Contributions to Discrete Mathematics

Volume 7, Number 2, Pages 30–43
ISSN 1715-0868

DOMINATION VALUE IN GRAPHS

EUNJEONG YI

Abstract. A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex not in D is adjacent to at least one vertex in D. A dominating set of G of minimum cardinality is called a $\gamma(G)$-set. For each vertex $v \in V(G)$, we define the *domination value* of v to be the number of $\gamma(G)$-sets to which v belongs. In this paper, we study some basic properties of the domination value function, thus initiating *a local study of domination* in graphs. Further, we characterize domination value for the Petersen graph, complete n-partite graphs, cycles, and paths.

1. Introduction

Let $G = (V(G), E(G))$ be a simple, undirected, and nontrivial graph with order $|V(G)|$ and size $|E(G)|$. For $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph of G induced by S. The *degree of a vertex* v in G, denoted by $\deg_G(v)$, is the number of edges that are incident to v in G; an *end-vertex* is a vertex of degree one, and a *support vertex* is a vertex that is adjacent to an end-vertex. We denote by $\Delta(G)$ *the maximum degree* of a graph G. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ of v is the set of all vertices adjacent to v in G, and the *closed neighborhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. A set $D \subseteq V(G)$ is a *dominating set* (DS) of G if for each $v \notin D$ there exists a $u \in D$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS in G; a DS of G of minimum cardinality is called a $\gamma(G)$-set. For earlier discussions on domination in graphs, see [1, 2, 4, 8, 11]. For a survey of domination in graphs, refer to [5, 6]. We generally follow [3] for notation and graph theory terminology. Throughout the paper, we denote by P_n, C_n, and K_n the path, the cycle, and the complete graph on n vertices, respectively.

In [12], Slater introduced the notion of the number of dominating sets of G, which he denoted by $\mathrm{HED}(G)$ in honor of Steve Hedetniemi; further, he also used $\#\gamma(G)$ to denote the number of $\gamma(G)$-sets. In this paper, we will use $\tau(G)$ to denote the total number of $\gamma(G)$-sets, and by $DM(G)$ the

Received by the editors October 26, 2010, and in revised form February 20, 2012.
2000 *Mathematics Subject Classification.* 05C69, 05C38.
Key words and phrases. Dominating set, domination value, a local study of domination, Petersen graph, complete n-partite graphs, cycles, paths.

collection of all $\gamma(G)$-sets. For each vertex $v \in V(G)$, we define the *domination value* of v, denoted by $DV_G(v)$, to be the number of $\gamma(G)$-sets to which v belongs; we often drop G when ambiguity is not a concern. See [9] for a discussion on *total domination value* in graphs. For a further work on domination value in graphs, see [13]. In this paper, we study some basic properties of the domination value function, thus initiating a *local study* of domination in graphs. When a real-world situation can be modeled by a graph, the locations (vertices) with high domination values are of interest. One can use domination value in selecting locations for fire departments or convenience stores, for example. Though numerous papers on domination have been published, no prior systematic local study of domination is known. However, in [10], Mynhardt characterized the vertices in a tree T whose domination value is 0 or $\tau(T)$. It should be noted that finding domination value of any given vertex in a given graph G can be an extremely difficult task, given the difficulty attendant to finding $\tau(G)$ or just $\gamma(G)$.

2. Basic properties of domination value: upper and lower bounds

In this section, we consider the lower and upper bounds of the domination value function for a fixed vertex v_0 and for $v \in N[v_0]$. Clearly, $0 \leq DV_G(v) \leq \tau(G)$ for any graph G and for any vertex $v \in V(G)$. We will say the bound is sharp if equality is obtained for a graph of some order in an inequality. We first make the following observations.

Observation 2.1. $\sum_{v \in V(G)} DV_G(v) = \tau(G) \cdot \gamma(G)$.

Observation 2.2. *If there is an isomorphism of graphs carrying a vertex v in G to a vertex v' in G', then $DV_G(v) = DV_{G'}(v')$.*

Examples of graphs that admit automorphisms are cycles, paths, and the Petersen graph. The Pertersen graph, which is often used as a counter-example for conjectures, is vertex-transitive ([7], p.27). Let $\mathcal{P}$ denote the Petersen graph with labeling as in Figure 1.

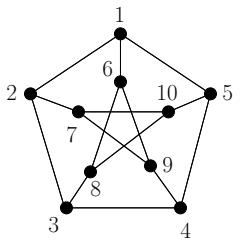


Figure 1. The Petersen graph

It is easy to check that $\gamma(\mathcal{P}) = 3$. We will show that $DV(v) = 3$ for each $v \in V(\mathcal{P})$. Since $\mathcal{P}$ is vertex-transitive, it suffices to compute $DV_{\mathcal{P}}(1)$. For the $\gamma(\mathcal{P})$-set Γ containing the vertex 1, one can easily check that no vertex in $N(1)$ belongs to Γ. Further, notice that no three vertices from $\{1, 2, 3, 4, 5\}$ form a $\gamma(\mathcal{P})$-set. Keeping these two conditions in mind, one can readily verify that the $\gamma(\mathcal{P})$-sets containing the vertex 1 are $\{1, 3, 7\}$, $\{1, 4, 10\}$, and $\{1, 8, 9\}$, and thus $DV(1) = 3 = DV(v)$ for each $v \in V(\mathcal{P})$.

