

KNESER-POULSEN CONJECTURE FOR A SMALL  
NUMBER OF INTERSECTIONS

IGORS GORBOVICKIS

ABSTRACT. The Kneser-Poulsen conjecture says that if a finite collection of balls in the Euclidean space  $\mathbb{E}^d$  is rearranged so that the distance between each pair of centers does not get smaller, then the volume of the union of these balls also does not get smaller. In this paper, we prove that if in the initial configuration the intersection of any two balls has common points with no more than  $d + 1$  other balls, then the conjecture holds.

## 1. INTRODUCTION

Given a positive integer  $d$ , we denote by  $\mathbb{E}^d$  the  $d$ -dimensional Euclidean space. Let  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  and  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  be two configurations of  $N$  points, where  $\mathbf{p}_i \in \mathbb{E}^d$  and  $\mathbf{q}_i \in \mathbb{E}^d$ , for each  $i = 1, \dots, N$ . Let  $|\dots|$  be the Euclidean norm. If for all  $1 \leq i < j \leq N$ ,  $|\mathbf{p}_i - \mathbf{p}_j| \leq |\mathbf{q}_i - \mathbf{q}_j|$ , we say that  $\mathbf{q}$  is an *expansion* of  $\mathbf{p}$  and  $\mathbf{p}$  is a *contraction* of  $\mathbf{q}$ . If  $\mathbf{p}_0 \in \mathbb{E}^d$ , we denote by  $B_d(\mathbf{p}_0, r)$  the closed  $d$ -dimensional ball of radius  $r$  in  $\mathbb{E}^d$  about the point  $\mathbf{p}_0$ . We define  $B_d(\mathbf{p}_0, r)$  to be an empty set, if  $r < 0$  or if  $r$  is not a real number. We also let  $\text{Vol}_d$  represent the  $d$ -dimensional volume.

The following conjecture was independently stated by Kneser [7] in 1955 and Poulsen [8] in 1954 for the case when  $r_1 = \dots = r_N$ .

**Conjecture 1.1.** *If  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$  is an expansion of  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  in  $\mathbb{E}^d$ , then for any vector of radii  $\mathbf{r} = (r_1, \dots, r_N)$ ,*

$$(1.1) \quad \text{Vol}_d \left[ \bigcup_{i=1}^N B_d(\mathbf{p}_i, r_i) \right] \leq \text{Vol}_d \left[ \bigcup_{i=1}^N B_d(\mathbf{q}_i, r_i) \right].$$

For  $d = 1$ , Conjecture 1.1 is obvious. In the case when  $d = 2$ , the conjecture was proved by K. Bezdek and R. Connelly in [2].

**Theorem 1.2** (Bezdek, Connelly [2]). *Conjecture 1.1 holds, when  $d = 2$ .*

---

Received by the editors June 23, 2010, and in revised form March 15, 2014.

2010 *Mathematics Subject Classification.* 52A20, 52A40, 52A38, 51M25, 51M16.

*Key words and phrases.* Kneser-Poulsen conjecture, volume inequalities.

Research supported in part by NSF Grant No. DMS-0209595 (USA).

For  $d \geq 3$  the conjecture currently remains open.

References to related results as well as the history of this conjecture can be found in [2] and [5]. In the current paper, we prove the following theorem which confirms Conjecture 1.1 for  $d \geq 3$  under some additional assumptions.

**Theorem 1.3.** *Consider a configuration of  $N$  closed  $d$ -dimensional balls defined by their centers  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  in  $\mathbb{E}^d$  and the corresponding radii  $\mathbf{r} = (r_1, \dots, r_N)$ . Assume that the intersection of every pair of these balls has common points with no more than  $d + 1$  other balls from the considered configuration. Then for any expansion  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N) \in (\mathbb{E}^d)^N$  of the centers  $\mathbf{p}$ , the inequality (1.1) holds.*

As the limiting case of this theorem, we get the following corollary.

