



ON CYCLE PACKINGS AND FEEDBACK VERTEX SETS

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ABSTRACT. For a graph G , let \mathbf{fvs} and \mathbf{cp} denote the minimum size of a feedback vertex set in G and the maximum size of a cycle packing in G , respectively. Kloks, Lee, and Liu conjectured that $\mathbf{fvs}(G) \leq 2 \mathbf{cp}(G)$ if G is planar. They proved a weaker inequality, replacing 2 by 5. We improve this, replacing 5 by 3, and verifying the resulting inequality for graphs embedded in surfaces of nonnegative Euler characteristic. We also generalize to arbitrary surfaces. We show that, if a graph G embeds in a surface of Euler characteristic $c \leq 0$, then $\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G) + 103(-c)$. Lastly, we consider what the best possible bound on \mathbf{fvs} might be, and give some open problems.

1. BACKGROUND

All graphs will be finite and undirected. We will allow loops and multiple edges. A *cycle packing* in a graph G is a set of vertex-disjoint cycles in G . We denote the maximum cardinality of a cycle packing in G by $\mathbf{cp}(G)$. A *feedback vertex set* in G is a set of vertices of G that meets every cycle in G . We denote the minimum cardinality of a feedback vertex set in G by $\mathbf{fvs}(G)$.

It is easily seen that $\mathbf{fvs}(G) \geq \mathbf{cp}(G)$. Thus, knowing the value of $\mathbf{cp}(G)$, places a lower bound on $\mathbf{fvs}(G)$. Does it also place an *upper* bound on $\mathbf{fvs}(G)$? Erdős and Pósa [1] answered this question in the affirmative by showing the following.

Theorem 1.1 (Erdős & Pósa 1965 [1]). *The maximum value of $\mathbf{fvs}(G)$, over all graphs G with $\mathbf{cp}(G) = k$, is $\Theta(k \log k)$.*

Kloks, Lee, and Liu [2] sought a better upper bound for planar graphs. They made the following conjecture, known as “Jones’ Conjecture”¹.

Conjecture 1.2 (Jones’ Conjecture—Kloks, Lee & Liu 2002 [2, Conj. 2]). *Let G be a planar graph. Then*

$$\mathbf{fvs}(G) \leq 2 \mathbf{cp}(G).$$

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¹“Jones” is an alternate name for C. M. Lee [3].

If this conjecture holds, then the inequality is sharp, as shown by K_4 —or, more generally, a disjoint union of copies of K_4 , possibly with pendant vertices of degree 1 added.

Kloks, Lee, and Liu [2, Thm. 11] verified Conjecture 1.2 for the special case of outerplanar graphs. For planar graphs in general, they proved a weaker result [2, Thm. 8]: $\mathbf{fvs}(G) \leq 5 \mathbf{cp}(G)$. We improve this last result, replacing 5 with 3. We also generalize to graphs on other surfaces, and we state a number of open problems.

We denote the vertex set of graph G by $V(G)$. We denote the degree of a vertex v by $d(v)$, and the length of a face f by $\ell(f)$. An embedding of a graph G in a surface is a *2-cell embedding* if the interior of each face of G is homeomorphic to an open disk.

2. SURFACES OF NONNEGATIVE EULER CHARACTERISTIC

We show that $\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G)$ if G embeds in a surface with nonnegative Euler characteristic.

We begin with a technical lemma showing the existence of certain configurations in a graph. We will make use of this lemma in our proof of the above result.

Lemma 2.1. *Let G be a connected simple graph, 2-cell embedded in a closed surface S with nonnegative Euler characteristic. Let G have minimum degree at least 3 and girth at least 4. Then one of the following holds.*

- (i) G has either
 - (a) a face of length 6 with all vertices having degree 3,
 - (b) a face of length 5 with at least 3 vertices of degree 3, counting multiplicities, or
 - (c) a face of length 4 with at least 1 vertex of degree 3.
- (ii) Every face of G has length 4, and every vertex of G has degree 4.

Proof. Let G and S be as in the statement of the result. Assign a charge ch to the vertices and faces of G as follows. For each vertex v , let $ch(v) = 4 - d(v)$. For each face f , let $ch(f) = 4 - \ell(f)$. Then the sum of all the vertex and face charges is $4V - 4E + 4F$ (where V , E , and F count the vertices, edges, and faces of G , respectively), which is 4 times the Euler characteristic of the surface S , and thus is nonnegative.

Now redistribute the vertex charge. For each vertex v , add $ch(v)/d(v)$ to the charge of each face on which v lies, counting multiplicities, and set the charge of v to 0. Denote this revised charge by ch' . Then for each face f ,

$$ch'(f) := 4 - \ell(f) + \sum_{v \in f} \frac{4 - d(v)}{d(v)},$$

where “ $v \in f$ ” means that v is a vertex in the facial walk of f , and the sum counts multiplicities.

The total ch' charge is the same as the total ch charge; in particular, it is nonnegative. Since $ch'(v) = 0$ for each vertex v , there must be a face f_+ with $ch'(f_+) \geq 0$.

Let k be the number of vertices of degree 3 lying on f_+ , counting multiplicities. Since vertices of degree 4 or more had nonpositive ch charge, we have

$$0 \leq ch'(f_+) \leq 4 - \ell(f_+) + \frac{1}{3}k,$$

and so

$$(2.1) \quad k \geq 3\ell(f_+) - 12.$$

If $\ell(f_+) > 6$, then inequality (2.1) implies that $k > \ell(f_+)$, which is impossible, and so we must have $4 \leq \ell(f_+) \leq 6$.