Observation 2.3. *Let G be the disjoint union of two graphs G_1 and G_2. Then $\gamma(G) = \gamma(G_1) + \gamma(G_2)$ and $\tau(G) = \tau(G_1) \cdot \tau(G_2)$. For $v \in V(G_1)$, $DV_G(v) = DV_{G_1}(v) \cdot \tau(G_2)$.*

Proposition 2.4. *For a fixed $v_0 \in V(G)$, we have*

$$\tau(G) \leq \sum_{v \in N[v_0]} DV_G(v) \leq \tau(G) \cdot \gamma(G),$$

and both bounds are sharp.

Proof. The upper bound follows from Observation 2.1. For the lower bound, note that every $\gamma(G)$-set Γ must contain a vertex in $N[v_0]$: otherwise Γ fails to dominate v_0.

For sharpness of the lower bound, take v_0 to be an end-vertex of P_{3k} for $k \geq 1$ (see Theorem 5.1 and Corollary 5.2). For sharpness of the upper bound, take as v_0 the central vertex of (A) in Figure 2. □

Proposition 2.5. *For any $v_0 \in V(G)$,*

$$\sum_{v \in N[v_0]} DV_G(v) \leq \tau(G) \cdot (1 + \deg_G(v_0)),$$

and the bound is sharp.

Proof. For each $v \in N[v_0]$, $DV_G(v) \leq \tau(G)$ and $|N[v_0]| = 1 + \deg_G(v_0)$. Thus,

$$\sum_{v \in N[v_0]} DV(v) \leq \sum_{v \in N[v_0]} \tau(G) = \tau(G) \sum_{v \in N[v_0]} 1 = \tau(G)(1 + \deg_G(v_0)).$$

The upper bound is achieved for a graph of order n for any $n \geq 1$. Let G_n be a graph on n vertices containing an isolated vertex. To see the sharpness of the upper bound, take as v_0 one of the isolates vertices, then the upper bound follows by Observation 2.3 and $\deg_G(v_0) = 0$. □

We will compare two examples, where each example attains the upper bound of Proposition 2.4 or Proposition 2.5, but not both. Let v_0 be the central vertex of degree 3, which is not a support vertex as in the graph (A) of Figure 2. Then $\sum_{v \in N[v_0]} DV_G(v) = 3$. Note that $\tau(G) = 1$, $\gamma(G) = 3$, and $\deg_G(v_0) = 3$. Proposition 2.4 yields the upper bound $\tau(G) \cdot \gamma(G) = 1 \cdot 3 = 3$, which is sharp. But, the upper bound provided by Proposition 2.5 is $\tau(G) \cdot (1 + \deg_G(v_0)) = 1 \cdot (1 + 3) = 4$, which is not sharp in this case.

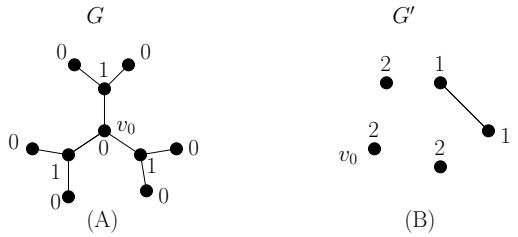


FIGURE 2. Examples of local domination values and their upper bounds

Now, let v_0 be an isolated vertex as labeled in the graph (B) of Figure 2. Then $\sum_{v\in N[v_0]} DV_{G'}(v) = 2$. Note that $\tau(G') = 2$, $\gamma(G') = 4$, and $\deg_{G'}(v_0) = 0$. Proposition 2.5 yields the upper bound $\tau(G') \cdot (1 + \deg_{G'}(v_0)) = 2 \cdot (1 + 0) = 2$, which is sharp. But, the upper bound provided by Proposition 2.4 is $\tau(G') \cdot \gamma(G') = 2 \cdot 4 = 8$, which is not sharp in this case.

Proposition 2.6. *Let H be a subgraph of G with $V(H) = V(G)$. If $\gamma(H) = \gamma(G)$, then $\tau(H) \leq \tau(G)$.*

Proof. By the first assumption, every DS for H is a DS for G. By $\gamma(H) = \gamma(G)$, it's guaranteed that every DS of minimum cardinality for H is also a DS *of minimum cardinality for G.* □

The *complement* $\overline{G} = (V(\overline{G}), E(\overline{G}))$ of a graph G is the graph such that $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. We recall the following.

Theorem 2.7. *Let G be any graph of order n. Then*

(i) ([8], *Jaegar and Payan*) $\gamma(G) + \gamma(\overline{G}) \leq n + 1$, *and*
(ii) ([2], *p.304*) $\gamma(G) \leq n - \Delta(G)$.

Proposition 2.8. *Let G be a graph on $n = 2m \geq 4$ vertices. If G or $\overline{G}$ is mK_2, then*

$$DV_G(v) + DV_{\overline{G}}(v) = n - 1 + 2^{n/2-1}.$$

Proof. Without loss of generality, assume $G = mK_2$ and label the vertices of G by $1, 2, \ldots, 2m$. Further assume that the vertex $2k - 1$ is adjacent to the vertex $2k$, where $1 \leq k \leq m$. Then $DV_G(1) = 2^{m-1}$, which consists of the vertex 1 and one vertex from each path K_2. By Observation 2.2 and Observation 2.3, $DV_G(v) = 2^{m-1} = 2^{n/2-1}$ for any $v \in V(G)$.

