_b$ is said to be atomic if each decomposition $\widetilde{P} = \widetilde{P}_{b_1} + \widetilde{P}_{b_2}$ implies either $b = b_1$ or $b = b_2$.

Lemma 4.3. *If $\widetilde{P}_b$ is an atomic fiber of the matrix A then P_b is atomic, too.*

Proof. Suppose $P_b = P_{b_1} + P_{b_2}$ (and $b_1, b_2 \neq 0$). Then we have: $\widetilde{P}_b = \operatorname{conv}(P_b) = \operatorname{conv}(P_{b_1} + P_{b_2}) = \operatorname{conv}(P_{b_1}) + \operatorname{conv}(P_{b_2}) = \widetilde{P}_{b_1} + \widetilde{P}_{b_2}$ which is a contradiction. □

Lemma 4.3 enables us to compute the atomic fibers $\widetilde{P}_b$ via Algorithm 4.5.

Algorithm 4.5 Computing the atomic fibers $\widetilde{P}_b$

Input: $F := \{b_1, \dots, b_s\}$ with P_{b_i} is an atomic fiber
Output: A set $G = \{\bar{b}_1, \dots, \bar{b}_t\}$ such that $\widetilde{P}_{\bar{b}_i}$ is an atomic fiber.

```
Set G := ∅.
for all b ∈ F do
    if P̃_b ≠ P̃_g + P̃_{b−g} for all b ≠ g ∈ F then
        G := G ∪ {b}
    end if
end for
return G
```

5. Preliminaries of the project-and-lift algorithm

The way of computing atomic fibers presented in Section 4, however, is pretty slow, since there are far more extended atomic fibers than atomic fibers. A similar behavior can be observed when one extracts the Hilbert basis of the cone $\{\, x : Ax = 0,\ x \in \mathbb{R}^n_+ \,\}$ from the Graver basis of A, as the Graver basis is usually much bigger than the Hilbert basis one is interested in. Hemmecke [8] showed that one can reduce this difference in sizes by a project-and-lift algorithm. With this algorithm, bigger Hilbert bases, even those with more than 500,000 elements, can be computed nowadays.

In this section and in the following one, we will present an algorithm to compute the atomic fibers of a given matrix $A \in \mathbb{Z}^{d\times n}$ which is significantly faster than Algorithm 4.4.

During the algorithm we consider partially extended fibers

$$Q_b^{(k-1)} = \left\{ z \in \mathbb{Z}_+^{k-1} \times \mathbb{Z}^{n-k+1} : Az = b \right\}$$

with varying $b \in \mathbb{Z}^m$ with respect to k-restricted Minkowski-sums. The algorithm proceeds in n individual steps. The k-th step is illustrated in Figure 2.

The k-th lifting step follows the arrows in the figure. It starts by performing a "preprocessing step" in which the input set is prepared for the main part of this lifting step. This process is illustrated by the dotted arrow and will be explained in more detail in Section 6.3.

The k-th lifting step continues as follows: it performs a completion step similar to the one we presented in Algorithm 4.2, which is illustrated by the solid arrow going from the bottom up. This step will be explained in more detail in Section 6.1.

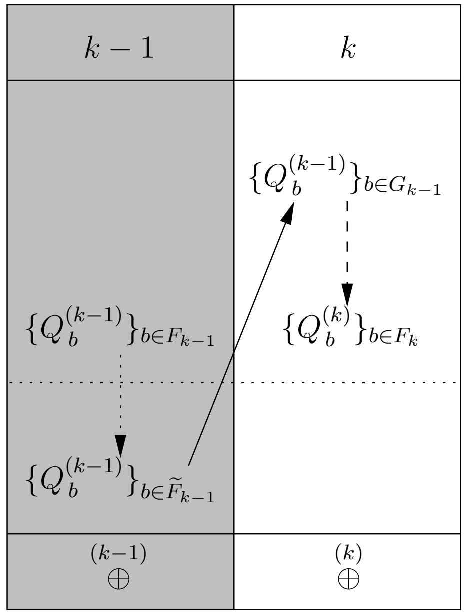

FIGURE 2. The k-th step of the project-and-lift algorithm

The dashed arrow, finally, stands for a step where we drop all elements of the fibers having a negative k-th component. It might happen that an atomic partially extended fiber becomes empty or reducible when processing this last step. Therefore we have to perform another reducibility test. The details of this subroutine will be given in Section 6.2.

Having performed the k-th lifting step we continue performing the $(k+1)$-st lifting step. The whole project-and-lift algorithm is presented in Algorithm 5.6 and is illustrated in Figure 3. After having performed n of these lifting steps we arrive at the finitely many atomic fibers of the matrix A.

Algorithm 5.6 The project-and-lift algorithm

Input: An integral matrix $A \in \mathbb{Z}^{d \times n}$.
Output: A set $G \subset \mathbb{Z}^d$ containing the right-hand sides of all atomic fibers.

$F_0 := \varnothing$
for $i = 1$ to n **do**
  Apply the procedure described in Section 6.3 to F_{i-1}. The output set is denoted by $\widetilde{F}_{i-1}$.
  Apply Algorithm 6.8 to the set $\widetilde{F}_{i-1}$. The output set is denoted by G_i.
  Apply Algorithm 6.9 to the set G_i. The output set is denoted by F_i.
end for
return $G := F_n$

In [5] the project-and-lift algorithm given in Table 5.6 has been generalized in such a way that one may compute the fibers of a matrix that are atomic

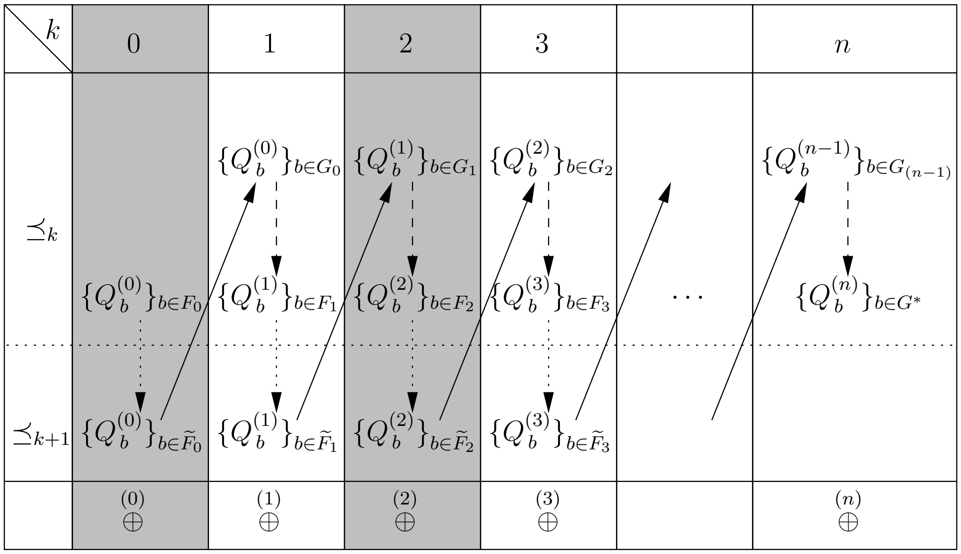

FIGURE 3. The scheme of the project-and-lift algorithm

with respect to a certain monoid. This generalization gives rise to new algorithmic subproblems, e.g. the equivalence relation on the set of right-hand sides of the fibers that is introduced in the following paragraph and studied in more detail in Subsection 6.3 is replaced by a preorder on this set of right-hand sides.