**Corollary 1.4.** *Consider a configuration of  $N$  closed  $d$ -dimensional balls defined by their centers  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  in  $\mathbb{E}^d$  and the corresponding radii  $\mathbf{r} = (r_1, \dots, r_N)$ . Assume that the intersection of every pair of these balls has common interior points with no more than  $d + 1$  other balls from the considered configuration. Then for any expansion  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N) \in (\mathbb{E}^d)^N$  of the centers  $\mathbf{p}$ , the inequality (1.1) holds.*

Corollary 1.4 can be viewed as a generalization of the following theorem proved in [2]

**Theorem 1.5** (Bezdek, Connelly [2]). *Conjecture 1.1 holds, if  $N \leq d + 3$ .*

The proof of Theorem 1.3 implements the following general idea which can also be found in other works on related subjects, such as [1], [2] and [3]. Namely, we embed the Euclidean space  $\mathbb{E}^d$  in a higher dimensional space and instead of considering  $d$ -dimensional ball configurations, we consider the corresponding higher dimensional objects. Viewing  $\mathbf{p}$  and  $\mathbf{q}$  as point configurations in a higher dimensional space, allows us to consider a piecewise smooth monotone expansion from  $\mathbf{p}$  to  $\mathbf{q}$ . At the same time, the higher dimensional ball configurations still carry some information about the  $d$ -dimensional ones. It appears that under the assumption of Theorem 1.3 we can use this information to obtain the inequality (1.1).

Since we will work in spaces of different dimensions, it will be convenient for the rest of the paper to fix  $d$  as in Theorem 1.3. Due to Theorem 1.2 by Bezdek and Connelly [2], we can assume that  $d \geq 3$ . We will use  $n$  to denote the dimension of an object in case we want to emphasize that this dimension is not necessarily equal to  $d$ .

## 2. VORONOI REGIONS

In this section, we first recall the definitions of truncated Voronoi regions and the walls between them. Then we formulate Csikós' formula (Theorem 2.2).

Let  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  be a configuration of points in  $\mathbb{E}^n$  with balls of radii  $\mathbf{r} = (r_1, \dots, r_N)$  centered at the corresponding points of the configuration.

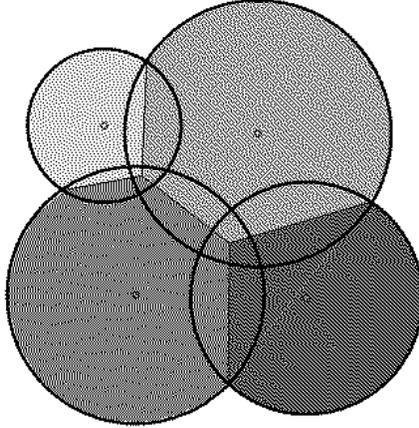


FIGURE 1. The truncated Voronoi region decomposition of a union of balls

The following sets are called (*extended*) *Voronoi regions*:

$$(2.1) \quad C_{n,i}(\mathbf{p}, \mathbf{r}) = \{\mathbf{p}_0 \in \mathbb{E}^n \mid \text{for all } j, |\mathbf{p}_0 - \mathbf{p}_i|^2 - r_i^2 \leq |\mathbf{p}_0 - \mathbf{p}_j|^2 - r_j^2\}.$$

*Remark 2.1.* It is easy to check that each Voronoi region  $C_{n,i}(\mathbf{p}, \mathbf{r})$  is a convex polyhedral set, and if  $(\mathbf{p}_i, r_i) \neq (\mathbf{p}_j, r_j)$  for  $i \neq j$ , then all of them together tile the whole Euclidean space  $\mathbb{E}^n$ .

We consider *truncated Voronoi regions*  $\hat{C}_{n,i}(\mathbf{p}, \mathbf{r}) = B_n(\mathbf{p}_i, r_i) \cap C_{n,i}(\mathbf{p}, \mathbf{r})$  and for each pair of distinct indices  $i \neq j$ , we define the *wall* between two truncated Voronoi regions as  $W_{n-1,ij}(\mathbf{p}, \mathbf{r}) = \hat{C}_{n,i}(\mathbf{p}, \mathbf{r}) \cap \hat{C}_{n,j}(\mathbf{p}, \mathbf{r})$ . Figure 1 gives an example of the truncated Voronoi region decomposition of a union of balls. The common boundaries of the shaded regions are the walls between the corresponding truncated Voronoi regions.