Now, consider $\overline{G}$ and the vertex labeled 1 for ease of notation. Since $\Delta(G) = n - 2$, $\gamma(\overline{G}) > 1$. Noting that $\{1, \alpha\}$ as α ranges from 2 to $2m$ enumerates all dominating sets of $\overline{G}$ containing the vertex 1, we have $\gamma(\overline{G}) = 2$ and $DV_{\overline{G}}(1) = n - 1$. By Observation 2.2, $DV_{\overline{G}}(v) = n - 1$ holds for any $v \in V(\overline{G})$. Therefore, $DV_G(v) + DV_{\overline{G}}(v) = n - 1 + 2^{n/2-1}$. □

Next we consider domination value of a graph G when $\Delta(G)$ is given.

Observation 2.9. *Let G be a graph of order $n \geq 2$ such that $\Delta(G) = n-1$. Then $\gamma(G) = 1$ and $DV(v) \leq 1$ for any $v \in V(G)$. Equality holds if and only if $\deg_G(v) = n-1$.*

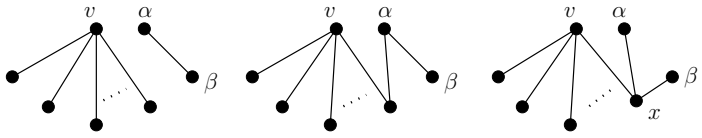
Proposition 2.10. *Let G be a graph of order $n \geq 3$ such that $\Delta(G) = n-2$. Then $\gamma(G) = 2$ and $DV(v) \leq n-1$ for any $v \in V(G)$. Further, if $\deg(v) = n-2$, then $DV(v) = |N[w]|$ where $vw \notin E(G)$.*

Proof. Let $\deg_G(v) = \Delta(G) = n-2$, then $\gamma(G) > 1$ and there is only one vertex w such that $vw \notin E(G)$. Clearly, $\{v, w\}$ is a $\gamma(G)$-set; so $\gamma(G) = 2$. Noticing that v dominates $N[v]$, we see that the number of $\gamma(G)$-sets containing v is $|N[w]|$; i.e., $DV(v) = |N[w]| \leq n-2$. □

Theorem 2.11. *Let G be a graph of order $n \geq 4$ and $\Delta(G) = n-3$. Fix a vertex v with $\deg_G(v) = \Delta(G)$.*

(i) *If G is disconnected, then $\gamma(G) = 2$ with $DV(v) = 2$ or $\gamma(G) = 3$ with $DV(v) \leq n-3$.*
(ii) *If G is connected, then $\gamma(G) = 2$ with $DV(v) \leq n-2$ or $\gamma(G) = 3$ with $DV(v) \leq ((n-1)/2)^2$.*

Proof. Since $\deg_G(v) = \Delta(G) = n-3$, there are two vertices, say α and β, such that $v\alpha, v\beta \notin E(G)$. We consider four cases.



(A) Case 1 (B) Case 2 (C) Case 3

FIGURE 3. Cases 1, 2, and 3 when $\Delta(G) = n-3$

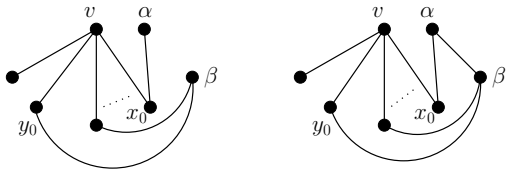
Case 1: Neither α nor β is adjacent to any vertex in $N[v]$.
Let $G' = \langle V(G) - \{\alpha, \beta\}\rangle$. Then $\deg_{G'}(v) = n-3$ with $|V(G')| = n-2$. By Observation 2.9, $\gamma(G') = 1$ and $DV_{G'}(v) = 1$. First suppose α and β are isolated vertices in G. (Consider (A) of Figure 3 with the edge $\alpha\beta$ being removed.) Observation 2.3, together with $\gamma(\langle\{\alpha, \beta\}\rangle) = 2$ and $\tau(\langle\{\alpha, \beta\}\rangle) = 1$, yields $\gamma(G) = 3$ and $DV_G(v) = 1$. Next assume that G has no isolated vertex, then $\alpha\beta \in E(G)$ (see (A) of Figure 3). Observation 2.3, together with $\gamma(\langle\{\alpha, \beta\}\rangle) = 1$ and $\tau(\langle\{\alpha, \beta\}\rangle) = 2$, yields $\gamma(G) = 2$ and $DV_G(v) = 2$.

Case 2: Exactly one of α and β is adjacent to a vertex in $N(v)$.
Without loss of generality, assume that α is adjacent to a vertex in $N(v)$. First suppose that G is not connected. Then $\alpha\beta \notin E(G)$. (Consider (B) of Figure 3 with the edge $\alpha\beta$ being removed.) Let $G' = \langle V(G) - \{\beta\}\rangle$. Then $\deg_{G'}(v) = n-3$ with $|V(G')| = n-1$. By Proposition 2.10,

$\gamma(G') = 2$ and $DV_{G'}(v) = |N[\alpha]| \leq n-3$. Observation 2.3, together with $\gamma(\langle\{\beta\}\rangle) = 1$ and $\tau(\langle\{\beta\}\rangle) = 1$, yields $\gamma(G) = 3$ and $DV_G(v) = |N[\alpha]| \leq n-3$. Next suppose that G is connected. Then $\alpha\beta \in E(G)$ and α is a support vertex of G. (See (B) of Figure 3.) Since $\Delta(G) < n-1$, $\gamma(G) > 1$. Since $\{v, \alpha\}$ is a $\gamma(G)$-set, $\gamma(G) = 2$. Noting that v dominates $V(G) - \{\alpha, \beta\}$, the number of $\gamma(G)$-sets containing v equals the number of vertices in G that dominates both α and β. Thus $DV_G(v) = 2$.