If $\ell(f_+) = 6$, then inequality (2.1) becomes $k \geq 6$, and so statement (i)(a) in our result holds, and we are done. If $\ell(f_+) = 5$, then inequality (2.1) becomes $k \geq 3$, and so statement (i)(b) holds, and again, we are done. We may thus assume that every face with a nonnegative ch' , has length 4. If some face of length 4 contains a vertex of degree 3, then statement (i)(c) holds, and we are done; we therefore suppose that there is no such face.

Suppose there is a face with a nonzero ch' . Since the sum of the face charges is nonnegative, there must be a face with a positive ch' . For a face of length 4, this can only happen if the face has a vertex of degree 3; however, we have already ruled this out.

We are left with the case in which every face has ch' charge 0, and hence, by our assumption above, length 4. No such face contains a vertex of degree 3; thus, every vertex in G must have degree 4, and so statement (ii) holds. \square

We will apply the following fact, which has been used by any number of authors, often without being stated formally.

Lemma 2.2. *Let G be a connected graph that embeds in a closed surface S of Euler characteristic c . Then G can be 2-cell embedded in some surface S^* of Euler characteristic $c^* \geq c$.*

Proof. If S is orientable, then let S^* be the orientable surface of minimum genus in which G can be embedded. Our conclusion then follows from a theorem of Youngs [8, Thm. 4.3], which states that every embedding of a connected graph in an orientable surface of minimum genus is a 2-cell embedding.

If S is nonorientable, and G is not a tree, then let S^* be the nonorientable surface of minimum genus in which G can be embedded. Our conclusion follows from a theorem of Parsons, Pica, Pisanski, and Ventre [6, Thm. 2], which states—among other things—that for a connected graph G , either G is a tree, or *some* embedding of G in a nonorientable surface of minimum genus is a 2-cell embedding.

Lastly, if G is a tree, then G is planar; let S^* be a sphere. \square

See Mohar & Thomassen [5, Sect. 3.4] for an exposition of the above ideas. Now we can prove our first theorem.

Theorem 2.3. *Let G be a graph that embeds in a closed surface of nonnegative Euler characteristic. Then*

$$\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G).$$

Proof. Our proof uses an improvement of a method of Kloks, Lee, and Liu [2, proof of Thm. 8].

Let G be a graph embedded in a closed surface of nonnegative Euler characteristic. We may assume that G is connected; if it is not, then add edges joining different components; this alters neither \mathbf{fvs} nor \mathbf{cp} . We may also assume that the embedding is a 2-cell embedding; if it is not, then, by Lemma 2.2, graph G can be 2-cell embedded in some surface of greater or equal (and thus still nonnegative) Euler characteristic.

We will construct a set \mathcal{P} of cycles of G , and $F \subseteq V(G)$, so that \mathcal{P} is a cycle packing, F is a feedback vertex set in G , and $|F| \leq 3|\mathcal{P}|$. This will suffice to prove our result.

We proceed by induction on the order of G . In the base case, G has order 0; we let \mathcal{P} and F be empty sets. Now suppose that G has order at least 1.

The remainder of this proof is divided into a number of cases. In each case, we will construct a graph G^* of strictly smaller order, apply the induction hypothesis to obtain sets \mathcal{P}^* and F^* meeting the requirements for G^* , and then use these to construct \mathcal{P} and F . If we form \mathcal{P} by adding 0 or more cycles to \mathcal{P}^* , then we will construct F by adding to F^* at most 3 vertices from each new cycle. Then it will suffice to verify, first, that each new cycle in \mathcal{P} is vertex-disjoint from every other cycle in \mathcal{P} , and second, letting T be the set of vertices that lie neither in $V(G^*)$ nor in F , that each vertex in T is adjacent to at most 1 vertex in $V(G^*) \cup T$ —and thus that G contains no cycles that fail to meet F .

Small Girth or Minimum Degree: Suppose that G has a cycle C of length at most 3 (this case includes a cycle of length 1 whose edge is a loop, as well as a cycle of length 2 containing parallel edges). Remove the vertices of C from G to obtain G^* , and apply the induction hypothesis to obtain \mathcal{P}^* and F^* . Let \mathcal{P} be \mathcal{P}^* along with C , and let F be F^* along with all the vertices of C , and we are done.

We may now assume that G has girth at least 4; in particular, G is simple.

Suppose that G has a vertex v of degree at most 2. If v has degree 0 or 1, then remove v from G to obtain G^* . Let $\mathcal{P} = \mathcal{P}^*$ and $F = F^*$, and we are done.

If v has degree 2—say v has neighbors x and y —then, because G has girth at least 4, vertices x and y must be nonadjacent. We obtain G^* by “unsubdividing an edge”: remove v from G and add an edge between x and y . Apply the induction hypothesis. Let \mathcal{P} contain the same cycles as \mathcal{P}^* , modified by subdividing, if necessary: if the cycle contains edge xy , then

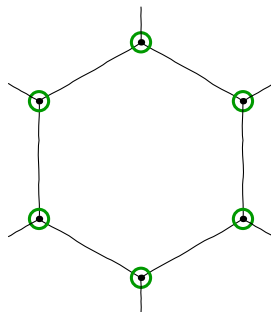


FIGURE 1. A face of length 6 with all vertices having degree 3. The small circle around each vertex emphasizes the fact that the vertex has degree 3.