Dealing with infinitely many atomic fibers. Let $A \in \mathbb{Z}^{d\times n}$ be an integral matrix. The project-and-lift algorithm will deal with partially extended fibers with respect to l-restricted Minkowski sums where $l \le n$. Recall from Definition 3.2 that $Q_b^{(k)}$ is atomic with respect to $\oplus^{(l)}$ if there is no decomposition

$$Q_b^{(k)} = Q_{b_1}^{(k)} \overset{(l)}{\oplus} Q_{b_2}^{(k)}$$

with $b_1, b_2 \in \mathbb{Z}^m$ and $\pi_l\big(Q_{b_1}^{(k)}\big), \pi_l\big(Q_{b_2}^{(k)}\big) \neq \pi_l\big(Q_0^{(k)}\big)$. Note that for $l < n$ there are usually some $\bar{b} \in \mathbb{Z}^m$ with $\pi_l\big(Q_{\bar{b}}^{(k)}\big) = \pi_l\big(Q_0^{(k)}\big)$. Therefore, if $Q_b^{(k)}$ is atomic with respect to $\oplus^{(l)}$ then so are $Q_{b+\bar{b}}^{(k)}$, $Q_{b+2\bar{b}}^{(k)}$, $\ldots$. This means that for $l < n$ we usually have infinitely many partially extended fibers which are atomic with respect to $\oplus^{(l)}$. It is clear that no terminating algorithm may compute the whole set of atomic partially extended fibers with respect to $\oplus^{(l)}$. Therefore we introduce an equivalence relation $\simeq_l$ (i.e., a reflexive, symmetric and transitive binary relation) on the set of right-hand side vectors $b \in M$ with non-empty partially extended fiber $Q_b^{(k)}$ and perform the l-th step of the project-and-lift algorithm with respect to the equivalence relation $\simeq_l$.

Definition 5.1. *Let $S^{(k)} \subseteq \mathbb{Z}^m$ be the subset of $\mathbb{Z}^m$ with $Q_b^{(k)} \neq \varnothing$ for $b \in S^{(k)}$. Let $A \in \mathbb{Z}^{d\times n}$, $0 \leq k \leq l \leq n$ and let $b, \bar b \in S^{(k)}$. We say that $b \simeq_l \bar b$ if $\bar b - b \in \{\, \lambda_{l+1} A_{l+1} + \cdots + \lambda_n A_n : \lambda_i \in \mathbb{Z} \,\}$.*

Note that $\bar b \simeq_l b$ is a different way of expressing $\pi_l\big(Q_{\bar b}^{(k)}\big) = \pi_l\big(Q_b^{(k)}\big)$ and $Q_b^{(k)} = Q_{\bar b}^{(k)} \oplus^{(l)} Q_{b-\bar b}^{(k)}$.

The relation $\simeq_l$ defines an equivalence relation on the set of right-hand sides $b \in \mathbb{Z}^m$ with non-empty partially extended fibers of order k.

Lemma 5.2. *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix.*

(i) *Let $0 \leq k \leq l \leq n$ and let $M \supseteq F = \{b_1, b_2, \dots\}$ be a set of vectors with $b_i \not\simeq_l b_j$ for all $i < j$. Then F is finite.*
(ii) *Let $0 \leq k \leq l \leq n$ and let $F = \{b_1, b_2, \dots\}$ be a set of right hand sides satisfying $Q_{b_i}^{(k)}$ is atomic with respect to $\oplus^{(l)}$ and $b_i \not\simeq_l b_j$ for all $b_i \neq b_j$. Then F is finite.*

Proof. (i): Let $b_i, b_j \in F$ with $i < j$ and let $\alpha^i, \alpha^j \in \mathbb{Z}_+^{2n}$ with

$$b_i = \sum\nolimits_{k=1}^n \alpha^i_{2k-1} A_k + \alpha^i_{2k} A_k \quad \text{and} \quad b_j = \sum\nolimits_{k=1}^n \alpha^j_{2k-1} A_k + \alpha^j_{2k} A_k.$$

Then $\big(\alpha^i, Q_{b_i}^{(k)}\big) \not\trianglelefteq \big(\alpha^j, Q_{b_j}^{(k)}\big)$. Suppose this is not the case. Then we have $\alpha^i \sqsubseteq \alpha^j$ and $Q_{b_j}^{(k)} = Q_{b_i}^{(k)} \oplus Q_{b_j - b_i}^{(k)}$ which implies that $Q_{b_j}^{(k)} = Q_{b_i}^{(k)} \oplus^{(l)} Q_{b_j-b_i}^{(k)}$. But this last relation contradicts the fact that $b_i \not\simeq_l b_j$. Therefore $\big(\alpha^i, Q_{b_i}^{(k)}\big) \not\trianglelefteq \big(\alpha^j, Q_{b_j}^{(k)}\big)$ for all $b_i, b_j \in F$ for $i < j$. Finiteness of F follows with Lemma 3.6(i).

(ii): This is a direct consequence of (i). □

Our algorithm will work with sets of vectors F which have the property claimed in Lemma 5.2. Additionally they will admit the following property: if $b \in M$ is the right-hand side of a partially extended fiber $Q_b^{(k)}$ which is atomic with respect to $\oplus^{(l)}$ then there is $\bar b \in F$ with $\bar b \simeq_l b$.

6. The k-th step of the project-and-lift algorithm

In the following subsections we will explain the individual steps the project-and-lift algorithm performs during one lifting step.

6.1. The completion procedure.

In this subsection we will explain the so-called "completion procedure" in the k-th step of the algorithm. This part is illustrated in Figure 4.

We denote by $S^{(k)} = \{\, b \in \mathbb{Z}^m : Q_b^{(k)} \neq \varnothing \,\}$ the subset of all right-hand sides $b \in \mathbb{Z}^m$ admitting non-empty partially extended fibers of order k. For $0 \leq l \leq n$, let $L^{(l)}$ denote the set $\{\, \lambda_{l+1} A_{l+1} + \cdots + \lambda_n A_n : \lambda_i \in \mathbb{Z} \,\}$.

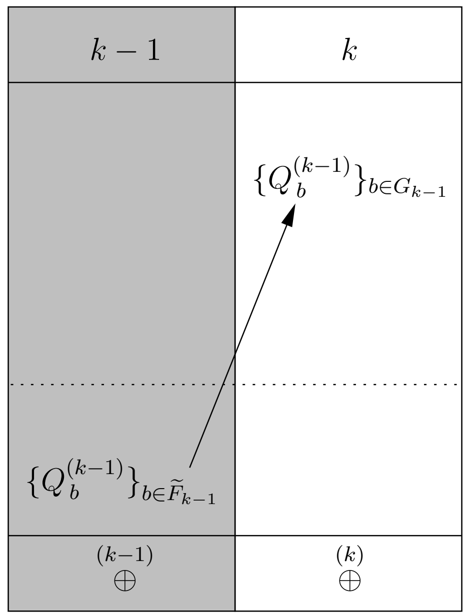

FIGURE 4. The completion procedure of the k-th lifting step

Definition 6.1. *We introduce a weight function ω_m for partially extended fibers $Q_b^{(k)}$ ($m \leq k \leq n$) by*

$$\omega_m\big(Q_b^{(k)}\big) = \min\Big\{\, ||\pi_m(v)||_1 : v \in Q_b^{(k)} \Big\}.$$

Remark: Actually, it suffices to determine $||\pi_m(v)||_1$ for $\sqsubseteq_m$-minimal elements v in $Q_b^{(k)}$ to determine the value of $\omega_m\big(Q_b^{(k)}\big)$. To see this, suppose there is $w \in Q_b^{(k)}$ non-minimal with respect to $\sqsubseteq_m$. Then there is $v \in Q_b^{(k)}$ with $v \sqsubseteq_m w$ and thus $0 \leq v^j \leq w^j$ for $j = 1, \ldots, k$. Therefore $||\pi_m(v)||_1 \leq ||\pi_m(w)||_1$.