Case 3: There exists a vertex in $N(v)$, say x, that is adjacent to both α and β.

Notice that $n \geq 6$ in this case, since $vx, \alpha x, \beta x \in E(G)$ and $\deg_G(v) = \Delta(G)$ (see (C) of Figure 3). Since $\{v, x\}$ is a $\gamma(G)$ set, $\gamma(G) = 2$. If $\alpha\beta \notin E(G)$, then $DV(v) = |N[\alpha] \cap N[\beta]| \leq n-3$. If $\alpha\beta \in E(G)$, then $|N[\alpha] \cap N[\beta]| \leq n-4$ since $\Delta(G) = n-3$. Noting both $\{v, \alpha\}$ and $\{v, \beta\}$ are $\gamma(G)$-sets, we have $DV(v) = 2 + |N[\alpha] \cap N[\beta]| \leq n-2$.



(A) Subcase 4.1 (B) Subcase 4.2

FIGURE 4. Subcases 4.1 and 4.2 when $\Delta(G) = n-3$

Case 4: There exist vertices in $N(v)$ that are adjacent to α and β, but no vertex in $N(v)$ is adjacent to both α and β.

Let $x_0 \in N(v) \cap N(\alpha)$ and $y_0 \in N(v) \cap N(\beta)$. We consider two subcases.

Subcase 4.1. $\alpha\beta \notin E(G)$ (see (A) of Figure 4).

First, assume $\gamma(G) = 2$. This is possible when $\{x_0, y_0\}$ is a $\gamma(G)$-set satisfying $N[x_0] \cup N[y_0] = V(G)$. Notice that there is no $\gamma(G)$-set containing v when $\gamma(G) = 2$ since there is no vertex in G that is adjacent to both α and β. Thus $DV(v) = 0$. Second, assume $\gamma(G) > 2$. Since $\{v, \alpha, \beta\}$ is a $\gamma(G)$-set, $\gamma(G) = 3$. Noticing that every $\gamma(G)$-set contains a vertex in $N[\alpha]$ and a vertex in $N[\beta]$ and that $N[\alpha] \cap N[\beta] = \emptyset$, we see

$$DV(v) = |N[\alpha]| \cdot |N[\beta]| \leq \left(\frac{|N[\alpha]| + |N[\beta]|}{2}\right)^2 \leq \left(\frac{n-1}{2}\right)^2,$$

where the first inequality is the arithmetic-geometric mean inequality (i.e., $(a+b)/2 \geq \sqrt{ab}$ for $a, b \geq 0$).

Subcase 4.2. $\alpha\beta \in E(G)$ (see (B) of Figure 4).

Since $\{v, \alpha\}$ is a $\gamma(G)$-set, $\gamma(G) = 2$. Since there is no vertex in $N(v)$ that is adjacent to both α and β, there are only two $\gamma(G)$-sets containing v, i.e., $\{v, \alpha\}$ and $\{v, \beta\}$. Thus $DV(v) = 2$.

□

Remark. In the proof of Theorem 2.11, we observe that one may have $DV(v) = 0$ even though $\deg_G(v) = \Delta(G) \leq n-3$. See Figure 5 for a graph of order n, $\deg_G(v) = \Delta(G) = n-3$, $\gamma(G) = 2$, and $DV(v) = 0$.

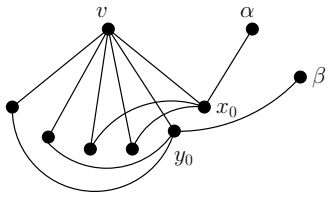


FIGURE 5. A graph of order 9, $\deg_G(v) = \Delta(G) = 6$, $DV(v) = 0$ with a unique γ-set $\{x_0, y_0\}$

3. Domination value on complete n-partite graphs

For a *complete* n-partite graph G, let $V(G)$ be partitioned into n-partite sets $V_1, V_2, \ldots, V_n$, and let $a_i = |V_i| \geq 1$ for each $1 \leq i \leq n$, where $n \geq 2$.

Proposition 3.1. *Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph with $a_i \geq 2$ for each i $(1 \leq i \leq n)$. Then*

$$\tau(G) = \frac{1}{2}\left[\left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2\right] \text{ and } DV(v) = \left(\sum_{i=1}^{n} a_i\right) - a_j \text{ if } v \in V_j.$$

Proof. Since $\Delta(G) < |V(G)| - 1$, $\gamma(G) > 1$. Any two vertices from different partite sets form a $\gamma(G)$-set, so $\gamma(G) = 2$. If $v \in V_j$, then

$$DV(v) = \deg_G(v) = \left(\sum_{i=1}^{n} a_i\right) - a_j. \tag{3.1}$$

From Observation 2.1 and (3.1), we have

$$\sum_{j=1}^{n}\sum_{v\in V_j} DV(v) = 2\tau(G) \iff \sum_{j=1}^{n}\left(a_j\sum_{i=1}^{n} a_i - a_j^2\right) = 2\tau(G)$$

$$\iff \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} a_j\right) - \sum_{j=1}^{n} a_j^2 = 2\tau(G),$$

and thus the formula for $\tau(G)$ follows. □

Proposition 3.2. *Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph such that $a_i = 1$ for some i, say $a_j = 1$ for $j = 1, 2, \ldots, k$, where $1 \leq k \leq n$. Then $\tau(G) = k$, and*

$$DV(v) = \begin{cases} 1 & \text{if } v \in V_j \ (1 \leq j \leq k), \\ 0 & \text{if } v \in V_j \ (k+1 \leq j \leq n). \end{cases}$$