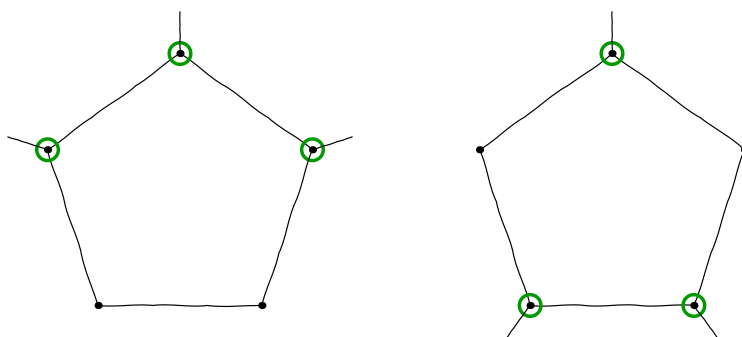


FIGURE 2. Faces of length 5 having at least 3 vertices of degree 3.

replace edge xy by vertex v and edges vx and vy . Let $F = F^*$. Clearly, we have $|F| \leq 3|\mathcal{P}|$. Since the cycles of G are precisely the cycles of G^* , with v added if the cycle in G^* uses edge xy , we see that F is a feedback vertex set in G , and we are done.

Applying Lemma 2.1: We may now assume that G has minimum degree at least 3. Since we are also assuming that G has girth at least 4, we may apply Lemma 2.1.

If statement (i) of the lemma holds for some face, then let C be the facial walk of this face. Suppose a vertex v appears more than once in C . Statement (i) requires facial length at most 6, so we can begin at vertex v , follow at most 3 edges of C , and end at v again; we have found a cycle of length 3 or less in G . But we have assumed that G has girth at least 4. Thus, no vertex is repeated; C is in fact a cycle. Remove C from G to obtain G^* . Let \mathcal{P} be \mathcal{P}^* along with C .

If C has length 6, and all its vertices have degree 3 (see Figure 1), then let F be F^* plus 3 vertices of C , no 2 of which are consecutive on C , and we are done.

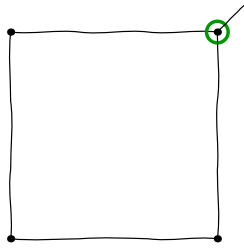


FIGURE 3. A face of length 4 having at least one vertex of degree 3.

If C has length 5 and at least 3 vertices of degree 3 (see Figure 2), then there must be 2 such vertices that are not consecutive on C ; let F be F^* plus the other 3 vertices of C , and, again, we are done.

If C has length 4 and at least 1 vertex of degree 3 (see Figure 3), then let F be F^* plus the other 3 vertices of C , and once again we are done.

Square Grids: It remains only to handle the case when statement (ii) of the lemma holds: every face of G has length 4, and every vertex of G has degree 4. (*Note: We observe that this is possible only when G is embedded in a surface with Euler characteristic 0. Thus, for surfaces of positive Euler characteristic, our proof is actually complete at this point.*)

We will make use of the following claim.

Claim. If G contains vertex-disjoint 4-cycles C_1, C_2 , and there are distinct, nonincident edges e_1, e_2 in G , each of which has one endpoint in each C_i , then we may construct \mathcal{P} and F as required.

To see that this claim holds, remove the vertices of C_1, C_2 from G to obtain G^* , and apply the induction hypothesis as usual. Let \mathcal{P} be \mathcal{P}^* plus C_1, C_2 , and let F be F^* plus all vertices of C_1, C_2 , except for the endpoint of e_1 in C_1 and the endpoint of e_2 in C_2 . Note that we form \mathcal{P} by adding cycles to \mathcal{P}^* , and we construct F by adding to F^* exactly three vertices from each new cycle.

We can now check that \mathcal{P} and F are the required sets by verifying the conditions given near the beginning of this proof. First, each new cycle is vertex-disjoint from every other cycle in \mathcal{P} . Second, let T be the set of vertices that lie neither in $V(G^*)$ nor in F ; so T contains the endpoint of e_1 in C_1 and the endpoint of e_2 in C_2 . Each vertex in T has 3 neighbors among those added to F . Every vertex of G has degree 4; hence each vertex in T is adjacent to at most 1 vertex in $V(G^*) \cup T$. Thus \mathcal{P} and F satisfy the required conditions. We conclude that the claim holds. (See Figure 4 for an illustration.)

In the remainder of this proof, we will proceed in a series of five steps. In each, we will apply the above claim if G meets certain conditions; otherwise we will proceed to the next step.

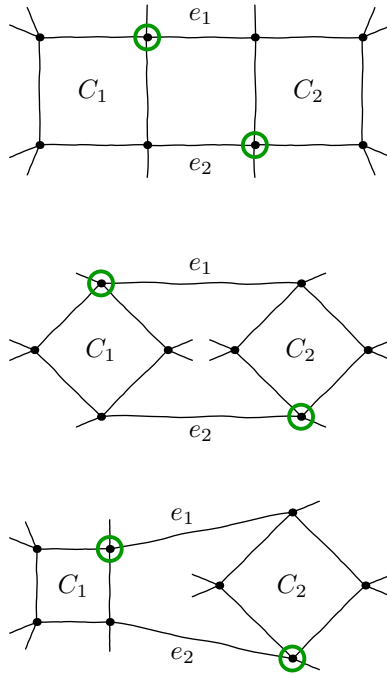


FIGURE 4. An illustration of the Claim in the “Square Grids” section of the proof of Theorem 2.3. Graph G contains vertex-disjoint 4-cycles C_1 and C_2 , and distinct, nonincident edges e_1 and e_2 , each of which has one endpoint in each C_i . Three possibilities are shown: the endpoints of e_1 and e_2 are consecutive in both cycles, in neither cycle, or in exactly one of the cycles. In each case, sets \mathcal{P} and F , as constructed in the proof, have the required properties. Circled vertices are elements of T , that is, the vertices *not* added to F . Note that each such vertex has at most 1 neighbor outside those pictured.