Algorithm 6.7 The normal-form algorithm

Input: $s, G^{\omega=0}, G^{\omega\geq 1}$
Output: A normal form of s with respect to $G^{\omega=0} \cup G^{\omega\geq 1}$

```
if ∃ g ∈ G^{ω≥1} with Q_s^{(k-1)} = Q_g^{(k-1)} ⊕^{(k)} Q_{s-g}^{(k-1)} then
   return 0
else
   while ∃ g ∈ G^{ω=0} with Q_s^{(k-1)} = Q_g^{(k-1)} ⊕^{(k)} Q_{s-g}^{(k-1)} do
      s := s − g
   end while
   return s
end if
```

Algorithm 6.8 The completion procedure to compute atomic partially extended fibers

Input: A set $\widetilde{F}_{k-1} \subseteq S^{(k-1)}$ with the following properties:

(i) For every right-hand side $b \in S^{(k-1)}$ there exists $\widetilde{b} \in \mathbb{Z}^m$ with $\widetilde{b} \simeq_k b$ and

$$Q_{\widetilde{b}}^{(k-1)} = \bigoplus\nolimits_i^{(k-1)} Q_{b_i}^{(k-1)} \quad \text{with } b_i \in \widetilde{F}_{k-1}. \tag{6.1}$$

(ii) $b_i \not\simeq_k b_j$ for $b_i, b_j \in \widetilde{F}_{k-1}$ with $b_i \neq b_j$.

Output: A set $G_{k-1} \subseteq S^{(k-1)}$ with the properties:

(i) For every right-hand side $b \in S^{(k-1)}$ of a partially extended fiber $Q_b^{(k-1)}$ which is atomic with respect to $\oplus^{(k)}$ there exists $\widetilde{b} \in G_{k-1}$ with $\widetilde{b} \simeq_k b$.

(ii) $b_i \not\simeq_k b_j$ for $b_i \neq b_j \in G_{k-1}$.

$\overline{G}^{\omega=0} := \{\, f \in \widetilde{F}_{k-1} : \omega_{k-1}(Q_f^{(k-1)}) = 0 \,\}$
$C^{\omega=0} := \bigcup_{f,g \in \overline{G}^{\omega=0}} \{f+g\}$
while $C^{\omega=0} \neq \varnothing$ **do**
  $s :=$ an element in $C^{\omega=0}$
  $C^{\omega=0} := C^{\omega=0} \setminus \{s\}$
  $f :=$ normal-form $(s, \overline{G}^{\omega=0}, \varnothing)$
  if $f \notin L^{(k)}$ **then**
    $\overline{G}^{\omega=0} := \overline{G}^{\omega=0} \cup \{f\}$
    $C^{\omega=0} := C^{\omega=0} \cup \bigcup_{g \in \overline{G}^{\omega=0}} \{f+g\}$
  end if
end while
$G^{\omega=0} := \varnothing$
for all $b \in \overline{G}^{\omega=0}$ **do**
  if $Q_b^{(k-1)} \neq Q_g^{(k-1)} \oplus^{(k)} Q_{b-g}^{(k-1)}$ for all $b \neq g \in \overline{G}^{\omega=0}$ **then**
    $G^{\omega=0} := G^{\omega=0} \cup \{b\}$
  end if
end for
$\overline{G}^{\omega\geq 1} := \{\, f \in \widetilde{F}_{k-1} : \omega_{k-1}(Q_f^{(k-1)}) > 0 \,\}$, $G^{\omega\geq 1} = \varnothing$
for all $g \in \overline{G}^{\omega\geq 1}$ **do**
  $G^{\omega\geq 1} = G^{\omega\geq 1} \cup$ normal-form $(g, G^{\omega=0}, \varnothing)$
end for
$C^{\omega\geq 1} := \bigcup_{f,g \in G^{\omega\geq 1}} \{f+g\}$
while $C^{\omega\geq 1} \neq \varnothing$ **do**
  $s :=$ an element in $C^{\omega\geq 1}$ with smallest weight $\omega_{k-1}(Q_s^{(k-1)})$
  $C^{\omega\geq 1} := C^{\omega\geq 1} \setminus \{s\}$

Algorithm 6.8 (cont.)

26: $\quad f :=$ normal-form $(s, G^{\omega=0}, G^{\omega\geq 1})$
27: $\quad$ **if** $f \neq 0$ **then**
28: $\quad\quad G^{\omega\geq 1} := G^{\omega\geq 1} \cup \{f\}$
29: $\quad\quad C^{\omega\geq 1} := C^{\omega\geq 1} \cup \bigcup_{g\in G^{\omega=0}\cup G^{\omega\geq 1}} \{f+g\}$
30: $\quad$ **end if**
31: **end while**
32: $G_{k-1} := G^{\omega=0} \cup G^{\omega\geq 1} \cup \{0\}$
33: **return** G_{k-1}

Lemma 6.3. *Algorithm 6.8 with input set $\widetilde{F}_{k-1} := \{b_1, \ldots, b_s\}$ terminates and computes a set $G_{k-1} = G^{\omega=0} \cup G^{\omega\geq 1} \cup \{0\}$ with properties (i) and (ii).*

For the proof of Lemma 6.3, we have to introduce some more notation.

Notation 6.4. *During the proof of Algorithm 6.8 we examine elements of partially extended fibers. These elements will be denoted*

$$Q_b^{(k-1)} \ni z = (z_1, z_2, z_3) \in \mathbb{Z}_+^{k-1} \times \mathbb{Z} \times \mathbb{Z}^{n-k},$$

i.e., $z_1 \in \mathbb{Z}_+^{(k-1)}$ denotes the first $k-1$ components, $z_2 \in \mathbb{Z}$ the k-th component and $z_3 \in \mathbb{Z}^{(n-k)}$ denotes the last $n-k$ components.

We are now in the position to prove Lemma 6.3.

Proof of Lemma 6.3. As the following proof will be slightly complex, consider the following outline of the proof first:

(1) We show that $G_{k-1} \subseteq S^{(k-1)}$.
(2) We show that the set $G^{\omega=0}$ is finite and that for all $b_i, b_j \in G^{\omega=0}$ with $b_i \neq b_j$ we have $b_i \not\simeq_k b_j$.
(3) We show that G_{k-1} is finite. This implies that Algorithm 6.8 terminates. At the same time we show that the output set admits property (ii), i.e., $b_i \not\simeq_k b_j$ for $b_i \neq b_j \in G_{k-1}$.
(4) We show that if $Q_b^{(k-1)}$ is an atomic partially extended fiber with respect to $\oplus^{(k)}$ then there is $\widetilde{b} \simeq_k b$ with $\widetilde{b} \in G_{k-1}$. This is property (i) of the output set.