Proof. Since $\Delta(G) = |V(G)| - 1$ by Observation 2.9, $\gamma(G) = 1$ and $DV(v)$ follows. By Observation 2.1, together with $\gamma(G) = 1$, we have $\tau(G) = \sum_{v \in V(G)} DV_G(v) = k$. □

If $a_i = 1$ for each i $(1 \leq i \leq n)$, then $G = K_n$. As an immediate consequence of Proposition 3.2, we have the following.

Corollary 3.3. *If $G = K_n$ $(n \geq 1)$, then $\tau(G) = n$ and $DV(v) = 1$ for each $v \in V(K_n)$.*

If $n = 2$, then $G = K_{a_1,a_2}$ is a complete bi-partite graph.

Corollary 3.4. *If $G = K_{a_1,a_2}$, then*

$$\tau(G) = \begin{cases} a_1 \cdot a_2 & \text{if } a_1, a_2 \geq 2, \\ 2 & \text{if } a_1 = a_2 = 1, \\ 1 & \text{if } \{a_1, a_2\} = \{1, x\}, \text{ where } x > 1. \end{cases}$$

If $a_1, a_2 \geq 2$, then

$$DV(v) = \begin{cases} a_2 & \text{if } v \in V_1, \\ a_1 & \text{if } v \in V_2. \end{cases}$$

If $a_1 = a_2 = 1$, $DV(v) = 1$ for any v in $K_{1,1}$. If $\{a_1, a_2\} = \{1, x\}$ with $x > 1$, say $a_1 = 1$ and $a_2 = x$, then

$$DV(v) = \begin{cases} 1 & \text{if } v \in V_1, \\ 0 & \text{if } v \in V_2. \end{cases}$$

4. Domination value on cycles

Let the vertices of the cycle C_n be labeled 1 through n consecutively in counterclockwise order, where $n \geq 3$. Observe that the domination value is constant on the vertices of C_n, for each n, by vertex-transitivity. Recall that $\gamma(C_n) = \lceil n/3 \rceil$ for $n \geq 3$ (see [3], p.364).

Example.

(1) $DM(C_4) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ since $\gamma(C_4) = 2$; so $\tau(C_4) = 6$ and $DV(i) = 3$ for each $i \in V(C_4)$.
(2) $\gamma(C_6) = 2$, $DM(C_6) = \{\{1,4\},\{2,5\},\{3,6\}\}$; so $\tau(C_6) = 3$ and $DV(i) = 1$ for each $i \in V(C_6)$.

Theorem 4.1. *For $n \geq 3$,*

$$\tau(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 3), \\ n\left(1 + \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor\right) & \text{if } n \equiv 1 \,(\mathrm{mod}\, 3), \\ n & \text{if } n \equiv 2 \,(\mathrm{mod}\, 3). \end{cases}$$

Proof. First, let $n = 3k$, where $k \geq 1$. Here $\gamma(C_n) = k$; a $\gamma(C_n)$-set Γ comprises k K_1's and Γ is fixed by the choice of the first K_1. There exists exactly one $\gamma(C_n)$-set containing the vertex 1, and there are two $\gamma(C_n)$-sets omitting the vertex 1 such as Γ containing the vertex 2 and Γ containing the vertex n. Thus $\tau(C_n) = 3$.

Second, let $n = 3k + 1$, where $k \geq 1$. Here $\gamma(C_n) = k + 1$; a $\gamma(C_n)$-set Γ is either comprised of $(k-1)$ K_1's and one K_2, or of $(k+1)$ K_1's.

Case 1: $\langle \Gamma \rangle \cong (k-1)K_1 \cup K_2$.

Note that Γ is fixed by the choice of the single K_2. Choosing a K_2 is the same as choosing its initial vertex in the counterclockwise order. Thus $\tau = 3k + 1$

Case 2: $\langle \Gamma \rangle \cong (k+1)K_1$.

Note that, since each K_1 dominates three vertices, there are exactly two vertices, say x and y, each of whom is adjacent to two distinct K_1's in Γ. And Γ is fixed by the placements of x and y. There are $n = 3k+1$ ways of choosing x. Consider the P_{3k-2} (a sequence of $3k-2$ slots) obtained as a result of cutting from C_n the P_3 centered about x. Vertex y may be placed in the first slot of any of the $\lceil (3k-2)/3 \rceil = k$ subintervals of the P_{3k-2}. As the order of selecting the two vertices x and y is immaterial, $\tau = k(3k+1)/2$.