Let f_0 be a face of G . Then f_0 has length 4, and every vertex on f_0 has degree 4. Furthermore, each face adjacent to f_0 has these same two properties. See Figure 5 for an illustration of this situation.

The central face in Figure 5 is f_0 . Because G is simple, the 4 vertices on f_0 are all distinct. These 4 vertices are also distinct from every other vertex in the figure, since otherwise G would contain a cycle of length at most 3. However, it is possible that not all vertices shown in the figure are distinct.

Step 1. Consider Figure 6. Each of the three drawings in this figure is a representation of the same vertices and edges as those in Figure 5.

Suppose that the circled vertices in the left-hand drawing in Figure 6 are all distinct. Then the two 4-cycles shown in the center drawing satisfy the requirements of our claim; we may thus apply the claim, and we are done.

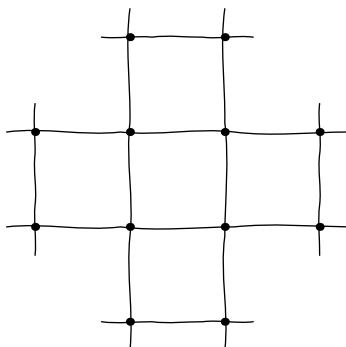


FIGURE 5. From the proof of Theorem 2.3, in the “Square Grids” section of the proof. The central face, denoted by f_0 , has length 4. Each neighboring face also has length 4, and each vertex has degree 4. The four central vertices in the figure are all distinct from each other and from the other vertices shown. However, the remaining eight vertices may not be all distinct.

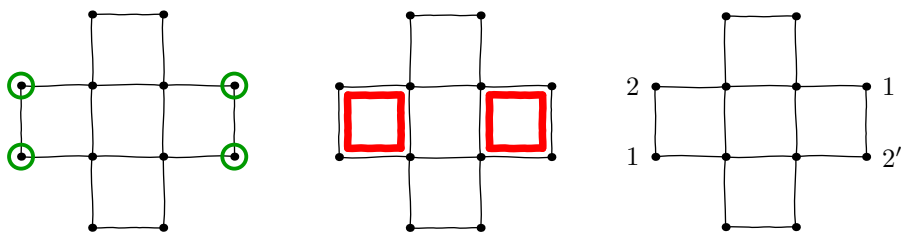


FIGURE 6. From the proof of Theorem 2.3. Square Grids, Step 1.

On the other hand, if the circled vertices in the left-hand drawing are not all distinct, then, because G has girth at least 4, some diagonally opposite pair of vertices must be identical. Without loss of generality, say those vertices labeled 1 in the right-hand drawing are identical. We label the other two vertices 2 and $2'$; these two vertices may or may not be identical.

Step 2. Consider Figure 7. The left-hand drawing represents the same situation as in the right-hand drawing in Figure 6, but with 4 vertices circled. Suppose that these circled vertices are all distinct. Then we may apply the claim to the two 4-cycles shown in the center drawing, and we are done. (Note that these really are cycles, since the two points labeled 1, actually represent the same vertex.)

On the other hand, if the circled vertices in the left-hand drawing are not all distinct, then we can see that the right-hand vertex at the bottom, must

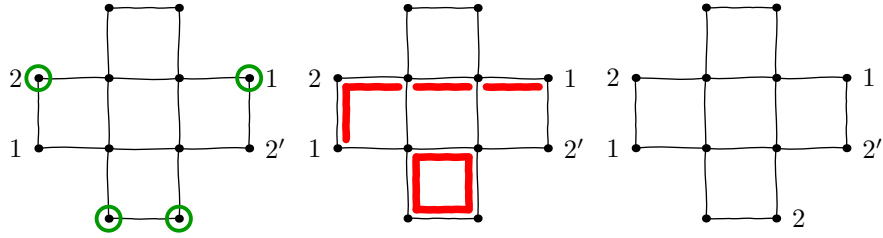


FIGURE 7. From the proof of Theorem 2.3. Square Grids, Step 2.

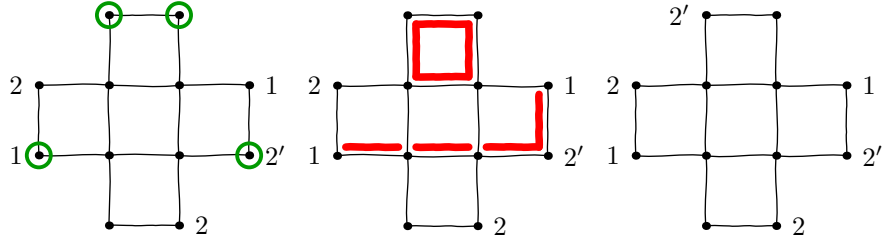


FIGURE 8. From the proof of Theorem 2.3. Square Grids, Step 3.

be identical to the vertex labeled 2; we label it 2 also (see the right-hand drawing). Since we now have 2 and 2' at distance two from each other in our diagram, these two vertices must be distinct.