Step 1.) It is clear that Algorithm 6.8 returns a set $G_{k-1} \subseteq S^{k-1}$ as all input vectors lie in this set.

Step 2.) We will now prove that the set $G^{\omega=0}$ is finite. To this aim we show finiteness of $\overline{G}^{\omega=0}$ first. Consider the sequence

$$\overline{G}^{\omega=0} \setminus \left\{ f \in \widetilde{F}_{k-1} : \omega_{k-1}\left(Q_f^{(k-1)}\right) = 0 \right\} = \{f_1, f_2, \ldots\}$$

produced in lines 1–11 of the algorithm. Clearly $f_i \in S^{(k-1)}$ for all i. Additionally $f_i \not\simeq_k f_j$ for all $i < j$. Finiteness of $\overline{G}^{\omega=0}$ follows with Lemma 5.2.

As $G^{\omega=0} \subseteq \overline{G}^{\omega=0}$ it is clear now that $G^{\omega=0}$ is finite. Additionally lines 12–17 of Algorithm 6.8 guarantee that $b_i \not\simeq_k b_j$ for all $b_i, b_j \in G^{\omega=0}$ with $b_i \neq b_j$.

Step 3.) Let $G^{\omega=\alpha} := \{\, b \in G_{k-1} : \omega_{k-1}\big(Q_b^{(k-1)}\big) = \alpha \,\}$ for $\alpha \in \mathbb{Z}_+$. Furthermore let $G^{\omega\leq\alpha} := \{\, b \in G_{k-1} : \omega_{k-1}\big(Q_b^{(k-1)}\big) \leq \alpha \,\}$ for $\alpha \in \mathbb{Z}_+$. We will show via induction that $b_i \not\simeq_k b_j$ for $b_i, b_j \in G^{\omega\leq\alpha}$ with $b_i \neq b_j$. Lemma 5.2 then yields that $G^{\omega\leq\alpha}$ is finite. Clearly we have $G_{k-1} = \bigcup_{\alpha\in\mathbb{Z}_+} G^{\omega\leq\alpha}$. Let $b_i, b_j \in G_{k-1}$ with $b_i \neq b_j$. Then there is $\alpha \in \mathbb{Z}_+$ with $b_i, b_j \in G^{\omega\leq\alpha}$ yielding $b_i \not\simeq_k b_j$. The set G_{k-1} admits property (ii) of the output set which together with Lemma 5.2 yields that G_{k-1} is finite.

We will show via induction that $G^{\omega\leq\alpha}$ is finite. With Step 2 of the proof we know that our claim is proved for $\alpha = 0$. Suppose that our assertions are true for all integers lower or equal than α. We will prove our claim for $\alpha + 1$. Let $b_i, b_j \in G^{\omega\leq\alpha+1}$ and suppose $b_i \simeq_k b_j$. There are several possible cases:

(i) $b_i, b_j \in \widetilde{F}_{k-1}$
This contradicts input property (ii) of the input set $\widetilde{F}_{k-1}$.

(ii) $b_i \in \widetilde{F}_{k-1}$, $b_j \notin \widetilde{F}_{k-1}$
This contradicts the if-clause of Algorithm 6.7 because b_i then is an appropriate reducer of b_j.

(iii) $b_i \notin \widetilde{F}_{k-1}$, $b_j \in \widetilde{F}_{k-1}$
As $\omega_{k-1}(Q_{b_j-b_i}^{(k-1)}) = 0$ and as $G^{\omega=0}$ is completed before $G^{\omega\geq1}$ we know that there is $\overline{b} \in G^{\omega=0}$ with $\overline{b} \simeq_k b_j - b_i \simeq_k b_j$. But this is a contradiction to lines 18–21 of Algorithm 6.8 because in this case b_j would not have been added to $G^{\omega\geq1}$ then.

(iv) $b_i, b_j \notin \widetilde{F}_{k-1}$
Depending on whether either b_i has been added to $G^{\omega=\alpha+1}$ before b_j was added or not we either have a contradiction to the if-clause of Algorithm 6.7 or to the else-clause of this algorithm.

We know via induction that $G^{\omega\leq\alpha}$ admits property (ii) of the output set. We will now show that this is also true for $G^{\omega\leq\alpha+1}$. Let $b_i, b_j \in G^{\omega\leq\alpha+1}$ and suppose $b_i \simeq_k b_j$. By induction, the previous discussion and monotonicity of the weight-function $\omega_k(\cdot)$: $b_i \in G^{\omega\leq\alpha}$ and $b_j \in G^{\omega=\alpha+1}$. But this contradicts the if-clause of Algorithm 6.7. Therefore $G^{\omega\leq\alpha+1}$ admits property (ii) of the output set which had to be proved.

Step 4.) Let $b \in S^{(k-1)}$ such that $Q_b^{(k-1)}$ is atomic with respect to $\oplus^{(k)}$ and weight $\omega_{k-1}\big(Q_b^{(k-1)}\big) =: \alpha_{\min}$ smallest such that there is no $\overline{b} \in G_{k-1}$ with $\overline{b} \sim b$. We know from input property (i) that there is $\widetilde{b} \in \mathbb{Z}^m$, $\widetilde{b} \simeq_k b$, admitting a representation (6.1):

$$Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{i \in I}^{(k-1)} Q_{b_i}^{(k-1)},$$

where $b_i \in \widetilde{F}_{k-1}$. We will show that $\widetilde{b} \in G_{k-1}$ yielding a contradiction. The above representation implies in particular that every $z = (z_1, z_2, z_3) \in Q_{\widetilde{b}}^{(k-1)}$ can be written as $(z_1, z_2, z_3) = \sum_{i \in I} (z_1^i, z_2^i, z_3^i)$ with $(z_1^i, z_2^i, z_3^i) \in Q_{b_i}^{(k-1)}$. In particular,

$$(z_1^i, z_2^i, z_3^i) \sqsubseteq_{k-1} (z_1, z_2, z_3) \tag{6.2}$$

for all i. If $Q_{b_i}^{(k-1)} \ni (z_1^i, z_2^i, z_3^i) \sqsubseteq_k (z_1, z_2, z_3)$ was valid this then would imply: $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{i \in I}^{(k)} Q_{b_i}^{(k-1)}$.

Let $R_{\widetilde{b},k}^{(k-1)} = \{(\overline{z}_1^1, \overline{z}_2^1, \overline{z}_3^1), \ldots, (\overline{z}_1^t, \overline{z}_2^t, \overline{z}_3^t)\}$ be the set of representatives of the $\sqsubseteq_k$-minimal elements in $Q_{\widetilde{b}}^{(k-1)}$ according to Definition 3.8. With Lemma 3.10 we know that it is sufficient to analyze the $\sqsubseteq_k$-minimal elements in a partially extended fiber to decide decomposability with respect to $\oplus^{(k)}$.