Summing over the two disjoint cases, we get

$$\tau(C_n) = (3k+1) + \frac{(3k+1)k}{2} = (3k+1)\left(1 + \frac{k}{2}\right) = n\left(1 + \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor\right).$$

Finally, let $n = 3k + 2$, where $k \geq 1$. Here $\gamma(C_n) = k + 1$; a $\gamma(C_n)$-set Γ comprises of only K_1's and is fixed by the placement of the only vertex which is adjacent to two distinct K_1's in Γ. Thus $\tau(C_n) = n$. □

Corollary 4.2. *Let $v \in V(C_n)$, where $n \geq 3$. Then*

$$DV(v) = \begin{cases} 1 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 3), \\ \frac{1}{2}\left\lceil \frac{n}{3} \right\rceil \left(1 + \left\lceil \frac{n}{3} \right\rceil\right) & \text{if } n \equiv 1 \,(\mathrm{mod}\, 3), \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 2 \,(\mathrm{mod}\, 3). \end{cases}$$

Proof. It follows by Observation 2.1, Observation 2.2, and Theorem 4.1. □

5. Domination value on paths

Let the vertices of the path P_n be labeled 1 through n consecutively. Recall that $\gamma(P_n) = \lceil n/3 \rceil$ for $n \geq 2$.

Example.

(1) $\gamma(P_4) = 2$, $DM(P_4) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$; so $\tau(P_4) = 4$ and $DV(i) = 2$ for each $i \in V(P_4)$.

(2) $\gamma(P_5) = 2$, $DM(P_5) = \{\{1,4\},\{2,4\},\{2,5\}\}$; so $\tau(P_5) = 3$, and

$$DV(i) = \begin{cases} 1 & \text{if } i = 1, 5, \\ 2 & \text{if } i = 2, 4, \\ 0 & \text{if } i = 3. \end{cases}$$

Remark. Since $P_n \subset C_n$ with the same vertex set, by Proposition 2.6, we have $\tau(P_n) \leq \tau(C_n)$ for $n \geq 3$, as one can verify from the theorem below.

Theorem 5.1. *For $n \geq 2$,*

$$\tau(P_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ n + \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) & \text{if } n \equiv 1 \pmod 3, \\ 2 + \left\lfloor \frac{n}{3} \right\rfloor & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. First, let $n = 3k$, where $k \geq 1$. Then $\gamma(P_n) = k$ and a $\gamma(P_n)$-set Γ comprises k K_1's. In this case, each vertex in Γ dominates three vertices, and no vertex of P_n is dominated by more than one vertex. Thus none of the end-vertices of P_n belongs to any Γ, which contains and is fixed by the vertex 2; hence $\tau(P_n) = 1$.

Second, let $n = 3k + 1$, where $k \geq 1$. Here $\gamma(P_n) = k + 1$; a $\gamma(P_n)$-set Γ is either comprised of $(k-1)$ K_1's and one K_2, or of $(k+1)$ K_1's.

Case 1: $\langle \Gamma \rangle \cong (k-1)K_1 \cup K_2$, where $k \geq 1$.

Note that Γ is fixed by the placement of the single K_2, and none of the end-vertices belong to any Γ, as each component with cardinality c in $\langle \Gamma \rangle$ dominates $c + 2$ vertices. Initial vertex of K_2 may be placed in one of the $n \equiv 2 \pmod 3$ slots. Thus $\tau = k$.

Case 2: $\langle \Gamma \rangle \cong (k+1)K_1$, where $k \geq 1$.

A Γ containing both end-vertices of the path is unique (no vertex is doubly dominated). The number of Γ containing exactly one of the end-vertices (one doubly dominated vertex) is $2\binom{k}{1} = 2k$. The number of Γ containing none of the end-vertices (two doubly dominated vertices) is $\binom{k}{2} = k(k-1)/2$. Thus $\tau = 1 + 2k + k(k-1)/2$.

Summing over the two disjoint cases, we get

$$\begin{aligned} \tau(P_n) &= k + \left(1 + 2k + \frac{k(k-1)}{2}\right) \\ &= 3k + 1 + \frac{k(k-1)}{2} \\ &= n + \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right). \end{aligned}$$

Finally, let $n = 3k + 2$, where $k \geq 0$. Here $\gamma(P_n) = k + 1$, and $\gamma(P_n)$-set Γ comprises of $(k+1)$ K_1's. Note that there is no Γ containing both end-vertices of P_n. The number of Γ's containing exactly one of the end-vertices (no doubly dominated vertex) of the path is two. The number of Γ's containing neither of the end-vertices (one doubly dominated vertex) is k. Summing the two disjoint cases, we have $\tau(P_n) = 2 + k = 2 + \lfloor n/3 \rfloor$. □

For the domination value of a vertex on P_n, note that $DV(v) = DV(n + 1 - v)$ for $1 \leq v \leq n$ as P_n admits the obvious automorphism carrying v to

$n+1-v$. More precisely, we have the classification result which follows. First, as an immediate consequence of Theorem 5.1, we have the following result.

Corollary 5.2. *Let $v \in V(P_{3k})$, where $k \geq 1$. Then*

$$DV(v) = \begin{cases} 0 & \text{if } v \equiv 0, 1 \,(\mathrm{mod}\, 3), \\ 1 & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases}$$

Proposition 5.3. *Let $v \in V(P_{3k+1})$, where $k \geq 1$. Write $v = 3q + r$, where q and r are non-negative integers such that $0 \leq r < 3$. Then, noting $\tau(P_{3k+1}) = (k^2+5k+2)/2$, we have*

$$DV(v) = \begin{cases} \dfrac{1}{2}q(q+3) & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ (q+1)(k-q+1) & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ \dfrac{1}{2}(k-q)(k-q+3) & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases}$$

Proof. Let Γ be a $\gamma(P_{3k+1})$-set for $k \geq 1$. We consider two cases.

Case 1: $\langle \Gamma \rangle \cong (k-1)K_1 \cup K_2$, where $k \geq 1$.