We define the function  $V_n(\mathbf{p}, \mathbf{r})$  to be the volume of the union of balls from the ball configuration determined by  $\mathbf{p}$  and  $\mathbf{r}$ , that is

$$(2.2) \quad V_n(\mathbf{p}, \mathbf{r}) = \text{Vol}_n \left[ \bigcup_{i=1}^N B_n(\mathbf{p}_i, r_i) \right].$$

According to Remark 2.1, the function  $V_n(\mathbf{p}, \mathbf{r})$  can also be expressed as

$$V_n(\mathbf{p}, \mathbf{r}) = \sum_{i=1}^N \text{Vol}_n \left[ \hat{C}_{n,i}(\mathbf{p}, \mathbf{r}) \right].$$

Consider a smooth (infinitely many times differentiable) motion  $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_N(t))$  of some configuration of  $N$  points in  $\mathbb{E}^n$ . Let  $d_{ij}(t) = |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ , and let  $d'_{ij}$  be the derivative of  $d_{ij}$  with respect to  $t$ . The following is Csikós' formula [4] for the derivative of the function  $V_n(\mathbf{p}(t), \mathbf{r})$ .

**Theorem 2.2.** *Let  $n \geq 2$  and let  $\mathbf{p}(t)$  be a smooth motion of a configuration of points in  $\mathbb{E}^n$  such that for each  $t$ , all the points are pairwise distinct. Then the function  $V_n(\mathbf{p}(t), \mathbf{r})$  is differentiable with respect to  $t$  and,*

$$\frac{d}{dt} V_n(\mathbf{p}(t), \mathbf{r}) = \sum_{1 \leq i < j \leq N} d'_{ij} \text{Vol}_{n-1} [W_{n-1,ij}(\mathbf{p}, \mathbf{r})].$$

### 3. VOLUME OF A POLYHEDRAL SET INTERSECTED WITH A BALL

We notice that both the truncated Voronoi regions and the walls between them can be viewed as intersections of some polyhedral sets with the corresponding balls. In this section, we give relevant statements about the volumes of such sets. It appears that the volume of such a set can be obtained from the volume of a certain higher dimensional polyhedral set intersected with a ball. This will play an important role in our argument.

The following lemma is a reformulation of Corollary 6 from [2].

**Lemma 3.1.** *Let  $P \subset \mathbb{E}^{n+2}$  be a polyhedral set, such that all its codimension 1 facets are orthogonal to some  $n$ -dimensional affine subspace  $X \subset \mathbb{E}^{n+2}$ . Consider a point  $\mathbf{p}_0 \in X$ . Then for every pair of real numbers  $r$  and  $s$ , the following derivative exists and satisfies*

$$\frac{d}{ds} \text{Vol}_{n+2} \left[ P \cap B_{n+2}(\mathbf{p}_0, \sqrt{r^2 + s}) \right] = \pi \text{Vol}_n \left[ X \cap P \cap B_{n+2}(\mathbf{p}_0, \sqrt{r^2 + s}) \right].$$

The following corollary can be proved by applying Lemma 3.1 several times.

**Corollary 3.2.** *Given a positive integer  $k$ , let  $P \subset \mathbb{E}^{n+2k}$  be a polyhedral set, such that all its codimension 1 facets are orthogonal to some  $n$ -dimensional affine subspace  $X \subset \mathbb{E}^{n+2k}$ . Consider a point  $\mathbf{p}_0 \in X$ . Then for every pair of real numbers  $r$  and  $s$ , the following derivative exists and satisfies*

$$\begin{aligned} \frac{d^k}{ds^k} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right] = \\ \pi^k \text{Vol}_n \left[ X \cap P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right]. \end{aligned}$$