From all representations $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j \in J}^{(k-1)} Q_{b_j}^{(k-1)}$ with $b_j \in G_{k-1}$ and $\pi_k\big(Q_{b_j}^{(k-1)}\big) \neq \pi_k\big(Q_0^{(k-1)}\big)$ and where the $\sqsubseteq_{k-1}$-minimal elements in $R_{\widetilde{b},k}^{(k-1)}$ are represented as $(\overline{z}_1^i, \overline{z}_2^i, \overline{z}_3^i) = \sum_{j \in J} (z_1^{i,j}, z_2^{i,j}, z_3^{i,j})$ with $(z_1^{i,j}, z_2^{i,j}, z_3^{i,j}) \in Q_{b_j}^{(k-1)}$ for $i = 1, \ldots, t$, choose a representation and elements $(z_1^{i,j}, z_2^{i,j}, z_3^{i,j})$ such that the sum

$$\sum_{i=1}^{t} \sum_{j \in J} \big\|(z_1^{i,j}, z_2^{i,j})\big\|_1 \tag{6.3}$$

is minimal.

By the triangle inequality we have

$$\sum_{i=1}^{t} \sum_{j \in J} \big\|(z_1^{i,j}, z_2^{i,j})\big\|_1 \geq \sum_{i=1}^{t} \big\|(\overline{z}_1^i, \overline{z}_2^i)\big\|_1 \tag{6.4}$$

Herein equality holds if and only if all $(z_1^{i,j}, z_2^{i,j})$ have the same sign pattern as $(\overline{z}_1^i, \overline{z}_2^i)$, $i = 1, \ldots, t$, that is, if and only if we have $(z_1^{i,j}, z_2^{i,j}, z_3^{i,j}) \sqsubseteq_k (\overline{z}_1^i, \overline{z}_2^i, \overline{z}_3^i)$ for all $j \in J$ and all $i = 1, \ldots, t$. Thus if we have equality in (6.4) for such a minimal representation $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j \in J}^{(k-1)} Q_{b_j}^{(k-1)}$ then by Lemma 3.10 $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j \in J}^{(k)} Q_{b_j}^{(k-1)}$. As $\pi_k\big(Q_{b_j}^{(k-1)}\big) \neq \pi_k\big(Q_0^{(k-1)}\big)$, as $Q_b^{(k-1)}$ (and therefore $Q_{\widetilde{b}}^{(k-1)}$) is atomic with respect to $\oplus^{(k)}$, the above representation must be trivial and we are done.

Therefore let us assume, that

$$\sum_{i=1}^{t}\sum_{j\in J}\left\|(z_1^{i,j},z_2^{i,j})\right\|_1 > \sum_{i=1}^{t}\left\|(\overline{z}_1^i,\overline{z}_2^i)\right\|_1 \tag{6.5}$$

In the following, we construct a new representation $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j'\in J'}^{(k-1)} Q_{b'_j}^{(k-1)}$ and elements $(\widetilde{z}_1^{i,j},\widetilde{z}_2^{i,j},\widetilde{z}_3^{i,j})$ whose corresponding sum (6.3) is smaller than the minimally chosen sum. This contradiction proves that we have indeed equality in (6.4) and our claim is proved.

From (6.5) and from (6.2), i.e., $z_1^{i,j} \sqsubseteq \overline{z}_1^i$ for all i and j, we conclude that there are indices i_0, j_1, j_2 such that $z_2^{i_0,j_1} \cdot z_2^{i_0,j_2} < 0$. As $b_{j_1}, b_{j_2} \in G^{\omega=0}$ the sum $b_{j_1}+b_{j_2}$ was built during the algorithm. There are two cases to consider:

(i) $\omega_{k-1}\left(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\right) = \omega_{k-1}\left(Q_{b_{j_1}}^{(k-1)}\right) + \omega_{k-1}\left(Q_{b_{j_2}}^{(k-1)}\right) = 0$ or

(ii) $\omega_{k-1}\left(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\right) = \omega_{k-1}\left(Q_{b_{j_1}}^{(k-1)}\right) + \omega_{k-1}\left(Q_{b_{j_2}}^{(k-1)}\right) > 0$

Consider case (i) first. We have

$$\omega_{k-1}\left(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\right) = \omega_{k-1}\left(Q_{b_{j_1}}^{(k-1)}\right) + \omega_{k-1}\left(Q_{b_{j_2}}^{(k-1)}\right) = 0$$

and thus there is no partially extended fiber with weight ω_{k-1} greater than 0 that reduces $Q_{b_{j_1}+b_{j_2}}^{(k-1)}$. Consequently the partially extended fiber $Q_{b_{j_1}+b_{j_2}}^{(k-1)}$ was either reduced to $Q_0^{(k-1)}$ by sets $Q_{b_{j''}}^{(k-1)}$, $j'' \in J''$, during the else-clause of the Algorithm 6.7 or the vector $b_{j_1}+b_{j_2}$ has been added to the set $\overline{G}^{\omega=0}$. Then either $b_{j_1}+b_{j_2} \in G^{\omega=0}$ or we find sets $Q_{b_{j''}}^{(k-1)}$, $j'' \in J''$, with

$$Q_{b_{j_1}+b_{j_2}}^{(k-1)} = \bigoplus_{j''\in J''}^{(k)} Q_{b_{j''}}^{(k-1)}$$

with $b_{j''} \in G^{\omega=0}$. In the former case, set $J'' := \{j''\}$ with $b_{j''} := b_{j_1}+b_{j_2}$.

Now consider case (ii). Monotonicity of the weight-function ω_{k-1} implies that $\omega_{k-1}\big(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\big) \le \alpha_{\min}$. Suppose that $\omega_{k-1}\big(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\big) < \alpha_{\min}$. As there are representatives of all equivalence classes of weight smaller than α_{min} included in G_{k-1} we find an index set J'' with $b_{j''} \in G_{k-1}$ and

$$Q_{b_{j_1}+b_{j_2}}^{(k-1)} = \bigoplus_{j''\in J''}^{(k)} Q_{b_{j''}}^{(k-1)}.$$

Therefore let us assume that $\omega_{k-1}\big(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\big) = \alpha_{\min}$. In this case as well, we either find an index set J'' with $b_{j''} \in G_{k-1}$ and

$$Q_{b_{j_1}+b_{j_2}}^{(k-1)} = \bigoplus_{j''\in J''}^{(k)} Q_{b_{j''}}^{(k-1)}$$

or $b_{j_1} + b_{j_2}$ has been added to the set $G^{\omega\geq 1}$. In the latter case we set $J'' := \{j''\}$ with $b_{j''} = b_{j_1} + b_{j_2}$.

