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DETERMINATION OF THE PRIME BOUND OF A GRAPH

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Abstract. Given a graph G, a subset M of $V(G)$ is a module of G if for each $v \in V(G) \setminus M$, v is adjacent to all the elements of M or adjacent to none of them. For instance, $V(G)$, $\emptyset$ and $\{v\}$ ($v \in V(G)$) are modules of G called trivial. Given a graph G, $\omega_M(G)$ (respectively $\alpha_M(G)$) denotes the largest integer m such that there is a module M of G which is a clique (respectively a stable) set in G with $|M| = m$. A graph G is prime if $|V(G)| \geq 4$ and if all its modules are trivial. The prime bound of G is the smallest integer $p(G)$ such that there is a prime graph H with $V(H) \supseteq V(G)$, $H[V(G)] = G$ and $|V(H) \setminus V(G)| = p(G)$. We establish the following. For every graph G such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$ and $\log_2(\max(\alpha_M(G), \omega_M(G)))$ is not an integer, $p(G) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil$. Then, we prove that for every graph G such that $\max(\alpha_M(G), \omega_M(G)) = 2^k$ where $k \geq 1$, $p(G) = k$ or $k+1$. Moreover $p(G) = k+1$ if and only if G or its complement admits exactly 2^k isolated vertices. Lastly, we show that $p(G) = 1$ for every non prime graph G such that $|V(G)| \geq 4$ and $\alpha_M(G) = \omega_M(G) = 1$.

1. Introduction

A *graph* $G = (V(G), E(G))$ is constituted by a finite *vertex set* $V(G)$ and an *edge set* $E(G) \subseteq \binom{V(G)}{2}$. Given a set finite S, $K_S = \left(S, \binom{S}{2}\right)$ is the *complete* graph on S whereas $(S, \emptyset)$ is the *empty* graph. Let G be a graph. With each $W \subseteq V(G)$ associate the *subgraph* $G[W] = \left(W, \binom{W}{2} \cap E(G)\right)$ of G induced by W. Given $W \subseteq V(G)$, $G[V(G) \setminus W]$ is also denoted by $G - W$ and by $G - w$ if $W = \{w\}$. A graph H is an *extension* of G if $V(H) \supseteq V(G)$ and $H[V(G)] = G$. Given $p \geq 0$, a p-extension of G is an extension H of G such that $|V(H) \setminus V(G)| = p$. The *complement* of G is the graph $\overline{G} = \left(V(G), \binom{V(G)}{2} \setminus E(G)\right)$. A subset W of $V(G)$ is a *clique* (respectively a *stable set*) in G if $G[W]$ is complete (respectively empty). The largest cardinality of a clique (respectively a stable set) in G is the *clique number* (respectively the *stability number*) of G, denoted by $\omega(G)$ (respectively $\alpha(G)$). Given $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v in G is the family $\{w \in V(G) : \{v, w\} \in E(G)\}$. We consider N_G as the function

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from $V(G)$ to $2^{V(G)}$ defined by $v \mapsto N_G(v)$ for each $v \in V(G)$. A vertex v of G is *isolated* if $N_G(v) = \varnothing$. The number of isolated vertices of G is denoted by $\iota(G)$.

We use the following notation. Let G be a graph. For $v \neq w \in V(G)$,

$$(v,w)_G = \begin{cases} 0, & \text{if } \{v,w\} \notin E(G), \\ 1, & \text{if } \{v,w\} \in E(G). \end{cases}$$

Given $W \subsetneq V(G)$, $v \in V(G) \setminus W$ and $i \in \{0,1\}$, $(v,W)_G = i$ means $(v,w)_G = i$ for every $w \in W$. Given $W, W' \subsetneq V(G)$, with $W \cap W' = \varnothing$, and $i \in \{0,1\}$, $(W,W')_G = i$ means $(w,W')_G = i$ for every $w \in W$. Given $W \subsetneq V(G)$ and $v \in V(G) \setminus W$, $v \longleftrightarrow_G W$ means that there is $i \in \{0,1\}$ such that $(v,W)_G = i$. The negation is denoted by $v \not\longleftrightarrow_G W$.

Given a graph G, a subset M of $V(G)$ is a *module* of G if for each $v \in V(G) \setminus M$, we have $v \longleftrightarrow_G M$. For instance, $V(G)$, $\varnothing$ and $\{v\}$ $(v \in V(G))$ are modules of G called *trivial*. Clearly, if $|V(G)| \leq 2$, then all the modules of G are trivial. On the other hand, if $|V(G)| = 3$, then G admits a nontrivial module. A graph G is then said to be *prime* if $|V(G)| \geq 4$ and if all its modules are trivial. For instance, given $n \geq 4$, the *path* $(\{1,\dots,n\}, \{\{p,q\} : |p-q| = 1\})$ is prime. Given a graph G, G and $\overline{G}$ share the same modules. Thus G is prime if and only if $\overline{G}$ is.

Given a set S with $|S| \geq 2$, K_S admits a prime $\lceil \log_2(|S|+1) \rceil$-extension (see Sumner [8, Theorem 2.45] or Lemma 3.2 below). This is extended to any graph in [3, Theorem 3.7] and [2, Theorem 3.2] as follows.

Theorem 1.1. *A graph G, with $|V(G)| \geq 2$, admits a prime $\lceil \log_2(|V(G)| + 1) \rceil$-extension.*

We now introduce the notion of prime bound. Let G be a graph. The *prime bound* of G is the smallest integer $p(G)$ such that G admits a prime $p(G)$-extension. Observe that $p(G) = p(\overline{G})$ for every graph G. By Theorem 1.1, $p(G) \leq \lceil \log_2(|V(G)| + 1) \rceil$. By considering the clique number and the stability number, Brignall [3, Conjecture 3.8] conjectured the following.

Conjecture 1.2. *For a graph G with $|V(G)| \geq 2$,*

$$p(G) \leq \lceil \log_2(\max(\alpha(G), \omega(G)) + 1) \rceil.$$

We answer the conjecture positively by refining the notions of clique number and of stability number as follows. Given a graph G, the *modular clique number* $\omega_M(G)$ of G is the largest cardinality of a clique in G which is also a module of G. The *modular stability number* of G is $\alpha_M(G) = \omega_M(\overline{G})$. The following lower bound is simply obtained.

Lemma 1.3. *For every graph G such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$,*

$$p(G) \geq \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil.$$

Theorem 3.2 of [2] is proved by induction on the number of vertices. Using the main arguments of this proof, we improve Theorem 1.1 as follows.

Theorem 1.4. *For every graph G such that* $\max(\alpha_M(G), \omega_M(G)) \geq 2$,

$$p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G)) + 1) \rceil.$$

Theorem 1.4 is proved using an induction argument as well. A direct construction of a suitable extension is provided in [1, Theorem 2]. The following is an immediate consequence of Lemma 1.3 and Theorem 1.4.

Corollary 1.5. *For every graph G such that* $\max(\alpha_M(G), \omega_M(G)) \geq 2$,

$$\lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil \leq p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G)) + 1) \rceil.$$

Let G be graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. On the one hand, it follows from Corollary 1.5 that

$$p(G) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil$$

when

$$\max(\alpha_M(G), \omega_M(G)) \notin \{2^k : k \geq 1\}.$$

On the other, if $\max(\alpha_M(G), \omega_M(G)) = 2^k$, where $k \geq 1$, then $p(G) = k$ or $k+1$. The next theorem allows us to determine this.

Theorem 1.6. *For every graph G such that* $\max(\alpha_M(G), \omega_M(G)) = 2^k$ *where* $k \geq 1$,

$$p(G) = k + 1 \text{ if and only if } \iota(G) = 2^k \text{ or } \iota(\overline{G}) = 2^k.$$

Lastly, we show that $p(G) = 1$ for every non prime graph G such that $|V(G)| \geq 4$ and $\alpha_M(G) = \omega_M(G) = 1$ (see Proposition 5.2).

2. Preliminaries

Given a graph G, the family of the modules of G is denoted by $\mathcal{M}(G)$. Furthermore set $\mathcal{M}_{\geq 2}(G) = \{M \in \mathcal{M}(G) : |M| \geq 2\}$. We begin with the well known properties of the modules of a graph (for example, see [4, Theorem 3.2, Lemma 3.9]).