Denote by $DV^1(v)$ the number of such Γ's containing v. Noting $\tau = k$ in this case, we have

$$DV^1(v) = \begin{cases} q & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ 0 & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ k-q & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.1}$$

We prove by induction on k. One can easily check (5.1) for $k = 1$. Assume that (5.1) holds for $G = P_{3k+1}$ and consider $G' = P_{3k+4}$. First, notice that each Γ of the k $\gamma(P_{3k+1})$-sets of G induces a $\gamma(P_{3k+4})$-set $\Gamma' = \Gamma \cup \{3k+3\}$ of G'. Additionally, G' has the $\gamma(P_{3k+4})$-set Γ^* that contains and is determined by $\{3k+2, 3k+3\}$, which does not come from any $\gamma(P_{3k+1})$-set of G. The presence of Γ^* implies that $DV^1_{G'}(v) = DV^1_G(v)+1$ for $v \equiv 2 \,(\mathrm{mod}\, 3)$, where $v \leq 3k+1$. Clearly, $DV^1_{G'}(3k+2) = 1$, $DV^1_{G'}(3k+3) = k+1$, and $DV^1_{G'}(3k+4) = 0$.

Case 2: $\langle \Gamma \rangle \cong (k+1)K_1$, where $k \geq 1$.

Denote by $DV^2(v)$ the number of such Γ's containing v. First, suppose both end-vertices belong to the unique Γ and denote by $DV^{2,1}(v)$ the number of such Γ's containing v. Then we have

$$DV^{2,1}(v) = \begin{cases} 0 & \text{if } v \equiv 0, 2 \,(\mathrm{mod}\, 3), \\ 1 & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.2}$$

Second, suppose exactly one end-vertex belongs to each Γ; denote by $DV^{2,2}(v)$ the number of such Γ's containing v. Then, noting $\tau = 2k$ in

this case, we have

$$DV^{2,2}(v) = \begin{cases} q & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ k & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ k-q & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.3}$$

We prove by induction on k. One can easily check (5.3) for $k = 1$. Assume that (5.3) holds for $G = P_{3k+1}$ and consider $G' = P_{3k+4}$. First, notice that each Γ of the k $\gamma(P_{3k+1})$-sets of G containing the left end-vertex 1 induces a $\gamma(P_{3k+4})$-set $\Gamma' = \Gamma \cup \{3k+3\}$ of G'. Second, each Γ of k $\gamma(P_{3k+1})$-sets of G containing the right end-vertex $3k+1$ induces a $\gamma(P_{3k+4})$-set $\Gamma' = \Gamma \cup \{3k+4\}$ of G'. Third, a $\gamma(P_{3k+1})$-set Γ of G containing 1 and $3k+1$ (both left and right end-vertices of G) induces a $\gamma(P_{3k+4})$-set $\Gamma^{*1} = \Gamma \cup \{3k+3\}$ of G' (making $3k+2$ the only doubly dominated vertex in G'). Additionally, $\Gamma^{*2} = \{v \in V(P_{3k+1}) \mid v \equiv 2 \,(\mathrm{mod}\, 3)\} \cup \{3k+2, 3k+4\}$ is a $\gamma(P_{3k+4})$-set for G', which does not come from any $\gamma(P_{3k+1})$-set of G. The presence of Γ^{*1} and Γ^{*2} imply that

$$DV^{2,2}_{G'}(v) = \begin{cases} DV^{2,2}_{G}(v) & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ \\ DV^{2,2}_{G}(v) + 1 & \text{if } v \equiv 1, 2 \,(\mathrm{mod}\, 3), \end{cases}$$

for $v \leq 3k+1$. Clearly, $DV^{2,2}_{G'}(3k+2) = 1$, $DV^{2,2}_{G'}(3k+3) = k+1$, and $DV^{2,2}_{G'}(3k+4) = k+1$.

Third, suppose no end-vertex belongs to Γ; denote by $DV^{2,3}(v)$ the number of such Γ's containing v. Then, noting $\tau = \binom{k}{2}$ in this case and setting $\binom{a}{b} = 0$ when $a < b$, we have

$$DV^{2,3}(v) = \begin{cases} \frac{1}{2}(q-1)q & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ q(k-q) & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ \frac{1}{2}(k-q-1)(k-q) & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.4}$$

Again, we prove by induction on k. Since $DV^{2,3}(v) = 0$ for each $v \in V(P_4)$, we consider $k \geq 2$. One can easily check (5.4) for the base case, $k = 2$. Assume that (5.4) holds for $G = P_{3k+1}$ and consider $G' = P_{3k+4}$, where $k \geq 2$. First, notice that each Γ of the $\binom{k}{2}$ $\gamma(P_{3k+1})$-sets of G containing neither end-vertices of G induces a $\gamma(P_{3k+4})$-set $\Gamma' = \Gamma \cup \{3k+3\}$ of G'. Additionally, each Γ_r of the k $\gamma(P_{3k+1})$-sets of G containing the right-end vertex $3k+1$ of G induces a $\gamma(P_{3k+4})$-set $\Gamma'_r = \Gamma_r \cup \{3k+3\}$ of G' (making $3k+2$ one of the two doubly-dominated vertices in G'): If we denote by $DV^r_G(v)$ the number of such Γ_r's containing v in G, then one can readily check

$$DV^r_G(v) = \begin{cases} 0 & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ q & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ k-q & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3), \end{cases}$$

again by induction on k. Thus, the presence of Γ'_r implies $DV^{2,3}_{G'}(v) = DV^{2,3}_{G}(v) + DV^{r}_{G}(v)$ for $v \leq 3k+1$. Clearly, $DV^{2,3}_{G'}(3k+2) = 0 = DV^{2,3}_{G'}(3k+4)$ and $DV^{2,3}_{G'}(3k+3) = \binom{k}{2} + k = k(k+1)/2$.