Step 3. Consider Figure 8. As before, the left-hand drawing represents the same situation as in the right-hand drawing in the previous figure, but with 4 vertices circled. Apply much the same reasoning as in the previous step. If the circled vertices in the left-hand drawing are all distinct, then apply the claim to the two 4-cycles in the center drawing, and we are done.

Otherwise, the left-hand vertex at the top must be identical to the vertex labeled 2'; we label it 2' also.

Step 4. Consider Figure 9. Once again, the left-hand drawing represents the same situation as in the right-hand drawing in the previous figure, but with 4 vertices circled. Apply much the same reasoning as in Step 1. If the circled vertices in the left-hand drawing are all distinct, then apply the claim to the two 4-cycles in the center drawing, and we are done.

Otherwise, the two vertices labeled 3 in the right-hand drawing must be identical; the common label indicates this fact.

Step 5. Consider Figure 10. This represents the same situation as in the right-hand drawing in Figure 9, but with two 4-cycles shown. Vertices 1, 2, 3, 2' do, in fact, form a 4-cycle, in that order, because of the adjacencies

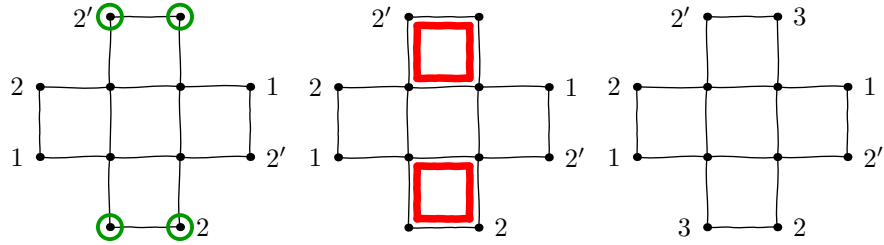


FIGURE 9. From the proof of Theorem 2.3. Square Grids, Step 4.

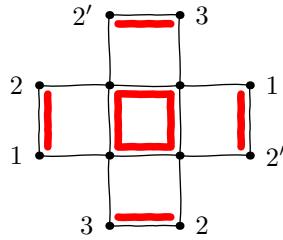


FIGURE 10. From the proof of Theorem 2.3. Square grids, step 5.

we have established between them. We may thus apply the claim to the two 4-cycles, and we are done. \square

Note that, in the final section of the above proof, where we deal with 4-regular graphs in which each face has length 4, we could, instead, have based the argument on a classification of the square grids on the torus and Klein bottle. Classifications of such graphs have been done—albeit under restrictions a bit too strong for our proof—by Thomassen [7, Thm. 4.1] and by Márquez, de Mier, Noy, and Revuelta [4, Thm. 1].

Theorem 2.3 is sharp—at least for graphs with one particular order—on the projective plane, torus, and Klein bottle, as shown by K_5 (and also $2K_5$, for the Klein bottle). However, we do not know whether it is sharp on the plane, other than the trivial case when graph G is a forest. If Jones’ Conjecture (Conjecture 1.2) is true, then Theorem 2.3 is *not* sharp on the plane.

3. ARBITRARY SURFACES

We can generalize Theorem 2.3 to arbitrary surfaces. We show that, for a graph G on a surface of genus g , we have $\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G) + O(g)$.

Theorem 3.1. *Let G be a graph that embeds in a closed surface with Euler characteristic $c \leq 0$. Then*

$$\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G) + 103(-c).$$

Proof. The early part of this proof uses reasoning very similar to that in the proof of Theorem 2.3. Arguments from that proof will not be repeated in their entirety here; the reader is referred to the earlier proof for details.

Let G , c be as in the statement of the result; we may assume that G is connected.

By Lemma 2.2, graph G has a 2-cell embedding in some surface with Euler characteristic $c^* \geq c$; proving our result for c^* will imply the desired statement for c . If $c^* \geq 0$, then apply Theorem 2.3, and we are done. Thus, we may assume that G is 2-cell embedded, and that $c = c^* < 0$.

As in the proof of Theorem 2.3, we proceed by induction on the order of G , which we denote by n . The base case is $n = 0$, in which case an empty cycle packing and an empty feedback vertex set show that our result holds. Suppose that $n \geq 1$.

We define the charge ch' just as in the proof of Lemma 2.1: each vertex has charge 0, while, for each face f ,

$$ch'(f) := 4 - \ell(f) + \sum_{v \in f} \frac{4 - d(v)}{d(v)},$$

where the sum counts multiplicities. As in Lemma 2.1, we have $\sum_f ch'(f) = 4c$.

If e is an edge in G , then e lies in the facial walk of two faces, counting multiplicities. These two faces are said to be *adjacent*. For each face f , there are $\ell(f)$ faces adjacent to f , counting multiplicities; some of these may be f itself.

As in the proof of Theorem 2.3, we may apply the induction hypothesis (and thus we are done) if G contains any of the following.

- A vertex of degree at most 2.
- A cycle of length at most 3.
- A face of length 6, all of whose vertices have degree 3 (see Figure 1).
- A face of length 5, at least 3 of whose vertices have degree 3, counting multiplicities (see Figure 2).
- A face of length 4 with at least 1 vertex of degree 3 (see Figure 3).
- A face of length 4 in which all vertices have degree 4, and each adjacent face also has these properties—length 4, all vertices having degree 4 (see Figure 5).