Proposition 2.1. *Let G be a graph.*

1. *Given $W \subseteq V(G)$, $\{M \cap W : M \in \mathcal{M}(G)\} \subseteq \mathcal{M}(G[W])$.*
2. *Given a module $M \in \mathcal{M}(G)$, $\mathcal{M}(G[M]) = \{N \in \mathcal{M}(G) : N \subseteq M\}$.*
3. *Given $M, N \in \mathcal{M}(G)$ with $M \cap N = \varnothing$, there is $i \in \{0,1\}$ such that $(M,N)_G = i$.*

Given a graph G, a partition P of $V(G)$ is a *modular partition* of G if $P \subseteq \mathcal{M}(G)$. Let P be such a partition. Given $M \neq N \in P$, there is $i \in \{0,1\}$ such that $(M,N)_G = i$ by (3) of Proposition 2.1. This justifies the following definition: The *quotient* of G by P is the graph G/P defined on $V(G/P) = P$ by $(M,N)_{G/P} = (M,N)_G$ for $M \neq N \in P$. We use the following properties of the quotient (for example, see [4, Theorems 4.1–4.3, Lemma 4.1]).

Proposition 2.2. *Given a graph G, consider a modular partition P of G.*

1. *Given $W \subseteq V(G)$, if $|W \cap X| = 1$ for each $X \in P$, then $G[W]$ and G/P are isomorphic.*

(2) *For every* $M \in \mathcal{M}(G)$, $\{X \in P : M \cap X \neq \varnothing\} \in \mathcal{M}(G/P)$.
(3) *For every* $Q \in \mathcal{M}(G/P)$, $\bigcup Q \in \mathcal{M}(G)$.

The following strengthening of the notion of module is introduced to present the modular decomposition theorem (see Theorem 2.4 below). Given a graph G, a module M of G is said to be *strong* provided that for every $N \in \mathcal{M}(G)$, if $M \cap N \neq \varnothing$, then $M \subseteq N$ or $N \subseteq M$. The family of the strong modules of G is denoted by $\mathcal{S}(G)$. Furthermore set

$$\mathcal{S}_{\geq 2}(G) = \{M \in \mathcal{S}(G) : |M| \geq 2\}.$$

We recall the following well known properties of the strong modules of a graph (for example, see [4, Theorem 3.3]).

Proposition 2.3. *Let G be a graph. For every $M \in \mathcal{M}(G)$,*

$$\mathcal{S}(G[M]) = \{N \in \mathcal{S}(G) : N \subsetneq M\} \cup \{M\}.$$

With each graph G, we associate the family $\Pi(G)$ of the maximal proper and nonempty strong modules of G under inclusion. For convenience set

$$\Pi_1(G) = \{M \in \Pi(G) : |M| = 1\} \text{ and } \Pi_{\geq 2}(G) = \{M \in \Pi(G) : |M| \geq 2\}.$$

The modular decomposition theorem is stated as follows.

Theorem 2.4 (Gallai [5, 6])**.** *For a graph G with $|V(G)| \geq 2$, the family $\Pi(G)$ realizes a modular partition of G. Moreover, the corresponding quotient $G/\Pi(G)$ is complete, empty or prime.*

Let G be a graph with $|V(G)| \geq 2$. As a direct consequence of the definition of a strong module, we obtain that the family $\mathcal{S}(G) \setminus \{\varnothing\}$ endowed with inclusion is a tree called the *modular decomposition tree* [7] of G. Given $M \in \mathcal{S}_{\geq 2}(G)$, it follows from Proposition 2.3 that $\Pi(G[M]) \subseteq \mathcal{S}(G)$. Furthermore, given $W \subseteq V(G)$, the family $\{M \in \mathcal{S}(G) : M \supseteq W\}$ endowed with inclusion is a total order. Its smallest element is denoted by $\widehat{W}$.

Let G be a graph with $|V(G)| \geq 2$. Using Theorem 2.4, we label $\mathcal{S}_{\geq 2}(G)$ by the function λ_G defined as follows. For each $M \in \mathcal{S}_{\geq 2}(G)$,

$$\lambda_G(M) = \begin{cases} \bullet & \text{if } G[M]/\Pi(G[M]) \text{ is complete,} \\ \circ & \text{if } G[M]/\Pi(G[M]) \text{ is empty,} \\ \sqsubset & \text{if } G[M]/\Pi(G[M]) \text{ is prime.} \end{cases}$$

3. Some prime extensions

Lemma 3.1. *Let S and S' be disjoint and finite sets such that $|S| \geq 2$ and $|S'| = \lceil \log_2(|S|+1) \rceil$. There exists a prime graph G defined on $V(G) = S \cup S'$ such that S and S' are stable sets in G.*

Proof. If $|S| = 2$, then $|S'| = 2$ and we can choose a path on 4 vertices for G. Assume that $|S| \geq 3$. As $|S'| = \lceil \log_2(|S|+1) \rceil$, $2^{|S'|-1} \leq |S|$ and hence $|S'| \leq |S|$. Thus there exists a bijection $\psi_{S'}$ from S' onto $S'' \subseteq S$. Consider the injection $f_{S''} : S'' \longrightarrow 2^{S'} \setminus \{\varnothing\}$ defined by $s'' \mapsto S' \setminus \{(\psi_{S'})^{-1}(s'')\}$. Since

$|S'| = \lceil \log_2(|S|+1) \rceil$, $|S| < 2^{|S'|}$ and there exists an injection f_S from S into $2^{S'} \setminus \{\varnothing\}$ such that $(f_S)_{\restriction S''} = f_{S''}$. Lastly, consider the graph G defined on $V(G) = S \cup S'$ such that S and S' are stable sets in G and $(N_G)_{\restriction S} = f_S$. We prove that G is prime. If $|S| = 3$, then $|S'| = 2$ and G is a path on 5 vertices which is prime. Assume that $|S| \geq 4$ and hence $|S'| \geq 3$. Let $M \in \mathcal{M}_{\geq 2}(G)$.

First, if $M \subseteq S$, then we would have $f_S(u) = f_S(v)$ for any $u \neq v \in M$. Thus $M \cap S' \neq \varnothing$.

Second, suppose that $M \subseteq S'$. Recall that for each $s \in S$, either $M \cap N_G(s) = \varnothing$ or $M \subseteq N_G(s)$. Given $u \in M$, consider the function $f : S \longrightarrow 2^{(S' \setminus M) \cup \{u\}} \setminus \{\varnothing\}$ defined by

$$f(s) = \begin{cases} N_G(s), & \text{if } M \cap N_G(s) = \varnothing, \\ (N_G(s) \setminus M) \cup \{u\}, & \text{if } M \subseteq N_G(s), \end{cases}$$

for every $s \in S$. Since $(N_G)_{\restriction S}$ is injective, f is also and we would obtain that $|S| < 2^{|S'|-1}$. It follows that $M \cap S \neq \varnothing$.

Third, suppose that $S' \setminus M \neq \varnothing$. We have $(S \cap M, S' \setminus M)_G = (S' \cap M, S' \setminus M)_G = 0$. Given $s' \in S' \cap M$, $N_G(\psi_{S'}(s')) = S' \setminus \{s'\}$. In particular $S' \setminus M \subseteq N_G(\psi_{S'}(s'))$ and hence $\psi_{S'}(s') \in S \setminus M$. Furthermore $(\psi_{S'}(s'), S' \cap M)_G = (\psi_{S'}(s'), S \cap M)_G = 0$. Therefore $S' \cap M = \{s'\}$. Similarly, we prove that $|S' \setminus M| = 1$ which would imply that $|S'| = 2$. It follows that $S' \subseteq M$.

Lastly, suppose that $S \setminus M \neq \varnothing$. For each $s \in S \setminus M \neq \varnothing$, we would have $(s, S')_G = (s, S \cap M)_G = 0$ and hence $N_G(s) = \varnothing$. It follows that $S \subseteq M$ and $M = S \cup S'$. □

Lemma 3.2. *Let C and S' be disjoint and finite sets such that $|C| \geq 2$ and $|S'| = \lceil \log_2(|C|+1) \rceil$. There exists a prime graph G defined on $V(G) = C \cup S'$ such that C is a clique and S' is a stable set in G.*

Proof. There exists a bijection $\psi_{S'}$ from S' onto $S'' \subseteq C$. Consider the injection $f_{S''} : S'' \longrightarrow 2^{S'} \setminus \{S'\}$ defined by $s'' \mapsto \{(\psi_{S'})^{-1}(s'')\}$. Let f_C be any injection from C into $2^{S'} \setminus \{S'\}$ such that $(f_C)_{\restriction S''} = f_{S''}$. Lastly, consider the graph G defined on $V(G) = C \cup S'$ such that C is a clique in G, S' is a stable set in G and $N_G(c) \cap S' = f_C(c)$ for each $c \in C$. We prove that G is prime. Let $M \in \mathcal{M}_{\geq 2}(G)$. As in the proof of Lemma 3.1, we have $M \cap C \neq \varnothing$ and $M \cap S' \neq \varnothing$.