Given a positive integer  $n$  and a non-negative integer  $k$ , let  $\mathcal{P}_n^k$  be the space of all polyhedral sets in  $\mathbb{E}^{n+2k}$  that are intersections of  $n$  half-spaces. Every half-space in  $\mathbb{E}^{n+2k}$  is uniquely defined by an inequality of the form  $\langle \mathbf{u}, \mathbf{x} \rangle \leq c$ , where  $\mathbf{u} \in \mathbb{S}^{n+2k-1}$  is a unit vector and  $c \in \mathbb{R}$  is a real number. This gives a bijection between the set  $\mathcal{H}_{n+2k}$  of all half-spaces in  $\mathbb{E}^{n+2k}$  and  $\mathbb{S}^{n+2k-1} \times \mathbb{R}$ . This bijection induces a topology on  $\mathcal{H}_{n+2k}$ . As the map  $(\mathcal{H}_{n+2k})^n \rightarrow \mathcal{P}_n^k$ , defined by  $(H_1, \dots, H_n) \mapsto H_1 \cap \dots \cap H_n$  is surjective,  $\mathcal{P}_n^k$  is a quotient space of  $(\mathcal{H}_{n+2k})^n$ , carrying a quotient topology.

**Lemma 3.3.** *For  $n \geq 3$ , consider a polyhedral set  $P \in \mathcal{P}_n^k$  and a point  $\mathbf{p}_0 \in \mathbb{E}^{n+2k}$ . Then for every pair of real numbers  $r$  and  $s$ , the following*

derivative exists and satisfies the inequality

$$(3.1) \quad 0 \leq \frac{d^{k+1}}{ds^{k+1}} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right] \leq \max \left\{ \frac{1}{2} \pi^k \sigma_{n-1} (r^2 + s)^{\frac{n-2}{2}}, 0 \right\},$$

where  $\sigma_{n-1}$  is the  $(n-1)$ -dimensional surface volume of the  $n$ -dimensional unit ball. Moreover, the derivative  $\frac{d^{k+1}}{ds^{k+1}} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right]$  depends continuously on  $P$  and  $s$  simultaneously.

*Proof.* Since  $P \in \mathcal{P}_n^k$ , it can be represented as  $P = \bigcap_{i=1}^n H_i$  for some half-spaces  $H_i \subset \mathbb{E}^{n+2k}$ . Hence there exists an  $n$ -dimensional affine subspace  $X$  that contains the point  $\mathbf{p}_0$  and is orthogonal to the boundary hyperplanes of the half-spaces  $H_1, \dots, H_n$ . Then Corollary 3.2 implies that

$$\begin{aligned} \frac{d^{k+1}}{ds^{k+1}} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right] &= \\ &= \pi^k \frac{d}{ds} \text{Vol}_n \left[ X \cap P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right]. \end{aligned}$$

Now according to the chain rule,

$$(3.2) \quad \frac{d^{k+1}}{ds^{k+1}} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right] = \frac{\pi^k}{2\sqrt{r^2 + s}} \frac{d}{d\tilde{r}} \text{Vol}_n \left[ X \cap P \cap B_{n+2k}(\mathbf{p}_0, \tilde{r}) \right] \Big|_{\tilde{r}=\sqrt{r^2 + s}}.$$

Note that  $X \cap P \cap B_{n+2k}(\mathbf{p}_0, \tilde{r})$  is the intersection of an  $n$ -dimensional polyhedral set  $X \cap P$  with a ball of radius  $\tilde{r}$ . The derivative of the volume of this set with respect to  $\tilde{r}$  is equal to the surface volume of the spherical part of its boundary. Since this surface volume is non-negative and not greater than the surface volume of the  $n$ -dimensional ball of radius  $\tilde{r}$ , we obtain the required inequalities (3.1).

Finally, we notice that the surface volume of the spherical part of the boundary considered in the previous paragraph, depends continuously on  $P \in \mathcal{P}_n^k$ ,  $\mathbf{p}_0$ , and  $s$  simultaneously. Indeed, if the normal vectors of the boundary hyperplanes  $\partial H_1, \dots, \partial H_n$  are linearly independent, then the affine subspace  $X$  is uniquely determined for the configuration  $(P, \mathbf{p}_0, s)$ , and for all nearby configurations. In that case the continuity assertion is obvious. In case the normal vectors of the boundary hyperplanes  $\partial H_1, \dots, \partial H_n$  are linearly dependent, then there are infinitely many choices for the affine subspace  $X$ . However, it follows from (3.2) that the considered surface volume does not depend on the choice of  $X$ . Thus, for a configuration  $(\tilde{P}, \tilde{\mathbf{p}}_0, \tilde{s})$  that is close to the configuration  $(P, \mathbf{p}_0, s)$ , we are allowed to choose the corresponding affine subspaces  $\tilde{X}$  and  $X$  also to be close to each other. This implies that the considered surface volumes for the configurations  $(P, \mathbf{p}_0, s)$