Hence we may assume that G contains none of the above configurations, which we will call *forbidden configurations*. In particular, G is simple, with minimum degree at least 3 and girth at least 4. Furthermore, G has no face f with $ch'(f) > 0$. Every face of G with $ch' = 0$ has length 4 and 4 vertices of degree 4. Lastly, for each face of G with $ch' = 0$, there is an adjacent face with $ch' < 0$.

Note: The final property in the preceding paragraph holds because otherwise, there is a face with $ch' = 0$ such that every adjacent face has $ch' = 0$. That is, there is a face of length 4 with 4 vertices of degree 4 such that each adjacent face is of this form as well. This is the last forbidden configuration in the list above. Such a configuration is forbidden since, if it exists, then we may apply the argument from the “Square Grids” section of the proof of Theorem 2.3.

In the proof of Theorem 2.3, we used the fact that $\sum ch' = 4c \geq 0$ to show that all the possibilities were covered. However, here we have $c < 0$, and so other cases remain.

We will handle all remaining cases as base cases in the induction, directly constructing a feedback vertex set and a cycle packing. Specifically, we will show that, for a graph G containing no forbidden configuration, we have $n \leq 103(-c)$. Thus, forming a feedback vertex set using every vertex of G , and forming a cycle packing using no cycles, we may conclude that G satisfies the required inequality.

For each face f of G , we define

$$\varphi(f) := \sum_{v \in f} \frac{1}{d(v)},$$

where the sum counts multiplicities. Then $\sum_f \varphi(f) = n$. (We may thus consider $\varphi(f)$ to be the contribution of face f to the order of G .)

Note that

$$\begin{aligned} -ch'(f) &= - \left[4 - \ell(f) + \sum_{v \in f} \frac{4 - d(v)}{d(v)} \right] \\ &= \ell(f) - 4 - \sum_{v \in f} \left[\frac{4}{d(v)} - 1 \right] \\ &= 2\ell(f) - 4 - 4 \sum_{v \in f} \frac{1}{d(v)} \\ &= 2\ell(f) - 4 - 4\varphi(f). \end{aligned}$$

Upper Bounds: Our goal in this section of the proof is to find upper bounds for $\varphi(f)/[-ch'(f)]$ and $\ell(f)/[-ch'(f)]$, for all faces f with $ch'(f) < 0$. In the following section, we will use these bounds to prove our result.

Using the above expansion of $-ch'(f)$, we obtain

$$(3.1) \quad \frac{\varphi(f)}{-ch'(f)} = \frac{\varphi(f)}{2\ell(f) - 4 - 4\varphi(f)}$$

and

$$(3.2) \quad \frac{\ell(f)}{-ch'(f)} = \frac{\ell(f)}{2\ell(f) - 4 - 4\varphi(f)}.$$

If $ch'(f)$ is negative and $\ell(f)$ is fixed, then each of the two quantities above increases as $\varphi(f)$ increases. Thus, for fixed $\ell(f)$, we can find upper bounds for the quantities by computing their values when $\varphi(f)$ is maximized.

In order to maximize $\varphi(f)$, we reason using the forbidden configurations, and the requirement that $ch'(f) < 0$, to bound the number of vertices on face f that have degree 3. We will usually maximize φ by assuming the maximum number of vertices of degree 3, and giving all remaining vertices degree 4. If there cannot be any vertices of degree 3, then we will assume the maximum number of vertices of degree 4, and give all remaining vertices degree 5.

Now we compute our upper bounds. We consider four cases: $\ell(f) \geq 7$, $\ell(f) = 6$, $\ell(f) = 5$, and $\ell(f) = 4$.

Case 1. If $\ell(f) \geq 7$, then all vertices on face f can have degree 3; this does not result in the existence of any forbidden configuration. Thus, the greatest possible value of $\varphi(f)$ is

$$\varphi(f) = \ell(f) \cdot \frac{1}{3}.$$

Using this value, along with equations (3.1) and (3.2), we obtain

$$\begin{aligned} \frac{\varphi(f)}{-ch'(f)} &= \frac{\ell(f)}{2\ell(f) - 12}; \\ \frac{\ell(f)}{-ch'(f)} &= \frac{3\ell(f)}{2\ell(f) - 12}. \end{aligned}$$

The right-hand sides above are both maximized when $\ell(f) = 7$. Thus, for each fixed $\ell(f) \geq 7$, we have

$$\begin{aligned} \frac{\varphi(f)}{-ch'(f)} &\leq \frac{7}{2 \cdot 7 - 12} = \frac{7}{2}; \\ \frac{\ell(f)}{-ch'(f)} &\leq \frac{3 \cdot 7}{2 \cdot 7 - 12} = \frac{21}{2}. \end{aligned}$$

Case 2. If $\ell(f) = 6$, then it is impossible for all vertices on face f to have degree 3; this would be a forbidden configuration. We maximize $\varphi(f)$ with 5 vertices of degree 3 and 1 vertex of degree 4. Thus, when $\ell(f) = 6$ and $\varphi(f)$ is maximized, we again apply equations (3.1) and (3.2) to obtain the following:

$$\begin{aligned} \varphi(f) &= 5 \cdot \frac{1}{3} + 1 \cdot \frac{1}{4} = \frac{23}{12}; \\ \frac{\varphi(f)}{-ch'(f)} &= \frac{23}{4}; \\ \frac{\ell(f)}{-ch'(f)} &= 18. \end{aligned}$$