Now, suppose that $S' \setminus M \neq \varnothing$. We have $(C \cap M, S' \setminus M)_G = (S' \cap M, S' \setminus M)_G = 0$. Given $t' \in S' \setminus M$, $N_G(\psi_{S'}(t')) \cap S' = \{t'\}$. Thus $\psi_{S'}(t') \in C \setminus M$. But $(\psi_{S'}(t'), S' \cap M)_G = (\psi_{S'}(t'), C \cap M)_G = 1$ which contradicts $N_G(\psi_{S'}(t')) \cap S' = \{t'\}$. It follows that $S' \subseteq M$.

Lastly, suppose that $C \setminus M \neq \varnothing$. For each $c \in C \setminus M \neq \varnothing$, we have $(c, S')_G = (c, C \cap M)_G = 1$ and hence $N_G(c) \cap S' = S'$. It follows that $C \subseteq M$ and $M = C \cup S'$. □

The question of prime extensions of a prime graph is not detailed enough in [2]. For instance, the number of prime 1-extensions of a prime graph given in [2] is not correct. Moreover, Corollary 3.4 below is used without a precise proof.

Lemma 3.3. *Let G be a prime graph. Given $a \notin V(G)$, there exist exactly*

$$2^{|V(G)|} - 2|V(G)| - 2$$

distinct prime extensions of G to $V(G) \cup \{a\}$.

Proof. Consider any graph H defined on $V(H) = V(G) \cup \{a\}$ such that $H[V(G)] = G$. We prove that H is not prime if and only if

$$N_H(a) \in \{\varnothing, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$

To begin, assume that $N_H(a) \in \{\varnothing, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}$. If $N_H(a) = \varnothing$ or $V(G)$, then $V(G)$ is a nontrivial module of H. If there is $v \in V(G)$ such that $N_H(a) \setminus \{v\} = N_G(v)$, then $\{a, v\}$ is a nontrivial module of H.

Conversely, assume that H admits a nontrivial module M. By Proposition 2.1.(1), $M \setminus \{a\} \in \mathcal{M}(G)$. As G is prime, $M \setminus \{a\} \neq \varnothing$ and $M \subsetneq V(H)$, either $|M \setminus \{a\}| = 1$ or $M = V(G)$. In the second instance, $N_H(a) = \varnothing$ or $V(G)$. In the first, there is $v \in V(G)$ such that $M = \{a, v\}$. Thus $N_H(a) = N_G(v)$ or $N_G(v) \cup \{v\}$. To conclude, observe that

$$|\{\varnothing, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}| = 2 + 2|V(G)|$$

because G is prime. □

Corollary 3.4. *Let G be a prime graph. For any $a \neq b \notin V(G)$, there exists a prime extension H of G to $V(G) \cup \{a, b\}$ such that $(a, b)_H = 0$.*

Proof. Since $|V(G)| \geq 4$, $2^{|V(G)|} - 2|V(G)| - 2 \geq 2$. Consequently there is an extension H of G to $V(G) \cup \{a, b\}$ such that $(a, b)_H = 0$, $N_H(a) \neq N_H(b)$ and

$$N_H(a), N_H(b) \notin \{\varnothing, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$

By the proof of Lemma 3.3, $H - a$ and $H - b$ are prime. We show that H is prime also. Let $M \in \mathcal{M}_{\geq 2}(H)$. By Proposition 2.1.(1), $M \setminus \{a\} \in \mathcal{M}(H - a)$. As $H - a$ is prime and $M \setminus \{a\} \neq \varnothing$, either $|M \setminus \{a\}| = 1$ or $M \setminus \{a\} = V(H) \setminus \{a\}$. In the first, there is $v \in V(G) \cup \{b\}$ such that $M = \{a, v\}$. If $v = b$, then $N_H(a) = N_H(b)$. If $v \in V(G)$, then $\{a, v\}$ would be a nontrivial module of $H - b$. Consequently $M \setminus \{a\} = V(H) \setminus \{a\}$. Since $H - b$ is prime, $a \not\leftrightarrow_H V(G)$ and hence $a \in M$. Thus $M = V(H)$. □

4. Proof of Theorem 1.4

Let G be a graph with $|V(G)| \geq 2$. By [2, Theorem 3.2], there exists a prime extension H of G such that

$$2 \leq |V(H) \setminus V(G)| \leq \lceil \log_2(|V(G)| + 1) \rceil$$

and $V(H)\setminus V(G)$ is a stable set in H. We can consider the smallest integer $q(G)$ such that $q(G)\geq 2$ and G admits a prime $q(G)$-extension H such that $V(H)\setminus V(G)$ is a stable set in H.

The results below, from Proposition 4.1 to Corollary 4.4, are suggested by the proof of [2, Theorem 3.2].

We introduce a basic construction. Consider a graph G and a modular partition P of G such that $P\subseteq \mathcal{S}(G)$ and $P\cap\mathcal{S}_{\geq 2}(G)\neq\varnothing$. Let $X\in P\cap\mathcal{S}_{\geq 2}(G)$ such that

$$q(G[X])=\max(\{q(G[Y]):Y\in P\cap\mathcal{S}_{\geq 2}(G)\}).$$

Consider a set S such that $S\cap V(G)=\varnothing$ and $|S|=q(G[X])$. There exists a prime $q(G[X])$-extension H_X of $G[X]$ to $X\cup S$ such that S is a stable set in H_X. Since X is not a module of H_X, there is $s_X\in S$ such that $s_X\not\longleftrightarrow_{H_X} X$. Furthermore, if there is $v\in S$ such that $(v,X)_{H_X}=0$, then $V(H_X)\setminus\{v\}$ would be a nontrivial module of H_X. Thus $\{v\in S: v\longleftrightarrow_{H_X} X\}=\{v\in S:(v,X)_{H_X}=1\}$. As S is a stable set in H_X, $\{v\in S:(v,X)_{H_X}=1\}$ is a module of H_X. It follows that

$$\begin{cases}\{v\in S: v\longleftrightarrow_{H_X} X\}=\{v\in S:(v,X)_{H_X}=1\},\\ |\{v\in S: v\longleftrightarrow_{H_X} X\}|\leq 1,\\ s_X\in S\setminus\{v\in S: v\longleftrightarrow_{H_X} X\}.\end{cases}$$

Now, for each $Y\in(P\cap\mathcal{S}_{\geq 2}(G))\setminus\{X\}$, there is a prime $q(G[Y])$-extension H_Y of $G[Y]$ to $Y\cup S_Y$ such that $\{v\in S: v\longleftrightarrow_{H_X} X\}\subseteq S_Y\subseteq S$ and S_Y is a stable set in H_Y. Consider the extension H of G and of H_X to $V(G)\cup S$ satisfying

- for each $Y\in(P\cap\mathcal{S}_{\geq 2}(G))\setminus\{X\}$, $H[Y\cup S_Y]=H_Y$;
- for each $v\in V(G)$ such that $\{v\}\in P$, $(v,S\setminus\{s_X\})_H=0$ and $(v,s_X)_H=1$.

Proposition 4.1. *Given a graph G, consider a modular partition P of G such that $P\subseteq\mathcal{S}(G)$ and $P\cap\mathcal{S}_{\geq 2}(G)\neq\varnothing$. If the corresponding extension H is not prime, then all the nontrivial modules of H are included in $\{v\in V(G): \{v\}\in P\}$.*

Proof. Let M be a nontrivial module of H. By Proposition 2.1.(1), $M\cap(X\cup S)\in\mathcal{M}(H[X\cup S])$. Since $H[X\cup S]$ is prime, we have $M\supseteq X\cup S$, $|M\cap(X\cup S)|=1$ or $M\cap(X\cup S)=\varnothing$.

For a first contradiction, suppose that $M\supseteq X\cup S$. Given $v\in V(G)$, if $\{v\}\in P$, then $v\not\longleftrightarrow_H S$ so that $v\in M$. Thus $\{v\in V(G):\{v\}\in P\}\subseteq M$. Let $Y\in P\cap\mathcal{S}_{\geq 2}(G)$. By Proposition 2.1.(1), $M\cap(Y\cup S_Y)\in\mathcal{M}(H[Y\cup S_Y])$. Since $H[Y\cup S_Y]$ is prime and since $S_Y\subseteq M\cap(Y\cup S_Y)$, $Y\subseteq M$. Therefore $\bigcup(P\cap\mathcal{S}_{\geq 2}(G))\subseteq M$ and we would have $M=V(H)$.

For a second contradiction, suppose that $|M\cap(X\cup S)|=1$. Consider $v\in S\cup X$ such that $M\cap(X\cup S)=\{v\}$. Suppose that $v\in X$. We have $M\subseteq V(G)$ and $M\in\mathcal{M}(G)$ by Proposition 2.1.(1). As $X\in\mathcal{S}(G)$ and $v\in X\cap M$, $X\subseteq M$ or $M\subseteq X$. In both cases, we would have $|M\cap(X\cup S)|\geq 2$.