and  $(\tilde{P}, \tilde{\mathbf{p}}_0, \tilde{s})$  are close and continuous dependence of these surface volumes on the configuration follows.

Now it follows from (3.2) and the above continuity assertion, that when  $r^2 + s \neq 0$ , the derivative  $\frac{d^{k+1}}{ds^{k+1}} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right]$  also depends continuously on  $P$  and  $s$  simultaneously. On the other hand, when  $r^2 + s$  approaches zero, both the lower and the upper bounds in (3.1) approach zero as well, which implies that when  $r^2 + s = 0$ , the derivative  $\frac{d^{k+1}}{ds^{k+1}} \text{Vol}_{n+2k} \left[ P \cap B_{n+2k}(\mathbf{p}_0, \sqrt{r^2 + s}) \right]$  is also continuous in  $P$  and  $s$  simultaneously.  $\square$

#### 4. THE VOLUMES OF THE WALLS AND THEIR DERIVATIVES

We return to the original setting where we have a configuration of  $N$  balls of radii  $r_1, \dots, r_N$  respectively.

**Definition 4.1.** Given a vector of radii  $\mathbf{r} = (r_1, \dots, r_N) \in \mathbb{R}^N$ , we say that a configuration of  $N$  points  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  in some Euclidean space  $\mathbb{E}^n$  is  $(d, \mathbf{r})$ -nice, if in the configuration of  $N$  balls of radii  $r_1, \dots, r_N$  centered at the corresponding points  $\mathbf{p}_1, \dots, \mathbf{p}_N$  the intersection of each pair of balls has common points with no more than  $d + 1$  other balls.

From now on we fix the radii  $\mathbf{r} = (r_1, \dots, r_N)$  and we consider a one parameter family

$$\mathbf{r}(s) = \left( \sqrt{r_1^2 + s}, \dots, \sqrt{r_N^2 + s} \right),$$

which coincides with the initial vector of radii  $\mathbf{r}$ , when  $s = 0$ .

**Proposition 4.2.** *Let  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N) \subset \mathbb{E}^n$  be a configuration of  $N$  distinct points. Then we have the following.*

(i) *Each Voronoi region  $C_{n,i}(\mathbf{p}, \mathbf{r}(s))$  is a convex polyhedral set completely determined by  $\mathbf{p}$  and  $\mathbf{r}$  and independent of  $s$ .*

(ii) *Each truncated Voronoi region  $\hat{C}_{n,i}(\mathbf{p}, \mathbf{r}(s))$  is an intersection of a fixed convex polyhedral set from part (i) and the ball  $B_n(\mathbf{p}_i, \sqrt{r_i^2 + s})$ .*

(iii) *Each wall  $W_{n-1,ij}(\mathbf{p}, \mathbf{r}(s))$  between two truncated Voronoi regions is an intersection of the ball  $B_n(\mathbf{p}_i, \sqrt{r_i^2 + s})$  with an  $(n-1)$ -dimensional convex polyhedral set independent of  $s$  and lying in the radical hyperplane of the balls  $B_n(\mathbf{p}_i, r_i)$  and  $B_n(\mathbf{p}_j, r_j)$ .*

*Proof.* (i) As it was noticed in Remark 2.1, the Voronoi region  $C_{n,i}(\mathbf{p}, \mathbf{r}(s))$  is a convex polyhedral set. Its independence of  $s$  follows from its definition (2.1).

Parts (ii) and (iii) immediately follow from part (i).  $\square$

Now we prove our key lemma.