Each of these cases gives representations

$$(z_1^{i,j_1}, z_2^{i,j_1}, z_3^{i,j_1}) + (z_1^{i,j_2}, z_2^{i,j_2}, z_3^{i,j_2}) = \sum_{j''\in J''} (\widetilde{z}_1^{i,j''}, \widetilde{z}_2^{i,j''}, \widetilde{z}_3^{i,j''}),$$

$$(\widetilde{z}_1^{i,j''}, \widetilde{z}^{i,j''}, \widetilde{z}_3^{i,j''}) \in Q_{b_{j''}}^{(k-1)},$$

$$(\widetilde{z}_1^{i,j''}, \widetilde{z}_2^{i,j''}, \widetilde{z}_3^{i,j''}) \sqsubseteq_k (z_1^{i,j_1}, z_2^{i,j_1}, z_3^{i,j_1}) + (z_1^{i,j_2}, z_2^{i,j_2}, z_3^{i,j_2})$$

for $i = 1, \ldots, t$. As all $\left(\widetilde{z}_1^{i,j''}, \widetilde{z}_2^{i,j''}\right)$ lie in the same orthant as $\left(z_1^{i,j_1}, z_2^{i,j_1}\right) + \left(z_1^{i,j_2}, z_2^{i,j_2}\right)$ we get

$$\begin{aligned}\Big\| \sum_{j''\in J''} (\widetilde{z}_1^{i,j''}, \widetilde{z}_2^{i,j''}) \Big\|_1 &= \left\|(z_1^{i,j_1}, z_2^{i,j_1}) + (z_1^{i,j_2}, z_2^{i,j_2})\right\|_1 \\ &\leq \left\|(z_1^{i,j_1}, z_2^{i,j_1})\right\|_1 + \left\|(z_1^{i,j_2}, z_2^{i,j_2})\right\|_1\end{aligned}$$

with strict inequality for $i = i_0$. So, by replacing in $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j\in J}^{(k-1)} Q_{b_j}^{(k-1)}$ the term $Q_{b_{j_1}}^{(k-1)} \oplus^{(k-1)} Q_{b_{j_2}}^{(k-1)}$ by $\bigoplus_{j''\in J''}^{(k)} Q_{b_{j''}}^{(k-1)}$ we arrive at a new representation $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j'\in J'}^{(k-1)} Q_{b_{j'}}^{(k-1)}$ whose corresponding sum (6.3) is at most

$$\sum_{i=1}^{t} \sum_{j'\in J'} \left\|(z_1^{i,j'}, z_2^{i,j'})\right\|_1 < \sum_{i=1}^{t} \sum_{j\in J} \left\|(z_1^{i,j}, z_2^{i,j})\right\|_1$$

contradicting the minimality of the representation $Q_{\widetilde{b}}^{(k-1)} = \bigoplus_{j\in J}^{(k-1)} Q_{b_j}^{(k-1)}$. Therefore we have equality in (6.4) and thus $\widetilde{b} \in G_{k-1}$ concluding our proof. $\square$

6.2. **Intersecting with the appropriate orthant and testing reducibility.** In this subsection we want to illustrate the step of the project-and-lift algorithm which follows the completion procedure in each lifting step. This "intersection and reducibility test" is illustrated by the dashed arrow in Figure 5.

Lemma 6.5. *Algorithm 6.9 with input set G_{k-1} terminates and computes a set $F_k \subseteq S^{(k)}$ with the properties (i) and (ii).*

Proof. Termination of Algorithm 6.9 is clear. We have to show correctness of the algorithm. If $Q_b^{(k)} \neq \varnothing$ then $b \in S^{(k)}$. Therefore $F_k \subseteq S^{(k)}$. If $Q_b^{(k)}$ is atomic with respect to $\oplus^{(k)}$, then $Q_b^{(k-1)}$ is atomic with respect to $\oplus^{(k)}$ as well, because $Q_b^{(k)} \subseteq Q_b^{(k-1)}$ and every decomposition of $Q_b^{(k-1)}$ would give a decomposition of $Q_b^{(k)}$. This characteristic immediately implies property (i) of the output set because we have property (i) of the input set.

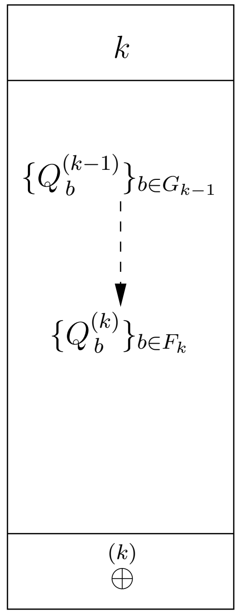

FIGURE 5. Intersecting with the appropriate orthant and dropping reducible partially extended fibers

Algorithm 6.9 Intersecting and testing reducibility

Input: A set $G_{k-1} \subseteq S^{(k-1)}$ with the properties:

(i) For every right-hand side $b \in S^{(k-1)}$ of a partially extended fiber $Q_b^{(k-1)}$ which is atomic with respect to $\oplus^{(k)}$ there exists $\widetilde{b} \in G_{k-1}$ with $\widetilde{b} \simeq_k b$.

(ii) $b_i \not\simeq_k b_j$ for $b_i, b_j \in G_{k-1}$ with $b_i \neq b_j$

Output: A set $F_k \subseteq S^{(k)}$ of right-hand sides with:

(i) For every right-hand side $b \in S^{(k)}$ of a partially extended fiber $Q_b^{(k)}$ which is atomic with respect to $\oplus^{(k)}$ there exists $\widetilde{b} \in F_k$ with $\widetilde{b} \simeq_k b$.

(ii) $b_i \not\simeq_k b_j$ for $b_i, b_j \in F_k$ with $b_i \neq b_j$.

$F_k := \varnothing$
for all $b \in G_{k-1}$ with $Q_b^{(k)} \neq \varnothing$ **do**
  if $Q_b^{(k)} \neq Q_g^{(k)} \oplus^{(k)} Q_{b-g}^{(k)}$ for all $b \neq g \in G_{k-1}$ **then**
    $F_k := F_k \cup \{b\}$
  end if
end for
return F_k

To see property (ii) of the output set, suppose that there are $b_1, b_2 \in F_k$ with $b_2 \simeq_k b_1$. Then, $Q_{b_1}^{(k)} = Q_{b_2}^{(k)} \oplus^{(k)} Q_{b_1-b_2}^{(k)}$ and $b_1 - b_2 \in L^{(k)}$. In this case either b_1 or b_2 would not have been added to F_k. This yields that $b_i \not\simeq_k b_j$ for all $b_i, b_j \in F_k$. Therefore Algorithm 6.9 is correct and terminates. $\square$

6.3. **Refining the equivalence relation.** There is one more step to explain in the k-th lifting step of the project-and-lift algorithm. This step is illustrated by the dotted arrow in Figure 6.

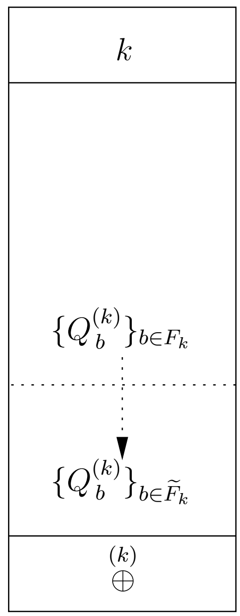

FIGURE 6. Refining the equivalence relation to prepare the $k+1$-st lifting step

Having a set $F_k \subseteq S^{(k)}$ at hand admitting the following properties:

(i) for every right-hand side $b \in S^{(k)}$ of a partially extended fiber $Q_b^{(k)}$ which is atomic with respect to $\oplus^{(k)}$ there exists $\widetilde{b} \in F_k$ with $\widetilde{b} \simeq_k b$.

(ii) $b_i \not\simeq_k b_j$ for $b_i, b_j \in F_k$,

we want to construct a set $\widetilde{F}_k \subseteq S^{(k)}$ with

(i) For every right-hand side $b \in S^{(k)}$ there is $\widetilde{b} \in \mathbb{Z}^m$ with $\widetilde{b} \simeq_{k+1} b$ and

$$Q_{\widetilde{b}}^{(k)} = \bigoplus_i^{(k)} Q_{b_i}^{(k)} \quad \text{where } b_i \in \widetilde{F}_k$$

(ii) $b_i \not\simeq_{k+1} b_j$ for all $b_i, b_j \in \widetilde{F}_k$ with $b_i \neq b_j$.