Suppose that $v \in S$. There is $Y \in P \setminus \{X\}$ such that $Y \cap M \neq \varnothing$. Let $y \in Y \cap M$. Since $y \longleftrightarrow_G X$, $v \longleftrightarrow_{H_X} X$ and hence $v \neq s_X$. If $Y \in P \cap \mathcal{S}_{\geq 2}(G)$, then $v \in S_Y$ and $M \cap (Y \cup S_Y)$ would be a nontrivial module of $H[Y \cup S_Y]$. If $Y = \{y\}$, then $(y, s_X)_H = 1$. Thus $(v, s_X)_H = 1$ and S would not be a stable set in H.

It follows that $M \cap (X \cup S) = \varnothing$. By Proposition 2.1.(1), $M \in \mathcal{M}(G)$. Suppose for a contradiction that there is $Y \in (P \cap \mathcal{S}_{\geq 2}(G)) \setminus \{X\}$ such that $Y \cap M \neq \varnothing$. As $Y \in \mathcal{S}(G)$, $Y \subseteq M$ or $M \subseteq Y$. In both cases, $M \cap (Y \cup S_Y)$ would be a nontrivial module of $H[Y \cup S_Y]$. It follows that $Y \cap M = \varnothing$. Therefore $M \subseteq \{v \in V(G) : \{v\} \in P\}$. □

Corollary 4.2. *Given a graph G such that $G/\Pi(G)$ is prime, we have*

$$q(G) \leq \begin{cases} 2, & \text{if } \ \Pi_{\geq 2}(G) = \varnothing, \\ \max(\{q(G[X]) : X \in \Pi_{\geq 2}(G)\}), & \text{if } \ \Pi_{\geq 2}(G) \neq \varnothing. \end{cases}$$

Proof. If G is prime, then $q(G) = 2$ by Corollary 3.4. Assume that G is not prime, that is, $\Pi_{\geq 2}(G) \neq \varnothing$. Let H be the extension of G associated with $\Pi(G)$. Suppose that H admits a nontrivial module M. By Proposition 4.1, $\{\{u\} : u \in M\} \subseteq \Pi_1(G)$. Thus $M \in \mathcal{M}(G)$ by Proposition 2.1.(1). By Proposition 2.2.(2), $\{\{u\} : u \in M\}$ would be a nontrivial module of $G/\Pi(G)$. □

Proposition 4.3. *Given a graph G such that $G/\Pi(G)$ is complete or empty, we have*

$$q(G) \leq \max(2, \lceil \log_2(|\Pi_1(G)| + 1) \rceil),$$

or

$$q(G) \leq \max(\{q(G[X]) : X \in \Pi_{\geq 2}(G)\}).$$

Proof. Assume that $G/\Pi(G)$ is empty. If $\Pi(G) = \Pi_1(G)$, then G is empty by Proposition 2.2.(1), and it suffices to apply Lemma 3.1. Assume that $\Pi_{\geq 2}(G) \neq \varnothing$ and set

$$W_2 = \bigcup \Pi_{\geq 2}(G).$$

Let H be the extension of G associated with $\Pi(G)$. Recall that $V(H) = V(G) \cup S$, $V(G) \cap S = \varnothing$ and $|S| = q(G[X])$ where $X \in \Pi_{\geq 2}(G)$ such that $q(G[X]) = \max(\{q(G[Y]) : Y \in \Pi_{\geq 2}(G)\})$. Moreover $H[X \cup S]$ is prime.

If $|\Pi_1(G)| \leq 1$, then H is prime by Proposition 4.1 so that $q(G) \leq \max (\{q(G[Y]) : Y \in \Pi_{\geq 2}(G)\})$. Assume that $|\Pi_1(G)| \geq 2$ and set

$$W_1 = V(G) \setminus W_2.$$

By Lemma 3.1, there exists a prime extension H_1 of $G[W_1]$ to $W_1 \cup S_1$ such that $|S_1| = \lceil \log_2(|W_1| + 1) \rceil$ and S_1 is stable in H_1. As $G/\Pi(G)$ is empty, $\Pi_{\geq 2}(G) \in \mathcal{M}(G/\Pi(G))$. By Proposition 2.2.(3), $W_2 \in \mathcal{M}(G)$. Thus $\Pi_{\geq 2}(G) \subseteq \mathcal{S}(G[W_2])$ by Proposition 2.3. It follows from Proposition 4.1 that $H[W_2 \cup S]$ is prime. We construct suitable extensions of G according to whether $|S_1| \leq |S|$ or not.

To begin, suppose $|S_1| \leq |S|$. We can assume that

$$\{v \in S : v \longleftrightarrow_{H[X \cup S]} X\} \subseteq S_1 \subseteq S$$

and consider an extension H' of H_1 and $H[W_2 \cup S]$ to $V(G) \cup S$. We show that H' is prime. Let $M \in \mathcal{M}_{\geq 2}(H')$. By Proposition 2.1.(1), $M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])$. Since $H[W_2 \cup S]$ is prime, $M \cap (W_2 \cup S) = \varnothing$, $|M \cap (W_2 \cup S)| = 1$ or $M \supseteq (W_2 \cup S)$.

- Suppose for a contradiction that $M \cap (W_2 \cup S) = \varnothing$. By Proposition 2.1.(1), M would be a nontrivial module of H_1.
- Suppose for a contradiction that $|M \cap (W_2 \cup S)| = 1$ and consider $w \in W_2 \cup S$ such that $M \cap (W_2 \cup S) = \{w\}$. First, suppose that $w \in W_2$ and consider $Y \in \Pi_{\geq 2}(G)$ such that $w \in Y$. By Proposition 2.1.(1), $M \in \mathcal{M}(G)$. As $Y \in \mathcal{S}(G)$ and $w \in X \cap M$, $X \subseteq M$ or $M \subseteq X$. In both cases, we would have $|M \cap (W_2 \cup S)| \geq 2$. Second, suppose that $w \in S$ and consider $v \in W_1 \cap M$. Since $v \longleftrightarrow_G X$, $w \longleftrightarrow_{H[W_2 \cup S]} X$ and hence $w \in S_1$. It follows from Proposition 2.1.(1) that M would be a nontrivial module of H_1.

Consequently $M \supseteq (W_2 \cup S)$. By Proposition 2.1.(1), $M \cap (W_1 \cup S_1) \in \mathcal{M}(H_1)$. As H_1 is prime and $M \cap (W_1 \cup S_1) \supseteq S_1$, $M \cap (W_1 \cup S_1) = (W_1 \cup S_1)$ so that $M = V(H')$.

Now, assume that $|S_1| > |S|$. We can assume that $S \subsetneq S_1$ and we consider the unique extension H'' of H_1 and $H[W_2 \cup S]$ to $V(G) \cup S_1$ such that

$$(W_2, S_1 \setminus S)_{H''} = 0. \tag{4.1}$$

We show that H'' is prime. Let $M \in \mathcal{M}_{\geq 2}(H'')$. We obtain $M \cap (W_1 \cup S_1) = \varnothing$, $|M \cap (W_1 \cup S_1)| = 1$ or $M \supseteq (W_1 \cup S_1)$. If $M \cap (W_1 \cup S_1) = \varnothing$, then M would be a nontrivial module of $H[W_2 \cup S]$.

Suppose for a contradiction that $|M \cap (W_1 \cup S)_1| = 1$ and consider $w \in W_1 \cup S_1$ such that $M \cap (W_1 \cup S_1) = \{w\}$. There is $v \in W_2 \cap M$. Let $Y \in \Pi_{\geq 2}(G)$ such that $v \in Y$.

- Suppose that $w \in W_1$. By Proposition 2.1.(1), $M \in \mathcal{M}(G)$. Since $Y \in \mathcal{S}(G)$ and since $Y \cap M \neq \varnothing$ and $w \in M \setminus Y$, $Y \subseteq M$. It follows from Proposition 2.1.(1) that $M \cap (W_2 \cup S)$ would be a nontrivial module of $H[W_2 \cup S]$.
- Suppose that $w \in S_1$. By Proposition 2.1.(1), $M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])$. As $H[W_2 \cup S]$ is prime, $v \in M \cap W_2$ and $M \cap S \subseteq \{w\}$, $M \cap (W_2 \cup S) = \{v\}$ hence $w \in S_1 \setminus S$. For every $u \in W_2 \setminus \{v\}$, we have $(u, v)_G = (u, w)_{H''} = 0$ by (4.1). Since $(v, W_1)_G = 0$, we would have $N_G(v) = \varnothing$ and hence $\{v\} \in \Pi_1(G)$.

It follows that $M \supseteq (W_1 \cup S_1)$. By Proposition 2.1.(1), $M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])$. As $H[W_2 \cup S]$ is prime and $M \cap (W_2 \cup S) \supseteq S$, $M \cap (W_2 \cup S) = (W_2 \cup S)$ so that $M = V(H'')$.