Summing over the three disjoint cases (5.2), (5.3), and (5.4) for $\langle \Gamma \rangle \cong (k+1)K_1$, we have

$$DV^2(v) = \begin{cases} q + \dfrac{1}{2}(q-1)q & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ 1 + k + q(k-q) & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ k - q + \dfrac{1}{2}(k-q-1)(k-q) & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.5}$$

Now, by summing over (5.1) and (5.5), i.e., $DV(v) = DV^1(v) + DV^2(v)$, we obtain the formula claimed in this proposition. □

Proposition 5.4. *Let $v \in V(P_{3k+2})$, where $k \geq 0$. Write $v = 3q + r$, where q and r are non-negative integers such that $0 \leq r < 3$. Then, noting $\tau(P_{3k+2}) = k+2$, we have*

$$DV(v) = \begin{cases} 0 & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ 1 + q & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ k + 1 - q & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases}$$

Proof. Let Γ be a $\gamma(P_{3k+2})$-set for $k \geq 0$. Then $\langle \Gamma \rangle \cong (k+1)K_1$. Note that no Γ contains both end-vertices of P_{3k+2}.

First, suppose Γ contains exactly one end-vertex, and denote by $DV'(v)$ the number of such Γ's containing v. Noting $\tau = 2$ in this case, for $v \in V(P_{3k+2})$, we have

$$DV'(v) = \begin{cases} 0 & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ 1 & \text{if } v \equiv 1, 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.6}$$

Next, suppose Γ contains no end-vertices (thus $k \geq 1$), and denote by $DV''(v)$ the number of such Γ's containing v. Noting $\tau = k$ in this case, we have

$$DV''(v) = \begin{cases} 0 & \text{if } v \equiv 0 \,(\mathrm{mod}\, 3), \\ q & \text{if } v \equiv 1 \,(\mathrm{mod}\, 3), \\ k - q & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{5.7}$$

We prove by induction on k. One can easily check (5.7) for the base case, $k = 1$. Assume that (5.7) holds for $G = P_{3k+2}$ and consider $G' = P_{3k+5}$. First, notice that each Γ of the k $\gamma(P_{3k+2})$-sets containing neither end-vertex of G induces a $\gamma(P_{3k+5})$-set $\Gamma' = \Gamma \cup \{3k+4\}$. Additionally, the only $\gamma(P_{3k+2})$-set Γ of G containing the right-end vertex $3k+2$ of G induces

a $\gamma(P_{3k+5})$-set $\Gamma^\star = \Gamma \cup \{3k+4\}$ of G' (making $3k+3$ the only doubly-dominated vertex). The presence of $\Gamma^\star$ implies that

$$DV''_{G'}(v) = \begin{cases} DV''_G(v) & \text{if } v \equiv 0, 1 \,(\mathrm{mod}\, 3), \\ DV''_G(v) + 1 & \text{if } v \equiv 2 \,(\mathrm{mod}\, 3), \end{cases}$$

for $v \leq 3k+2$. Clearly, $DV''_{G'}(3k+3) = 0 = DV''_{G'}(3k+5)$ and $DV''_{G'}(3k+4) = k+1$.

Now, by summing over the two disjoint cases (5.6) and (5.7), i.e., $DV(v) = DV'(v) + DV''(v)$, we obtain the formula claimed in this proposition. □

Acknowledgements

The author thanks Cong X. Kang for suggesting the concept of *domination value* and his valuable comments and suggestions. The author also thanks the referee for a couple of helpful comments.

References

1. C. Berge, *Theory of graphs and its applications*, Methuen, London, 1962.
2. ———, *Graphs and hypergraphs*, North-Holland, Amsterdam, 1973.
3. G. Chartrand and P. Zhang, *Introduction to graph theory*, McGraw-Hill, Kalamazoo, MI, 2004.
4. E. J. Cockayne and S. T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977), 247–261.
5. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (editors), *Domination in graphs: Advanced topics*, Marcel Dekker, New York, 1998.
6. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.
7. D. A. Holton and J. Sheehan, *The Petersen graph*, Cambridge University Press, 1993.
8. F. Jaegar and C. Payan, *Relations du type Nordhaus-Gaddum pour le numbre d'absorption d'un graphe simple*, C. R. Acad. Sci. Paris, Series A **274** (1972), 728–730.
9. C. X. Kang, *Total domination value in graphs*, Util. Math., to appear.
10. C. M. Mynhardt, *Vertices contained in every minimum dominating set of a tree*, J. Graph Theory **31** (1999), 163–177.
11. O. Ore, *Theory of graphs*, vol. 38, Amer. Math. Soc. Colloq. Publ., Providence, 1962.
12. P. J. Slater, *The Hedetniemi number of a graph*, Congr. Numer. **139** (1999), 65–75.
13. E. Yi, *Domination value in $P_2 \Box P_n$ and $P_2 \Box C_n$*, J. Combin. Math. Combin. Comput. **82** (2012), 59–75.

Texas A&M University at Galveston
Galveston, TX 77553, USA
E-mail address: yie@tamug.edu