Case 3. If $\ell(f) = 5$, then, again avoiding forbidden configurations, we maximize $\varphi(f)$ with 2 vertices of degree 3 and 3 vertices of degree 4. Proceeding as above, we obtain

$$\begin{aligned}\varphi(f) &= 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{4} = \frac{17}{12}; \\ \frac{\varphi(f)}{-ch'(f)} &= \frac{17}{4}; \\ \frac{\ell(f)}{-ch'(f)} &= 15.\end{aligned}$$

Case 4. If $\ell(f) = 4$, then vertices of degree 3 are forbidden; we maximize $\varphi(f)$ with 3 vertices of degree 4 and 1 vertex of degree 5. Then we have

$$\begin{aligned}\varphi(f) &= 3 \cdot \frac{1}{4} + 1 \cdot \frac{1}{5} = \frac{19}{20}; \\ \frac{\varphi(f)}{-ch'(f)} &= \frac{19}{4}; \\ \frac{\ell(f)}{-ch'(f)} &= 20.\end{aligned}$$

Conclusion. In each of the four cases above, the following inequalities hold.

$$(3.3) \quad \frac{\varphi(f)}{-ch'(f)} \leq \frac{23}{4};$$

$$(3.4) \quad \frac{\ell(f)}{-ch'(f)} \leq 20.$$

We conclude that, for *every* face f of G with $ch'(f) < 0$, inequalities (3.3) and (3.4) both hold.

(The rather unlikely-looking value of 103 in the statement of our theorem, will turn out to be 4 times the sum of the above two bounds; the 4 comes from the fact that the sum of all the charge is $4c$.)

Moving Charge Again: Next we define a new charge ch'' on the faces of G . We will place an upper bound on $\varphi(f)/[-ch''(f)]$, over all faces f of G . Multiplying this upper bound by the sum of all the charge, we will obtain an upper bound for n .

Below, t is some real number in the open interval $(0, 1)$; we will restrict t to a particular value later.

Rearrange the charge on G as follows. For each face f with $ch'(f) < 0$, and each adjacent face f^* with $ch'(f^*) = 0$, transfer charge $[t/\ell(f)] \cdot ch'(f)$ from face f to face f^* . Call the resulting charge ch'' . As noted earlier, by the absence of forbidden configurations, for each face with $ch' = 0$, there must be an adjacent face with $ch' < 0$. Thus we have $ch''(f) < 0$ for *every* face f of G .

We wish to place an upper bound on $\varphi(f)/[-ch''(f)]$. We do this first for faces with $ch' < 0$, and then for faces with $ch' = 0$.

If $ch'(f) < 0$, then $ch''(f) \leq (1-t) \cdot ch'(f) < 0$, and we have

$$\begin{aligned} 0 < \frac{\varphi(f)}{-ch''(f)} &\leq \frac{1}{1-t} \cdot \frac{\varphi(f)}{-ch'(f)} \\ &\leq \frac{1}{1-t} \cdot \frac{23}{4}, \end{aligned}$$

by inequality (3.3).

On the other hand, if $ch'(f) = 0$, then $ch''(f) \leq [t/\ell(f^*)] \cdot ch'(f^*)$, where f^* is some face adjacent to f , such that $ch'(f^*) < 0$. Further, f has length 4 and 4 vertices of degree 4, so that $\varphi(f) = 4 \cdot (1/4) = 1$. Thus, we have

$$\begin{aligned} 0 < \frac{\varphi(f)}{-ch''(f)} &\leq \frac{1}{\frac{t}{\ell(f^*)} \cdot [-ch'(f^*)]} \\ &= \frac{1}{t} \cdot \frac{\ell(f^*)}{-ch'(f^*)} \\ &\leq \frac{1}{t} \cdot 20, \end{aligned}$$

by inequality (3.4).

Our work so far is valid for every $t \in (0, 1)$. We now choose t so that the above two bounds (that is, $\frac{1}{1-t} \cdot \frac{23}{4}$ and $\frac{1}{t} \cdot 20$) are equal. This happens when $t = 80/103$, in which case both bounds become $103/4$. Defining our charge ch'' using this value of t , we have

$$\frac{\varphi(f)}{-ch''(f)} \leq \frac{103}{4},$$

for every face f of G .

Now we can bound the order of G .

$$\begin{aligned} n &= \sum_f \varphi(f) \\ &= \sum_f \left(\frac{\varphi(f)}{-ch''(f)} \cdot [-ch''(f)] \right) \\ &\leq \max_f \left[\frac{\varphi(f)}{-ch''(f)} \right] \cdot \sum_f [-ch''(f)]. \end{aligned}$$

Using our bound on $\varphi(f)/[-ch''(f)]$, and noting that $\sum_f [-ch''(f)] = -\sum_f ch(f) = -4c$, we see that

$$n \leq \frac{103}{4} \cdot (-4c) = 103(-c).$$

Hence, G has at most $103(-c)$ vertices. As noted earlier, we can form a feedback vertex set using every vertex of G , and a cycle packing using

no cycles. Thus, our graph G satisfies the required inequality; the desired result follows. \square

We can restate Theorems 2.3 and 3.1 using genus. An orientable surface with genus g has Euler characteristic $2(1 - g)$, while a nonorientable surface with genus g has characteristic $2 - g$. Thus we obtain the following result.