Finally, observe that when $G/\Pi(G)$ is complete, we can proceed as previously by replacing (4.1) by $(W_2, S_1 \setminus S)_{H''} = 1$. □

The next result follows from Corollary 4.2 and Proposition 4.3 by induction on the number of vertices.

Corollary 4.4. *Given a graph G with $|V(G)| \geq 2$,*

- $q(G) = 2$ *if for every $X \in \mathcal{S}_{\geq 2}(G)$ such that $\lambda_G(X) \in \{\bigcirc, \bullet\}$, we have* $|\Pi_1(G[X])| \leq 1$;
- $q(G) \leq \max(\{\lceil \log_2(|\Pi_1(G[Y])| + 1)\rceil : Y \in \mathcal{S}_{\geq 2}(G), \lambda_G(Y) \in \{\bigcirc, \bullet\}\})$ *if there is $X \in \mathcal{S}_{\geq 2}(G)$ such that $\lambda_G(X) \in \{\bigcirc, \bullet\}$ and $|\Pi_1(G[X])| \geq 2$.*

Given the second assertion of Corollary 4.4, Theorem 1.4 follows from the next transcription in terms of the modular decomposition tree. Let G be a graph. Denote by $\mathbb{M}(G)$ the family of the maximal elements of $\mathcal{M}_{\geq 2}(G)$ under inclusion which are cliques or stable sets in G.

Proposition 4.5. *Let G be a graph. Given $M \subseteq V(G)$, we have $M \in \mathbb{M}(G)$ if and only if $M \in \mathcal{M}_{\geq 2}(G)$, $\lambda_G(\widehat{M}) \in \{\bigcirc, \bullet\}$ and $M = \{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\}$.*

Proof. To begin, consider $M \in \mathbb{M}(G)$ and assume that M is a stable set in G. By Proposition 2.1.(1), $M \in \mathcal{M}(G[\widehat{M}])$. Set

$$Q = \{X \in \Pi(G[\widehat{M}]) : X \cap M \neq \varnothing\}.$$

By definition of $\widehat{M}$, $|Q| \geq 2$ and hence $M = \bigcup Q$ because $Q \subseteq \mathcal{S}(G[\widehat{M}])$. Furthermore, $Q \subseteq \mathcal{S}(G[M])$ by Proposition 2.3. As all the strong modules of an empty graph are trivial, we obtain $|X| = 1$ for each $X \in Q$, that is,

$$M \subseteq \{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\}.$$

By Proposition 2.2.(2), $Q \in \mathcal{M}(G[\widehat{M}]/\Pi(G[\widehat{M}]))$. For a contradiction, suppose that $\lambda_G(\widehat{M}) = \sqsubset$. Since $Q \in \mathcal{M}_{\geq 2}(G[\widehat{M}]/\Pi(G[\widehat{M}]))$, $Q = \Pi(G[\widehat{M}])$ and hence $M = \widehat{M}$. As $|X| = 1$ for each $X \in Q$, $G[\widehat{M}]/\Pi(G[\widehat{M}])$ and $G[\widehat{M}]$ are isomorphic by Proposition 2.2.(1). It would follow that $G[M]$ is prime. Consequently $\lambda_G(\widehat{M}) \in \{\bigcirc, \bullet\}$. Given $v \neq w \in M$, we have $(\{v\}, \{w\})_{G[\widehat{M}]/\Pi(G[\widehat{M}])} = (v, w)_G = 0$. Thus

$$\lambda_G(\widehat{M}) = \bigcirc.$$

Since $\lambda_G(\widehat{M}) = \bigcirc$, we have $\Pi_1(G[\widehat{M}]) \in \mathcal{M}(G[\widehat{M}]/\Pi(G[\widehat{M}]))$. By Proposition 2.2.(3), $\bigcup \Pi_1(G[\widehat{M}]) \in \mathcal{M}(G[\widehat{M}])$ and hence $\bigcup \Pi_1(G[\widehat{M}]) \in \mathcal{M}(G)$ by Proposition 2.1.(2). Given $v \neq w \in \bigcup \Pi_1(G[\widehat{M}])$, we have

$$(v, w)_G = (\{v\}, \{w\})_{G[\widehat{M}]/\Pi(G[\widehat{M}])} = 0.$$

Therefore $\bigcup \Pi_1(G[\widehat{M}])$ is a stable set of G. As $M \subseteq \bigcup \Pi_1(G[\widehat{M}])$, $M = \bigcup \Pi_1(G[\widehat{M}])$ by maximality of M. It follows that

$$M = \{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\}.$$

Conversely, consider $M \in \mathcal{M}_{\geq 2}(G)$ such that $\lambda_G(\widehat{M}) = \bigcirc$ and $M = \{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\}$. As $\lambda_G(\widehat{M}) = \bigcirc$, $\Pi_1(G[\widehat{M}]) \in \mathcal{M}(G[\widehat{M}]/\Pi(G[\widehat{M}]))$.

By Proposition 2.2.(3), $M = \bigcup \Pi_1(G[\widehat{M}]) \in \mathcal{M}(G[\widehat{M}])$ and hence $M \in \mathcal{M}(G)$ by Proposition 2.1.(2). Since $(v,w)_G = (\{v\},\{w\})_{G[\widehat{M}]/\Pi(G[\widehat{M}])} = 0$ for all $v \neq w \in M$, M is a stable set in G. There is $N \in \mathbb{M}(G)$ such that $N \supseteq M$. As M is a stable set in G, N is as well. By what precedes, $N = \{v \in \widehat{N} : \{v\} \in \Pi(G[\widehat{N}])\}$. We have $\widehat{M} \subseteq \widehat{N}$ because $M \subseteq N$. Furthermore $\widehat{M} \in \mathcal{S}(G[\widehat{N}])$ by Proposition 2.3. Given $v \in M$, we obtain $\{v\} \subsetneq \widehat{M} \subseteq \widehat{N}$. Since $\{v\} \in \Pi(G[\widehat{N}])$, $\widehat{M} = \widehat{N}$. Therefore $M = N$ because $M = \{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\}$ and $N = \{v \in \widehat{N} : \{v\} \in \Pi(G[\widehat{N}])\}$. □

Let G be a graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. Consider $M \in \mathbb{M}(G)$. By Proposition 4.5, $\lambda_G(\widehat{M}) \in \{\bigcirc, \bullet\}$ and $|\Pi_1(G[\widehat{M}])| = |M| \geq 2$. By Corollary 4.4,

$$p(G) \leq q(G) \leq \max(\{\lceil \log_2(|\Pi_1(G[Y])|+1)\rceil : Y \in \mathcal{S}_{\geq 2}(G), \lambda_G(Y) \in \{\bigcirc, \bullet\}\}).$$

By Proposition 4.5,

$$\max(\{\lceil \log_2(|\Pi_1(G[Y])|+1)\rceil : Y \in \mathcal{S}_{\geq 2}(G), \lambda_G(Y) \in \{\bigcirc, \bullet\}\})$$

equals

$$\max(\{\lceil \log_2(|M|+1)\rceil : M \in \mathbb{M}(G)\}).$$

Clearly

$$\max(\{\lceil \log_2(|M|+1)\rceil : M \in \mathbb{M}(G)\}) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))+1)\rceil$$

and consequently we recover Theorem 1.4,

$$p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G))+1)\rceil.$$

To obtain Corollary 1.5, we prove Lemma 1.3.

Proof of Lemma 1.3. Let G be a graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. There exists $S \in \mathcal{M}(G)$ such that $|S| = \max(\alpha_M(G), \omega_M(G))$ and S is a clique or a stable set in G. Given an integer $p < \log_2(\max(\alpha_M(G), \omega_M(G)))$, consider any p-extension H of G. We must prove that H is not prime. We have $2^{|V(H) \smallsetminus V(G)|} < |S|$ so that the function $S \longrightarrow 2^{V(H) \smallsetminus V(G)}$, defined by $s \mapsto N_H(s) \cap (V(H) \smallsetminus V(G))$ is not injective. There are $s \neq t \in S$ such that $v \longleftrightarrow_H \{s,t\}$ for every $v \in V(H) \smallsetminus V(G)$. As S is a module of G, we have $v \longleftrightarrow_H \{s,t\}$ for every $v \in V(G) \smallsetminus S$. Since S is a clique or a stable set in G, $\{s,t\}$ is a nontrival module of H. □

When a graph or its complement admits isolated vertices, we obtain the following.

Lemma 4.6. *Given a graph G, if $\iota(G) \neq 0$ or $\iota(\overline{G}) \neq 0$, then*

$$p(G) \geq \lceil \log_2(\max(\iota(G), \iota(\overline{G}))+1)\rceil.$$

Proof. By interchanging G and $\overline{G}$, assume that $\iota(G) \geq \iota(\overline{G})$. Given $p < \lceil \log_2(\iota(G)+1) \rceil$, consider any p-extension H of G. We have $2^{|V(H)\setminus V(G)|} \leq \iota(G)$ and we verify that H is not prime.