Lemma 6.6. *Let $Q_b^{(k)}$ be atomic with respect to $\oplus^{(k)}$. Then there is $\overline{b} \in F_k$ and $\lambda_b \in \mathbb{Z}$ such that $\overline{b} + \lambda_b A_{k+1} \simeq_{k+1} b$.*

Proof. Let $Q_b^{(k)}$ be atomic w.r.t. $\oplus^{(k)}$. Then there is $\overline{b} \in F_k$ with $\overline{b} \simeq_k b$. If $\overline{b} \simeq_{k+1} b$ we set $\lambda_b := 0$ and we are done. Now suppose $\overline{b} \not\simeq_{k+1} b$. Consider

$b - \overline{b} = \sum_{i \geq k+1} \lambda_i A_i$, where $\lambda_i \in \mathbb{Z}$ and $\lambda_{k+1} \neq 0$. Setting $\lambda_b := \lambda_{k+1}$ yields $\overline{b} + \lambda_b A_{k+1} \simeq_{k+1} b$. □

Now we are in the position to define the set $\widetilde{F}_k$. If $A_{k+1}, -A_{k+1} \not\simeq_{k+1} 0$, we set $\widetilde{F}_k := F_k \cup \{\pm A_{k+1}\}$. If $A_{k+1} \not\simeq_{k+1} 0$, $-A_{k+1} \not\simeq_{k+1} 0$ or $\pm A_{k+1} \not\simeq_{k+1} 0$ we set $\widetilde{F}_k = F_k \cup \{-A_{k+1}\}$, $\widetilde{F}_k = F_k \cup \{A_{k+1}\}$ or $\widetilde{F}_k = F_k$.

Lemma 6.7. *Let $b \in S^{(k)}$. There exists $\widetilde{b} \simeq_{k+1} b$ with*

$$Q^{(k)}_{\widetilde{b}} = \overset{(k)}{\bigoplus_i} Q^{(k)}_{b_i} \quad \textit{with } b_i \in \widetilde{F}_k.$$

Proof. Consider a decomposition of $Q^{(k)}_b$ into partially extended fibers that are atomic w.r.t. $\oplus^{(k)}$:

$$Q^{(k)}_b = \overset{(k)}{\bigoplus_i} Q^{(k)}_{b_i}.$$

With Lemma 6.6 we know that for each b_i in this decomposition there is $\overline{b}_i \in F_k$ and $\lambda_{b_i} \in \mathbb{Z}$ such that $b_i \simeq_{k+1} \overline{b}_i + \lambda_{b_i} A_{k+1}$. We set $\widetilde{b} := \sum_i (\overline{b}_i + \lambda_{b_i} A_{k+1})$. Then clearly $\widetilde{b} \simeq_{k+1} b$ and furthermore

$$Q^{(k)}_{\widetilde{b}} = \overset{(k)}{\bigoplus_i} \left(Q^{(k)}_{\overline{b}_i} \overset{(k)}{\oplus} \lambda_{b_i} Q^{(k)}_{A_{k+1}} \right) = \overset{(k)}{\bigoplus_i} Q^{(k)}_{\overline{b}_i} \overset{(k)}{\oplus} \Big(\sum_i \lambda_{b_i} \Big) Q^{(k)}_{A_{k+1}}.$$

This proves our claim. □

7. Comparing the two algorithms

We have created an implementation of the "project-and-lift" algorithm. The implementation is written in Allegro Common Lisp 8.0 and C. For the computation of the minimal elements of partially extended fibers, we use the library `libzsolve`, which is a part of `4ti2` [1], version 1.3.1. In this section, we report on the computational experience with this code on several test problems. All computation times are given in CPU seconds on a Sun Fire V440 with UltraSPARC-IIIi processors running at 1.6 GHz.

7.1. **Results for number-partitioning problems.** We first consider the problem of partitioning a natural number n into given *parts* (natural numbers) $a_1, \ldots, a_k$ (with possible multiplicity). To this end, consider the set

$$P_n = \Big\{ (x_1, \ldots, x_k) \in \mathbb{Z}^k_+ : n = \sum_{i=1}^{k} x_i \cdot a_i \Big\}. \tag{7.1}$$

We are interested in a minimal set $\{n_1, \ldots, n_q\}$ of natural numbers such that the set P_n of partitions of every number n is the Minkowski sum of some of the sets P_{n_j}. Thus we are interested in the atomic fibers corresponding to the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_k \end{pmatrix}. \tag{7.2}$$

We consider this problem for various sets of numbers $a_1, \ldots, a_k$. The results are shown in Table 1.

TABLE 1. Results for number-partitioning problems

		Time (s)	
Parts	Atomic fibers	Algorithm 4.4	Algorithm 5.6
1	1	1	1
1 2	2	1	1
1 2 3	4	1	1
1 2 3 4	9	1	1
1 2 3 4 5	32	?	311
1 2 3 4 5 6	41	?	30618
2 3	3	1	1
2 3 5	14	1	1
3 5	1	1	1
3 5 7	30	16	1
5 7 11	62	221	19
5 7 13	62	409	12

7.2. **Results for homogeneous number-partitioning problems.** We consider the problem of partitioning a natural number n into given natural numbers $a_1, \ldots, a_k$ (with possible multiplicity), where we prescribe the number of summands. To this end, we consider the set

$$P_n^m = \left\{ (x_1, \ldots, x_k) \in \mathbb{Z}_+^k : n = \sum_{i=1}^k x_i \cdot a_i,\ m = \sum_{i=1}^k x_i \right\}. \tag{7.3}$$

We are interested in a minimal set $\{(m_1, n_1), \ldots, (m_q, n_q)\}$ of pairs (m, n) such that the set P_n^m of partitions of every number n into m summands is the Minkowski sum of some of the sets $P_{n_j}^{m_j}$. Thus we are interested in the atomic fibers corresponding to the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}. \tag{7.4}$$

Again we consider the problem for various sets of numbers $a_1, \ldots, a_k$. The results are shown in Table 2. We remark that the problem data $(1, 2, 3, 4)$ correspond to a problem equivalent to the one from Example 3.7.

7.3. **Results for Steinberger's sums of roots of unity.** One example that appears and was solved in [18] is the computation of the atomic fibers

Table 2. Results for homogeneous number-partitioning problems

		Time (s)	
Parts	Atomic fibers	Algorithm 4.4	Algorithm 5.6
1	1	1	1
1 2	2	1	1
1 2 3	4	1	1
1 2 3 4	18	2	1
1 2 3 4 5	79	5511	12
2 3	2	1	1
2 3 5	4	1	1
2 3 5 7	26	172	1
1 2 3 5	12	3	2
1 2 3 6	35	858	4
1 2 3 7	19	199	1
1 2 3 8	58	63861	89
1 2 3 9	28	6707	3
1 2 3 10	87	>2000000	1211
1 2 3 11	39	135375	18
1 2 3 13	52	?	119
1 2 3 15	67	?	770
1 2 3 17	84	?	6331

of the matrix

$$\begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix}.$$

This matrix corresponds to a certain problem on 3×3 tables and has in fact 31 atomic fibers and 79 extended atomic fibers. The atomic fibers can be computed with our implementation in less than one CPU second.