**Lemma 4.3.** (i) Let  $d \geq 2$ , and let  $k$  be a non-negative integer. Consider a  $(d, \mathbf{r})$ -nice configuration of  $N$  distinct points  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$  in  $\mathbb{E}^{d+2k}$ . Then for every pair of indices  $i \neq j$ , the  $(d + 2k - 1)$ -dimensional volume  $\text{Vol}_{d+2k-1} [W_{d+2k-1,ij}(\mathbf{p}, \mathbf{r}(s))]$  of the wall between truncated Voronoi regions is at least  $k$  times differentiable as a function of  $s$  in a sufficiently small neighborhood  $U$  of the point  $s = 0$ . Also for each  $s \in U$ , the partial derivatives

$$(4.1) \quad \frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1,ij}(\mathbf{p}, \mathbf{r}(s))]$$

are non-negative and locally bounded as functions of  $(\mathbf{p}, s)$ .

(ii) If in addition to the conditions of the first part the points of  $\mathbf{p}$  are affinely independent, then the partial derivatives (4.1) are locally continuous in  $\mathbf{p}$  and  $s$  simultaneously.

*Proof.* For  $k = 0$  the result is obvious, so we can assume that  $k > 0$ .

We denote by  $H$  the radical hyperplane of the balls  $B_{d+2k}(\mathbf{p}_i, r_i)$  and  $B_{d+2k}(\mathbf{p}_j, r_j)$ . Then according to part (iii) of Proposition 4.2, the hyperplane  $H$  contains the wall  $W_{d+2k-1,ij}(\mathbf{p}, \mathbf{r}(s))$  which corresponds to these two balls. Let the point  $\mathbf{p}_0$  be the orthogonal projection of the point  $\mathbf{p}_i$  onto the hyperplane  $H$  and define  $h = |\mathbf{p}_i - \mathbf{p}_0|$ .

Since the point configuration  $\mathbf{p}$  is  $(d, \mathbf{r})$ -nice and the set of  $(d, \mathbf{r})$ -nice point configurations is open, there exists a neighborhood of the origin  $U \subset \mathbb{R}$ , such that for all  $s \in U$  the configuration  $\mathbf{p}$  is  $(d, \mathbf{r}(s))$ -nice. This implies that for all  $s \in U$ , the wall  $W_{d+2k-1,ij}(\mathbf{p}, \mathbf{r}(s))$  can be viewed as the intersection of the ball  $B(\mathbf{p}_0, \sqrt{r_i^2 - h^2 + s})$  with  $d+1$  half-spaces  $H_1, \dots, H_{d+1}$  in  $H$ . This observation together with Lemma 3.3 proves part (i) of Lemma 4.3.

We notice that if the points of the configuration  $\mathbf{p}$  are affinely independent, then both the hyperplane  $H$  and the half-spaces  $H_1, \dots, H_{d+1}$  depend locally continuously on  $\mathbf{p}$ . Thus Lemma 3.3 implies part (ii) of Lemma 4.3.  $\square$

## 5. A PATH BETWEEN $\mathbf{p}$ AND $\mathbf{q}$

The proof of Theorem 1.3 is essentially based on choosing an appropriate piecewise smooth path in the space of sufficiently high dimension that connects the configurations  $\mathbf{p}$  and  $\mathbf{q}$ . More detailed arguments follow.

**Lemma 5.1.** If  $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_N(t))$  is a piecewise smooth motion of a configuration of centers in  $\mathbb{E}^{d+2k}$  with  $d \geq 2$  and  $t \in [0, 1]$ , such that  $\mathbf{p}(t)$  is  $(d, \mathbf{r})$ -nice for all  $t \in [0, 1]$  and its points are affinely independent for all but finitely many values of  $t$  in  $[0, 1]$ , then the following identity holds:

$$(5.1) \quad \frac{\partial^k}{\partial s^k} (V_{d+2k}(\mathbf{p}(1), \mathbf{r}(s)) - V_{d+2k}(\mathbf{p}(0), \mathbf{r}(s))) \Big|_{s=0} = \int_0^1 \sum_{1 \leq i < j \leq N} d'_{ij} \frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1,ij}(\mathbf{p}(t), \mathbf{r}(s))] \Big|_{s=0} dt,$$

where the function  $V_{d+2k}$  is defined as in (2.2) and  $d_{ij}(t) = |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ .

*Proof.* It is obvious that

$$\frac{\partial^k}{\partial s^k} (V_{d+2k}(\mathbf{p}(1), \mathbf{r}(s)) - V_{d+2k}(\mathbf{p}(0), \mathbf{r}(s))) \Big|_{s=0} = \frac{\partial^k}{\partial s^k} \int_0^1 \frac{\partial}{\partial t} V_{d+2k}(\mathbf{p}(t), \mathbf{r}(s)) dt \Big|_{s=0}.$$

Now according to Csikós' formula (Theorem 2.2) we get

$$\frac{\partial^k}{\partial s^k} \int_0^1 \frac{\partial}{\partial t} V_{d+2k}(\mathbf{p}(t), \mathbf{r}(s)) dt \Big|_{s=0} = \frac{\partial^k}{\partial s^k} \int_0^1 \sum_{1 \leq i < j \leq N} d'_{ij} \text{Vol}_{d+2k-1} [W_{d+2k-1, ij}(\mathbf{p}(t), \mathbf{r}(s))] dt \Big|_{s=0}.$$

Finally, it follows from Lemma 4.3 that we can change the order of differentiation and integration in the last expression to obtain

$$\frac{\partial^k}{\partial s^k} \int_0^1 \sum_{1 \leq i < j \leq N} d'_{ij} \text{Vol}_{d+2k-1} [W_{d+2k-1, ij}(\mathbf{p}(t), \mathbf{r}(s))] dt \Big|_{s=0} = \int_0^1 \sum_{1 \leq i < j \leq N} d'_{ij} \frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1, ij}(\mathbf{p}(t), \mathbf{r}(s))] \Big|_{s=0} dt.$$

□

**Corollary 5.2.** *If  $d \geq 2$  and  $\mathbf{p} \subset \mathbb{E}^{d+2k}$  is a  $(d, \mathbf{r})$ -nice configuration of  $N$  points, where  $N \leq d + 2k + 1$ , then the function  $\frac{\partial^k}{\partial s^k} V_{d+2k}(\mathbf{p}, \mathbf{r}(s)) \Big|_{s=0}$  is locally continuous in the variable  $\mathbf{p}$ .*

*Proof.* Since  $\mathbf{p}$  is  $(d, \mathbf{r})$ -nice, all point configurations that are sufficiently close to  $\mathbf{p}$ , are also  $(d, \mathbf{r})$ -nice. If  $\mathbf{p}' \subset \mathbb{E}^{d+2k}$  is a configuration of centers that is sufficiently close to  $\mathbf{p}$ , then we can connect the configurations  $\mathbf{p}$  and  $\mathbf{p}'$  with a piecewise smooth path  $\mathbf{p}(t)$  that satisfies Lemma 5.1. Then Corollary 5.2 follows from the fact that the functions

$$\frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1, ij}(\mathbf{p}(t), \mathbf{r}(s))] \Big|_{s=0}$$

in the right hand side of (5.1) are bounded, as was shown in Lemma 4.3. □

**Corollary 5.3.** *If  $d \geq 2$  and  $\mathbf{p}, \mathbf{q} \subset \mathbb{E}^{d+2k}$  are two configurations of  $N$  affinely independent points,  $\mathbf{q}$  is an expansion of  $\mathbf{p}$ , and the configuration  $\mathbf{p}$  is  $(d, \mathbf{r})$ -nice, then*

$$\frac{\partial^k}{\partial s^k} V_{d+2k}(\mathbf{q}, \mathbf{r}(s)) \Big|_{s=0} \geq \frac{\partial^k}{\partial s^k} V_{d+2k}(\mathbf{p}, \mathbf{r}(s)) \Big|_{s=0}.$$