For each $x \in V(G)$ such that $N_G(x) = \varnothing$, we have $N_H(x) \subseteq V(H) \setminus V(G)$. Thus $(N_H)_{\restriction \{v\in V(G):N_G(v)=\varnothing\}}$ is a function from $\{v \in V(G) : N_G(v) = \varnothing\}$ to $2^{V(H)\setminus V(G)}$. As observed in the proof of Lemma 3.1, if $(N_H)_{\restriction \{v\in V(G):N_G(v)=\varnothing\}}$ is not injective, then $\{x, y\}$ is a nontrivial module of H when $x \neq y \in \{v \in V(G) : N_G(v) = \varnothing\}$ with $N_H(x) = N_H(y)$. So assume that

$$(N_H)_{\restriction \{v\in V(G):N_G(v)=\varnothing\}} \text{ is injective.}$$

As $2^{|V(H)\setminus V(G)|} \leq \iota(G)$, we obtain that $(N_H)_{\restriction \{v\in V(G):N_G(v)=\varnothing\}}$ is bijective. Thus there is $x \in \{v \in V(G) : N_G(v) = \varnothing\}$ such that $N_H(x) = \varnothing$. Therefore $V(H) \setminus \{x\}$ is a nontrivial module of H and H is not prime. □

The next result is a simple consequence of Proposition 4.5 which is useful in proving Theorem 1.6.

Corollary 4.7. *Given a graph G such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$, the elements of $\mathbb{M}(G)$ are pairwise disjoint.*

Proof. Consider $M, N \in \mathbb{M}(G)$ such that $M \cap N \neq \varnothing$. Let $v \in M \cap N$. Since $\widehat{M}, \widehat{N} \in \mathcal{S}(G)$ and $v \in \widehat{M} \cap \widehat{N}$, $\widehat{M} \subseteq \widehat{N}$ or $\widehat{N} \subseteq \widehat{M}$. For instance, assume that $\widehat{M} \subseteq \widehat{N}$. By Proposition 2.3, $\widehat{M} \in \mathcal{S}(G[\widehat{N}])$. Furthermore $\{v\} \in \Pi(G[\widehat{N}])$ by Proposition 4.5. As $\{v\} \subsetneq \widehat{M} \subseteq \widehat{N}$, we obtain $\widehat{M} = \widehat{N}$. Lastly, $M = \{w \in \widehat{M} : \{w\} \in \Pi(G[\widehat{M}])\}$ and $N = \{w \in \widehat{N} : \{w\} \in \Pi(G[\widehat{N}])\}$ by Proposition 4.5. Thus $M = N$. □

5. Proof of Theorem 1.6

Given a graph G, denote by $\mathbb{P}(G)$ the family of $M \in \mathcal{M}(G)$ such that $G[M]$ is prime. For every $M \in \mathbb{P}(G)$, $M \in \mathcal{S}(G)$ because $G[M]$ is prime. It follows that the elements of $\mathbb{P}(G)$ are pairwise disjoint. Thus the elements of $\mathbb{M}(G) \cup \mathbb{P}(G)$ are also by Corollary 4.7. Set

$$I(G) = V(G) \setminus ((\bigcup \mathbb{M}(G)) \cup (\bigcup \mathbb{P}(G))).$$

We prove Theorem 1.6 when $\max(\alpha_M(G), \omega_M(G)) = 2$.

Proposition 5.1. *For every graph G such that $\max(\alpha_M(G), \omega_M(G)) = 2$,*

$$p(G) = 2 \textit{ if and only if } \iota(G) = 2 \textit{ or } \iota(\overline{G}) = 2.$$

Proof. It follows from Lemma 1.3 and Theorem 1.4 that $p(G) = 1$ or 2. To begin, assume that $\iota(G) = 2$ or $\iota(\overline{G}) = 2$. By Lemma 4.6, $p(G) \geq 2$ and hence $p(G) = 2$. Conversely, assume that $p(G) = 2$. Let $a \notin V(G)$. As $\max(\alpha_M(G), \omega_M(G)) = 2$, $|N| = 2$ for each $N \in \mathbb{M}(G)$. Let $N_0 \in \mathbb{M}(G)$. For $N \in \mathbb{P}(G)$, $G[N]$ is prime. By Lemma 3.3, $G[N]$ admits a prime extension H_N defined on $N \cup \{a\}$. We consider any 1-extension H of G to $V(G) \cup \{a\}$ satisfying the following.

(1) For each $N \in \mathbb{M}(G)$, $a \not\leftrightarrow_H N$.
(2) For each $N \in \mathbb{P}(G)$, $H[N \cup \{a\}] = H_N$.
(3) Let $v \in I(G)$. There is $i \in \{0,1\}$ such that $(v, N_0)_G = i$. We require that $(v,a)_H \neq i$.

To begin, we prove that $\mathcal{S}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$. Given $M \in \mathcal{S}_{\geq 2}(G)$, we have to verify that $a \not\leftrightarrow_H M$. Let N be a minimal element under inclusion of $\{N' \in \mathcal{S}_{\geq 2}(G) : N' \subseteq M\}$. By Proposition 2.3, $\Pi(G[N]) \subseteq \mathcal{S}(G)$. By minimality of N, $\Pi(G[N]) = \Pi_1(G[N])$ so that $G[N]$ and $G[N]/\Pi(G[N])$ are isomorphic by Proposition 2.2.(1). We distinguish the following two cases.

- Assume that $\lambda_G(N) = \sqsubset$. We obtain that $G[N]$ is prime, that is, $N \in \mathbb{P}(G)$. As $H[N \cup \{a\}]$ is prime, $a \not\leftrightarrow_H N$.
- Assume that $\lambda_G(N) \in \{\bigcirc, \bullet\}$. By Proposition 4.5, $N \in \mathbb{M}(G)$. Thus $|N| = 2$ and $a \not\leftrightarrow_H N$ by definition of H.

In both cases, $a \not\leftrightarrow_H N$ and hence $a \not\leftrightarrow_H M$.

Now we prove that $\mathcal{M}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$. Let $M \in \mathcal{M}_{\geq 2}(G)$. Since $\mathcal{S}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$, assume that $M \notin \mathcal{S}_{\geq 2}(G)$. Set $Q = \{X \in \Pi(G[\widehat{M}]) : X \cap M \neq \varnothing\}$. By Proposition 2.1.(1), $M \in \mathcal{M}(G[\widehat{M}])$. By definition of $\widehat{M}$, $|Q| \geq 2$. Thus $M = \bigcup Q$ because $\Pi(G[\widehat{M}]) \subseteq \mathcal{S}(G[\widehat{M}])$. Furthermore $Q \neq \Pi(G[\widehat{M}])$ because $M \notin \mathcal{S}_{\geq 2}(G)$. By Proposition 2.2.(2), $Q \in \mathcal{M}(G[\widehat{M}]/\Pi(G[\widehat{M}]))$. As $2 \leq |Q| < |\Pi(G[\widehat{M}])|$, $\lambda_G(\widehat{M}) \in \{\bigcirc, \bullet\}$. If there is $X \in Q \cap \Pi_{\geq 2}(G[\widehat{M}])$, then $a \not\leftrightarrow_H X$ by what precedes and hence $a \not\leftrightarrow_H M$. Assume that $Q \subseteq \Pi_1(G[\widehat{M}])$. We obtain that M is a clique or a stable set in G. Since $\max(\alpha_M(G), \omega_M(G)) = 2$, $M \in \mathbb{M}(G)$ and $a \not\leftrightarrow_H M$ by definition of H.

As $p(G) = 2$, H admits a nontrivial module M_H. We have $a \in M_H$ because $\mathcal{M}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$.