The next higher problem on 4×4 tables leads to the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Our implementation was able to compute the 12,675 atomic fibers for this matrix within 6.5 CPU days.

7.4. **Network instances.** Fibers of matrices arising from multicommodity flow problems have been used by [5, 6] to reformulate nonlinear network design problem as integer linear programs. The matrices corresponding to these problems arise from fixed underlying digraphs with a prescribed commodity structure. Each commodity gives rise to a submatrix consisting of a node-arc incidence matrix (modeling demand and flow conservation constraints) and an identity matrix (modeling capacity constraints). As the capacity constraints are inequalities, slack variables are introduced for all arcs of the underlying digraph to formulate the capacity constraints as equality constraints. These slack variables give rise to a submatrix consisting of an all-zero matrix (with respect to the demand and flow-conservation constraints) and an identity matrix (with respect to the capacity constraints).

We computed the set of atomic fibers for a few network instances (see Table 3), the underlying graphs of which are depicted in Figure 7. The right-hand sides stem from the lattice that is induced by the flow-conservation constraints.

Note that circulations of flow are truncated in these examples, i.e., the fibers consist just of those multicommodity flows that are circulation-free. During the project-and-lift algorithm, elements in partially extended fibers are truncated if they admit a circulation of flow on variables that are already lifted, i.e., that are non-negative. Correctness of the project-and-lift algorithm follows from the following observation: let f be a circular solution in $Q_b^{(k)}$. Then f is a representative of the set of solutions $f + Q_0^{(k)}$. The non-negativity on the first k components implies that every element in this set admits a circulation of flow. Therefore, no circulation-free element is removed by truncating the representative f.

For some network instances, we bounded the demand vector of the multicommodity flow problems. This means that we computed only those atomic fibers with a demand vector less or equal than the bounding demand vector. The bounding demand vector is given in the column entitled "Demand bounds" of Table 3. The column entitled "Commodities" gives the number of commodities of the underlying multicommodity flow problem.

8. Conclusions

Our computational study in Section 7 shows that the development of the algorithmic theory of lifting partially extended atomic fibers has led to a dramatic improvement upon basic completion-type algorithms.

This algorithmic and computational study is only the beginning. We hope that the new computational tools introduced in this paper will have an impact not only on our own applications in optimization, but also on the computer algebra community through the increased computational power that is now available for constructing strong SAGBI bases.

An extension of the algorithm to compute the atomic fibers with respect to monoids is introduced in [5].

TABLE 3. Results for network instances

Instance	Nodes	Arcs	Commodities	Demand bounds	Atomic fibers	Time (s)	
						Algorithm 4.2	Algorithm 5.6
N1	3	4	1		13	5	1
N2	4	5	1		13	694	1
N3	5	7	2		19	>70000	1
N4	3	4	2		13	>70000	1
N5	4	6	1		40	>70000	3
N6	5	8	1		41	>70000	3
N7	8	12	2		29	>70000	3
N8	4	6	1		12	>70000	5
N9	4	8	2	(1, 1)	142	>70000	5
N10	4	7	1		101	>70000	9
N11	8	14	2	(1, 1)	273	>70000	22
N12	4	10	2	(1, 1)	509	>70000	23
N13	5	8	1		102	>70000	49
N14	8	16	2	(1, 1)	1606	>70000	507
N15	11	18	3	(1, 1, 1)	1119	>70000	449
N16	11	19	3	(1, 1, 1)	9195	>70000	37871

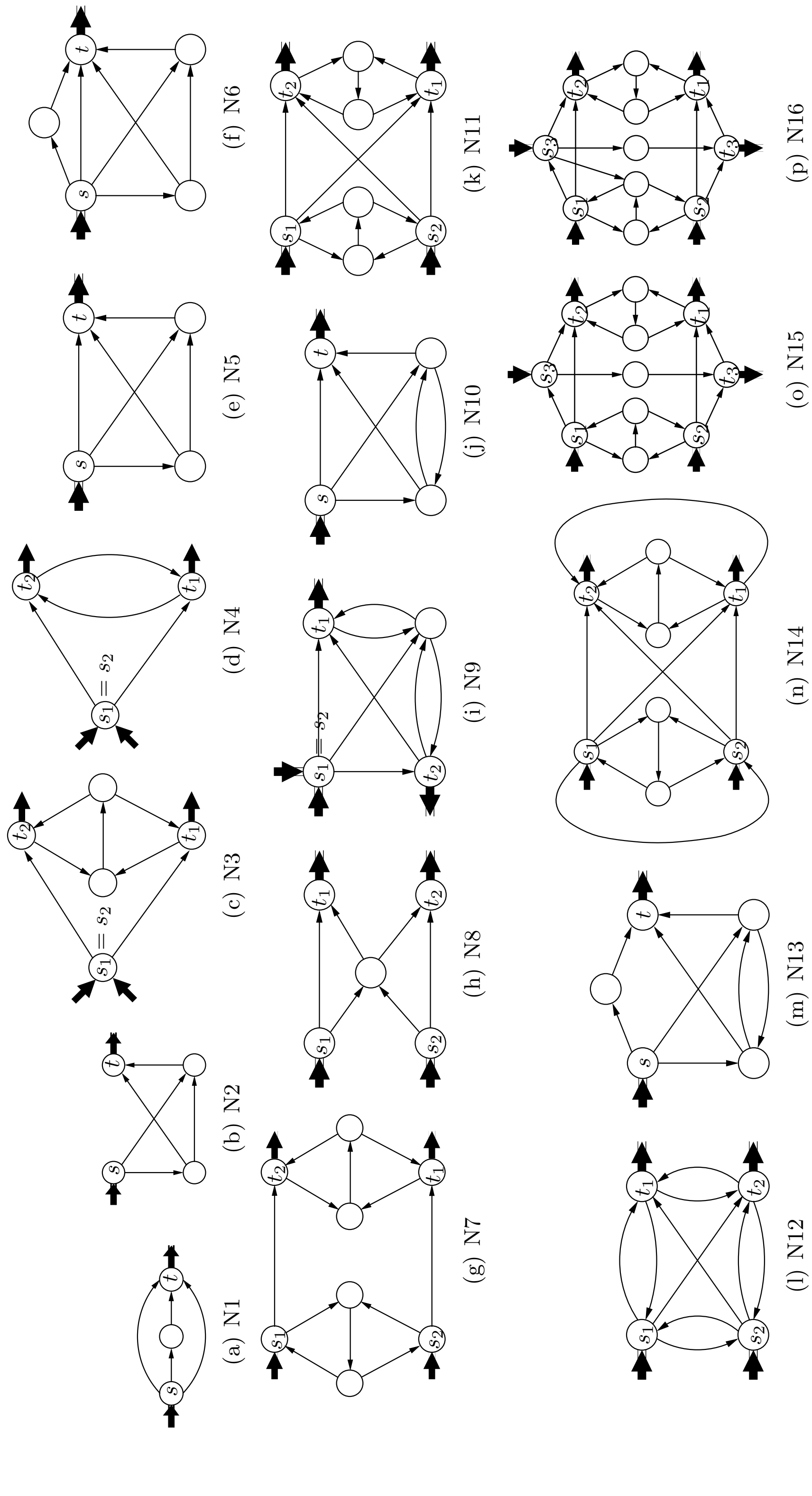

Figure 7. The network instances.

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