Corollary 3.2. *Let G be a graph embedded in a surface S .*

- (a) *If S is an orientable surface of genus at most 1, or a nonorientable surface of genus at most 2, then*

$$\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G).$$

- (b) *If S is an orientable surface of genus $g \geq 1$, then*

$$\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G) + 206(g - 1).$$

- (c) *If S is a nonorientable surface of genus $g \geq 2$, then*

$$\mathbf{fvs}(G) \leq 3 \mathbf{cp}(G) + 103(g - 2).$$

We expect that a somewhat cleverer analysis could reduce the value 103 in the preceding results. However, it seems unlikely that the coefficient of \mathbf{cp} (i.e., 3) can be reduced without using a significantly different proof technique.

4. WHAT IS THE BEST BOUND?

We consider how small an upper bound for $\mathbf{fvs}(G)$ can be.

Theorem 4.1. *Let $g(G)$ denote either the orientable or the nonorientable genus of graph G . Suppose that there exist a constant k and a function f , such that the following holds for every graph G :*

$$\mathbf{fvs}(G) \leq k \mathbf{cp}(G) + f(g(G)).$$

Then all of the following hold.

- (a) $k \geq 2$.
 (b) $f(g)$ is $\Omega(g)$.
 (c) If $k = 2$, then $f(g) \geq g$.

Proof. (a) Suppose for a contradiction that $k < 2$. Let s be a positive integer greater than $f(0)/(2 - k)$. Then $2s > ks + f(0)$. Let G be the disjoint union of s copies of K_4 ; we have $\mathbf{fvs}(G) = 2s$, $\mathbf{cp}(G) = s$, and $g(G) = 0$. Hence, $\mathbf{fvs}(G) > k \mathbf{cp}(G) + f(g(G))$, which is a contradiction.

(b) Let k, f be as in the statement of the result. Let f^* be the smallest function so that $\mathbf{fvs} \leq k \mathbf{cp} + f^*(g)$ holds for every graph. Clearly, f^* is nondecreasing. It follows from the Erdős-Pósa result (Theorem 1.1) that there exists a graph H such that $\mathbf{fvs}(H) > k \mathbf{cp}(H)$. Let $q = \mathbf{fvs}(H) - k \mathbf{cp}(H)$, and note that $q > 0$.

For each positive integer t , let tH denote the disjoint union of t copies of graph H . We have $g(tH) = tg(H)$, $\mathbf{cp}(tH) = t\mathbf{cp}(H)$, and $\mathbf{fvs}(tH) = t\mathbf{fvs}(H)$. Thus,

$$\begin{aligned} f^*(tg(H)) &= f^*(g(tH)) \\ &\geq \mathbf{fvs}(tH) - k\mathbf{cp}(tH) \\ &= t\mathbf{fvs}(H) - tk\mathbf{cp}(H) \\ &= tq. \end{aligned}$$

Since f^* is nondecreasing, it follows that $f^*(g)$ is $\Omega(g)$, and so is $f(g)$.

(c) In this case we have a specific example; we do not need to apply the Erdős-Pósa result. The graph consisting of g disjoint copies of K_5 has genus g , $\mathbf{cp} = g$, and $\mathbf{fvs} = 3g$. The desired statement follows. \square

Applying Corollary 3.2 and Theorem 4.1(b), we obtain the following.

Corollary 4.2. *Let $g(G)$ denote either the orientable or the nonorientable genus of graph G . Let $f(g)$ be the smallest value so that*

$$\mathbf{fvs}(G) = 3\mathbf{cp}(G) + f(g(G))$$

holds for every graph G . Then $f(g)$ is $\Theta(g)$.

It seems likely that the best coefficient for \mathbf{cp} is actually 2. Thus we propose the following generalization of Jones' Conjecture.

Conjecture 4.3. *Let $g(G)$ denote either the orientable or the nonorientable genus of graph G . Then there exists a function $f(g) = \Theta(g)$, so that*

$$\mathbf{fvs}(G) \leq 2\mathbf{cp}(G) + f(g(G))$$

holds for every graph G .

If the above conjecture holds, then we can set $f(0) = 0$; this follows from Theorem 4.1(c) and the fact that the disjoint union of planar graphs is planar. Hence, Conjecture 4.3 implies Jones' Conjecture (Conjecture 1.2).

If Conjecture 4.3 holds, then, by Theorem 4.1(c), we must have $f(g) \geq g$. Is it possible that $f(g) = g$?

Question 4.4. *Can we set $f(g) = g$ in Conjecture 4.3?*

If the answer is "yes", then the inequality $\mathbf{fvs} \leq 2\mathbf{cp} + g$ is sharp for graphs of arbitrarily high order, on both orientable and nonorientable surfaces, as shown by the disjoint union of g copies of K_5 and t copies of K_4 , for arbitrarily high values of t .

There must be a positive number k such that the statement $\mathbf{fvs}(G) \leq 3\mathbf{cp}(G) + kg(G)$ (with $g(G)$ appropriately defined) fails to hold for some graph G . In particular, this follows from Corollary 4.2. However, we cannot currently exhibit any such k .

Problem 4.5. *Find a positive number k such that the following statement fails to hold for some graph G : $\mathbf{fvs}(G) \leq 3\mathbf{cp}(G) + kg(G)$.*

More specifically, while we have $\mathbf{fvs}(K_5) = 3 \mathbf{cp}(K_5)$, we do not know of any example of a graph with $\mathbf{fvs} > 3 \mathbf{cp}$. The Erdős-Pósa result (Theorem 1.1) implies that such graphs exist, but does not provide a construction.

Problem 4.6. *Find an explicit construction of a graph G such that $\mathbf{fvs}(G) > 3 \mathbf{cp}(G)$.*

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