First, we show that $N \subseteq M_H$ for each $N \in \mathbb{P}(G)$. By Proposition 2.1.(1), $M_H \cap (N \cup \{a\}) \in \mathcal{M}(H[N \cup \{a\}])$. Since $H[N \cup \{a\}]$ is prime and $a \in M_H \cap (N \cup \{a\})$, we obtain either $(M_H \setminus \{a\}) \cap N = \varnothing$ or $N \subseteq M_H \setminus \{a\}$. Suppose for a contradiction that $(M_H \setminus \{a\}) \cap N = \varnothing$. By Proposition 2.1.(1), $M_H \setminus \{a\} \in \mathcal{M}(G)$. There is $i \in \{0,1\}$ such that $(M_H \setminus \{a\}, N)_G = i$ by Proposition 2.1.(3). Therefore $(a, N)_H = i$ which contradicts the fact that $H[N \cup \{a\}]$ is prime. It follows that $N \subseteq M_H$. Thus

$$\bigcup \mathbb{P}(G) \subseteq M_H. \tag{5.1}$$

Second, we show that $N \cap M_H \neq \varnothing$ for each $N \in \mathbb{M}(G)$. Otherwise consider $N \in \mathbb{M}(G)$ such that $N \cap M_H = \varnothing$. There is $i \in \{0,1\}$ such that $(M_H \setminus \{a\}, N)_G = i$. Thus $(a, N)_H = i$ which contradicts $a \not\leftrightarrow_H N$. Therefore

$$N \cap M_H \neq \varnothing \quad \text{for each} \quad N \in \mathbb{M}(G). \tag{5.2}$$

Third, let $v \in I(G)$. By (5.2), $N_0 \cap M_H \neq \varnothing$. Since $(v, N_0 \cap M_H)_G \neq (v, a)_H$, $v \in M_H$. Hence

$$I(G) \subseteq M_H. \tag{5.3}$$

By (5.1) and (5.3),

$$V(G) \smallsetminus M_H \subseteq \mathbb{M}(G). \tag{5.4}$$

To conclude, consider $v \in V(H) \smallsetminus M_H$. By (5.4), there is $N_v \in \mathbb{M}(G)$ such that $v \in N_v$. By interchanging G and $\overline{G}$, assume that N_v is a stable set in G. Since $v \longleftrightarrow_H M_H$ and $(v, N_v \cap M_H)_G = 0$, we obtain $(v, M_H)_H = 0$. Let $N \in \mathbb{M}(G) \smallsetminus \{N_v\}$. By Corollary 4.7, $N \cap N_v = \varnothing$. As $N \cap M_H \neq \varnothing$ by (5.2), we have $(v, N \cap M_H)_G = 0$ and hence $(v, N)_G = 0$. It follows that $N_G(v) = \varnothing$. Therefore $(N_v, V(G) \smallsetminus N_v)_G = 0$ because $N_v \in \mathcal{M}(G)$. Since N_v is a stable set in G, we obtain $N_v \subseteq \{u \in V(G) : N_G(u) = \varnothing\}$. Clearly $\{u \in V(G) : N_G(u) = \varnothing\} \in \mathcal{M}(G)$ and $\{u \in V(G) : N_G(u) = \varnothing\}$ is a stable set in G. Thus $\iota(G) \leq \max(\alpha_M(G), \omega_M(G)) = 2$. Consequently $N_v = \{u \in V(G) : N_G(u) = \varnothing\}$. □

Proof of Theorem 1.6. Consider a graph G such that

$$\max(\alpha_M(G), \omega_M(G)) = 2^k$$

where $k \geq 1$. It follows from Corollary 1.5 that $p(G) = k$ or $k+1$.

To begin, assume that $\iota(G) = 2^k$ or $\iota(\overline{G}) = 2^k$. By Lemma 4.6, $p(G) \geq k+1$ and hence $p(G) = k+1$.

Conversely, assume that $p(G) = k+1$. If $k = 1$, then it suffices to apply Proposition 5.1. Assume that $k \geq 2$. For convenience set

$$\mathbb{M}_{\max}(G) = \{N \in \mathbb{M}(G) : |N| = \max(\alpha_M(G), \omega_M(G))\}.$$

With each $N \in \mathbb{M}_{\max}(G)$ associate $w_N \in N$. Set $W = \{w_N : N \in \mathbb{M}_{\max}(G)\}$.

We prove that $\max(\alpha_M(G-W), \omega_M(G-W)) = 2^k - 1$. Let $N \in \mathbb{M}_{\max}(G)$. By Corollary 4.7, the elements of $\mathbb{M}_{\max}(G)$ are pairwise disjoint. Thus $N \smallsetminus W = N \smallsetminus \{w_N\}$. Clearly $N \smallsetminus \{w_N\}$ is a clique or a stable set in $G - W$. Furthermore $N \smallsetminus \{w_N\} \in \mathcal{M}(G-W)$. Therefore $2^k - 1 = |N \smallsetminus \{w_N\}| \leq \max(\alpha_M(G-W), \omega_M(G-W))$. Now consider $N' \in \mathbb{M}_{\max}(G-W)$. We show that $N' \in \mathcal{M}(G)$. We have to verify that for each $N \in \mathbb{M}_{\max}(G)$, $w_N \longleftrightarrow_G N'$. Let $N \in \mathbb{M}_{\max}(G)$. First, asume that there is $v \in (N \smallsetminus \{w_N\}) \smallsetminus N'$. We have $v \longleftrightarrow_G N'$. As N is a clique or a stable set in G, $\{v, w_N\} \in \mathcal{M}(G[N])$. By Proposition 2.1.(2), $\{v, w_N\} \in \mathcal{M}(G)$. Thus $w_N \longleftrightarrow_G N'$. Second, assume that $N \smallsetminus \{w_N\} \subseteq N'$. Clearly $w_N \longleftrightarrow_G N'$ when $N \smallsetminus \{w_N\} = N'$. Assume that $N' \smallsetminus (N \smallsetminus \{w_N\}) \neq \varnothing$. By interchanging G and $\overline{G}$, assume that N' is a clique in $G - W$. As $N \smallsetminus \{w_N\} \subseteq N'$ and $|N \smallsetminus \{w_N\}| \geq 2$, we obtain that N is a clique in G. Since $(N \smallsetminus \{w_N\}, N' \smallsetminus N)_G = 1$ and since $N \in \mathcal{M}(G)$, we have $(w_N, N' \smallsetminus N)_G = 1$. Furthermore $(w_N, N \smallsetminus \{w_N\})_G = 1$ because N is a clique in G. Therefore $(w_N, N')_G = 1$. Consequently $N' \in \mathcal{M}(G)$. As N' is a clique in G, there is $M \in \mathbb{M}(G)$ such that $M \supseteq N'$. If $M \notin \mathbb{M}_{\max}(G)$, then $|N'| \leq |M| < \max(\alpha_M(G), \omega_M(G))$. If $M \in \mathbb{M}_{\max}(G)$, then $N' \subseteq M \smallsetminus \{w_M\}$ and hence $|N'| < |M| = \max(\alpha_M(G), \omega_M(G))$. In both cases, we have $|N'| = \max(\alpha_M(G-W), \omega_M(G-W)) < \max(\alpha_M(G), \omega_M(G))$. It follows that $\max(\alpha_M(G-W), \omega_M(G-W)) = 2^k - 1$.

By Corollary 1.5, $p(G-W) = k$ and hence there exists a prime k-extension H' of $G-W$. We extend H' to $V(H') \cup W$ as follows. Let $N \in \mathbb{M}_{\max}(G)$. Consider the function $f_N : N \setminus \{w_N\} \longrightarrow 2^{V(H') \setminus V(G-W)}$ defined by $v \mapsto N_{H'}(v) \setminus V(G-W)$ for $v \in N \setminus \{w_N\}$. Since H' is prime, f_N is injective. As $|N \setminus \{w_N\}| = 2^k - 1$ and $|2^{V(H') \setminus V(G-W)}| = 2^k$, there is a unique $X_N \subseteq V(H') \setminus V(G-W)$ such that $f_N(v) \neq X_N$ for every $v \in N \setminus \{w_N\}$. Let H be the extension of H' to $V(H') \cup W$ such that $N_H(w_N) \cap (V(H') \setminus V(G-W)) = X_N$ for each $N \in \mathbb{M}_{\max}(G)$. As $p(G) = k+1$, H is not prime. Consider a nontrivial module M_H of H.

Observe the following. Given $N \neq N' \in \mathbb{M}_{\max}(G)$,

$$\left.\begin{array}{r} N \cap M_H \neq \varnothing \\ \text{and} \\ N' \cap M_H \neq \varnothing \end{array}\right\} \Longrightarrow M_H \supseteq V(H'). \tag{5.5}$$

Indeed, by Proposition 2.1.(1), $M_H \cap V(G) \in \mathcal{M}(G)$. Since $\widehat{N}, \widehat{N'} \in \mathcal{S}(G)$ and since $(M_H \cap V(G)) \cap \widehat{N} \neq \varnothing$ and $(M_H \cap V(G)) \cap \widehat{N'} \neq \varnothing$, $M_H \cap V(G)$ is comparable to $\widehat{N}$ and $\widehat{N'}$ under inclusion. Suppose for a contradiction that $M_H \cap V(G) \subsetneq \widehat{N}$ and $M_H \cap V(G) \subsetneq \widehat{N'}$. It follows that $N' \cap \widehat{N} \neq \varnothing$ and $N \cap \widehat{N'} \neq \varnothing$. As $\widehat{N'} \in \mathcal{S}(G)$, $\widehat{N'} \subsetneq N$ or $N \subseteq \widehat{N'}$. In the first instance, it follows from Proposition 2.3 that $\widehat{N'}$ would be a nontrivial strong module of $G[N]$ which contradicts the fact that N is a clique or a stable set in G. Thus $N \subseteq \widehat{N'}$ and hence $\widehat{N} \subseteq \widehat{N'}$. Similarly $N' \subseteq \widehat{N}$ and $\widehat{N'} \subseteq \widehat{N}$. Therefore $\widehat{N} = \widehat{N'}$ and it would follow from Proposition 4.5 that $N = N'$. Consequently $\widehat{N} \subseteq (M_H \cap V(G))$ or $\widehat{N'} \subseteq (M_H \cap V(G))$. For instance, assume that $\widehat{N} \subseteq (M_H \cap V(G))$. By Proposition 2.1.(1), $M_H \cap V(H') \in \mathcal{M}(H')$. Furthermore $(M_H \cap V(H')) \supseteq (N \setminus W)$ and $N \setminus W = N \setminus \{w_N\}$ by Corollary 4.7. Since H' is prime, we have $V(H') \subseteq M_H$. It follows that (5.5) holds.

As H' is prime and $M_H \cap V(H') \in \mathcal{M}(H')$, we have either $|M_H \cap V(H')| \leq 1$ or $M_H \supseteq V(H')$. For a contradiction, suppose that $|M_H \cap V(H')| \leq 1$. There is $N \in \mathbb{M}_{\max}(G)$ such that $w_N \in M_H$. It follows from (5.5) that $M_H \cap W = \{w_N\}$. Thus there is $v \in V(H')$ such that $M_H \cap V(H') = \{v\}$. Clearly $M_H = \{v, w_N\}$ and we distinguish the following two cases to obtain a contradiction.

- Suppose that $v \in V(G-W)$. By Proposition 2.1.(1), $\{v, w_N\} \in \mathcal{M}(G)$. Therefore there is $N' \in \mathbb{M}_{\max}(G)$ such that $N' \supseteq \{v, w_N\}$. By Corollary 4.7, $N = N'$ and we would obtain $N_H(w_N) \cap (V(H') \setminus V(G-W)) = f_N(v)$.
- Suppose that $v \in V(H') \setminus V(G-W)$. There is $i \in \{0,1\}$ such that $(w_N, N \setminus \{w_N\})_G = i$. We obtain $(v, N \setminus \{w_N\})_{H'} = i$ because $\{v, w_N\} \in \mathcal{M}(H)$. Since f_N is injective, the function $g_N : N \setminus \{w_N\} \longrightarrow 2^{((V(H') \setminus V(G-W)) \setminus \{v\})}$, defined by $g_N(u) = f_N(u) \setminus \{v\}$ for $u \in N \setminus \{w_N\}$, is injective as well. We would obtain $2^k - 1 \leq 2^{k-1}$.

Consequently $V(H') \subseteq M_H$. As M_H is a nontrivial module of H, there exists $N \in \mathbb{M}_{\max}(G)$ such that $w_N \notin M$. By interchanging G and $\overline{G}$, assume that N is a stable set in G. We have $(w_N, N \setminus \{w_N\})_G = 0$ and hence $(w_N, V(H'))_H = 0$. In particular $(w_N, V(G - W))_G = 0$. Given $N' \in \mathbb{M}_{\max}(G) \setminus \{N\}$, we obtain $(w_N, N' \setminus \{w_{N'}\})_G = 0$. Since $N' \in \mathcal{M}(G)$, $(w_N, w_{N'})_G = 0$. It follows that $N_G(w_N) = \varnothing$. As at the end of the proof of Proposition 5.1, we conclude by $N = \{u \in V(G) : N_G(u) = \varnothing\}$. □

Lastly, we examine the non prime graphs G such that

$$\alpha_M(G) = \omega_M(G) = 1.$$

Proposition 5.2. *For every non prime graph G such that $|V(G)| \geq 4$ and $\alpha_M(G) = \omega_M(G) = 1$, we have $p(G) = 1$.*

Proof. Consider a minimal element $N_{\min}$ of $\mathcal{S}_{\geq 2}(G)$. By Proposition 2.3, $\Pi(G[N_{\min}]) \subseteq \mathcal{S}(G)$. By minimality of $N_{\min}$, $\Pi(G[N_{\min}]) = \Pi_1(G[N_{\min}])$. Thus $G[N_{\min}]$ and $G[N_{\min}]/\Pi(G[N_{\min}])$ are isomorphic by Proposition 2.2.(1). If $\lambda_G(N_{\min}) \in \{\bigcirc, \bullet\}$, then $N_{\min}$ is a clique or a stable set in G and there would be $N \in \mathbb{M}(G)$ such that $N \supseteq N_{\min}$. Therefore $\lambda_G(N_{\min}) = \sqsubset$ and $N_{\min} \in \mathbb{P}(G)$.

Let $a \notin V(G)$. For each $N \in \mathbb{P}(G)$, $G[N]$ is prime. By Lemma 3.3, $G[N]$ admits a prime 1-extension H_N to $N \cup \{a\}$. We consider the 1-extension H of G to $V(G) \cup \{a\}$ satisfying the following.

(1) For each $N \in \mathbb{P}(G)$, $H[N \cup \{a\}] = H_N$.
(2) Let $v \in I(G)$. There is $i \in \{0, 1\}$ such that $(v, N_{\min})_G = i$. We require that $(v, a)_H \neq i$.

We proceed as in the proof of Proposition 5.1, to show that $\mathcal{M}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$. To begin, we prove that $\mathcal{S}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$. Given $M \in \mathcal{S}_{\geq 2}(G)$, we have to verify that $a \not\leftrightarrow_H M$. Let N be a minimal element under inclusion of $\{N' \in \mathcal{S}_{\geq 2}(G) : N' \subseteq M\}$. We obtain that $\Pi(G[N]) = \Pi_1(G[N])$ so that $G[N]$ and $G[N]/\Pi(G[N])$ are isomorphic by Proposition 2.2.(1). If $\lambda_G(N) \in \{\bigcirc, \bullet\}$, then N is a clique or a stable set in G and there would be $N' \in \mathbb{M}(G)$ such that $N' \supseteq N$. Thus $\lambda_G(N) = \sqsubset$. We obtain that $G[N]$ is prime, that is, $N \in \mathbb{P}(G)$. Since $H[N \cup \{a\}]$ is prime, $a \not\leftrightarrow_H N$ and hence $a \not\leftrightarrow_H M$.

Now we prove that $\mathcal{M}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$. Let $M \in \mathcal{M}_{\geq 2}(G)$. Since $\mathcal{S}_{\geq 2}(G) \cap \mathcal{M}(H) = \varnothing$, assume that $M \notin \mathcal{S}_{\geq 2}(G)$. Set $Q = \{X \in \Pi(G[\widehat{M}]) : X \cap M \neq \varnothing\}$. We obtain that $M = \bigcup Q$, $|Q| \geq 2$ and $\lambda_G(\widehat{M}) \in \{\bigcirc, \bullet\}$. If $|\Pi_1(G[\widehat{M}])| \geq 2$, then we would have $\{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\} \in \mathbb{M}(G)$ by Proposition 4.5. Consequently $|\Pi_1(G[\widehat{M}])| \leq 1$ and there is $X \in Q \cap \Pi_{\geq 2}(G[\widehat{M}])$. By what precedes $a \not\leftrightarrow_H X$ and hence $a \not\leftrightarrow_H M$.

Lastly, we establish that H is prime. Let $M_H \in \mathcal{M}_{\geq 2}(H)$. As previously shown, $a \in M_H$. We show that $N \subseteq M_H$ for each $N \in \mathbb{P}(G)$. By Proposition 2.1.(1), $M_H \cap (N \cup \{a\}) \in \mathcal{M}(H[N \cup \{a\}])$. Since $H[N \cup \{a\}]$ is prime and $a \in M_H \cap (N \cup \{a\})$, we obtain either $(M_H \setminus \{a\}) \cap N = \varnothing$

or $N \subseteq M_H \setminus \{a\}$. Suppose for a contradiction that $(M_H \setminus \{a\}) \cap N = \varnothing$. By Proposition 2.1.(1), $M_H \setminus \{a\} \in \mathcal{M}(G)$. There is $i \in \{0,1\}$ such that $(M_H \setminus \{a\}, N)_G = i$ by Proposition 2.1.(3). Therefore $(a, N)_H = i$ which contradicts the fact that $H[N \cup \{a\}]$ is prime. It follows that $N \subseteq M_H$ for each $N \in \mathbb{P}(G)$. In particular $N_{\min} \subseteq M_H$. Let $v \in I(G)$. As $(v, N_{\min})_G \neq (v, a)_H$, $v \in M_H$. Consequently $M_H = V(H)$. □

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