*Proof.* According to [1], configurations  $\mathbf{p}$  and  $\mathbf{q}$  can be connected by a piecewise smooth motion  $\mathbf{p}(t)$  so that  $\mathbf{p}(0) = \mathbf{p}$ ,  $\mathbf{p}(1) = \mathbf{q}$ , the distances  $d_{ij}(t) = |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$  are weakly increasing in  $t$  and for each  $t$ , the points of  $\mathbf{p}(t)$  are affinely independent. It follows from Kirszbraun's theorem [6] that since the distances  $d_{ij}(t)$  are weakly increasing and  $\mathbf{p}(0)$  is  $(d, \mathbf{r})$ -nice,  $\mathbf{p}(t)$  is  $(d, \mathbf{r})$ -nice for all  $t \in [0, 1]$ , and we can apply Lemma 5.1 to get

$$(5.2) \quad \left. \frac{\partial^k}{\partial s^k} (V_{d+2k}(\mathbf{q}, \mathbf{r}(s)) - V_{d+2k}(\mathbf{p}, \mathbf{r}(s))) \right|_{s=0} = \int_0^1 \sum_{1 \leq i < j \leq N} d'_{ij} \frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1, ij}(\mathbf{p}(t), \mathbf{r}(s))] \Big|_{s=0} dt.$$

By Lemma 4.3, the derivatives

$$\left. \frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1, ij}(\mathbf{p}(t), \mathbf{r}(s))] \right|_{s=0}$$

are always non-negative, and since  $d'_{ij} \geq 0$ , the expression under the integral in (5.2) is also non-negative.  $\square$

*Proof of Theorem 1.3.* Without loss of generality we may assume that the points in the configuration  $\mathbf{p}$  are distinct. Indeed, if a pair of points coincide, then we can exclude the point that corresponds to a smaller radius. This does not change the volume of the initial configuration, but can decrease the volume of the final configuration. Thus, the general case of Theorem 1.3 follows from the case when all points of  $\mathbf{p}$  are distinct.

Let  $k$  be a positive integer, such that  $d + 2k \geq N - 1$ , and we regard  $\mathbb{E}^d$  as the subset  $\mathbb{E}^d = \mathbb{E}^d \times \{0\} \subset \mathbb{E}^d \times \mathbb{E}^{2k} = \mathbb{E}^{d+2k}$ . We can view  $\mathbf{p}$  and  $\mathbf{q}$  as point configurations lying either in  $\mathbb{E}^d$  or in  $\mathbb{E}^{d+2k}$  and consider corresponding  $d$ -dimensional and  $(d + 2k)$ -dimensional volumes  $V_d(\mathbf{p}, \mathbf{r})$ ,  $V_d(\mathbf{q}, \mathbf{r})$ ,  $V_{d+2k}(\mathbf{p}, \mathbf{r})$  and  $V_{d+2k}(\mathbf{q}, \mathbf{r})$ .

Note that the sets  $\bigcup_{i=1}^N B_{d+2k}(\mathbf{p}_i, r_i)$  and  $\bigcup_{i=1}^N B_{d+2k}(\mathbf{q}_i, r_i)$  are unions of non-overlapping truncated Voronoi regions and according to part (ii) of Proposition 4.2, we can apply Corollary 3.2 to them. As a result, we obtain the following identity.

$$(5.3) \quad \pi^k (V_d(\mathbf{q}, \mathbf{r}) - V_d(\mathbf{p}, \mathbf{r})) = \left. \frac{\partial^k}{\partial s^k} (V_{d+2k}(\mathbf{q}, \mathbf{r}(s)) - V_{d+2k}(\mathbf{p}, \mathbf{r}(s))) \right|_{s=0}.$$

By Kirszbraun's theorem, since  $\mathbf{p}$  is  $(d, \mathbf{r})$ -nice and  $\mathbf{q}$  is its expansion,  $\mathbf{q}$  is also  $(d, \mathbf{r})$ -nice. Hence according to Corollary 5.2, the right hand side of (5.3) depends locally continuously on  $\mathbf{p}$  and  $\mathbf{q}$ . Let  $\mathbf{p}', \mathbf{q}' \subset \mathbb{E}^{d+2k}$  be small perturbations of  $\mathbf{p}$  and  $\mathbf{q}$  respectively, such that the configurations  $\mathbf{p}'$  and  $\mathbf{q}'$  consist of affinely independent points,  $\mathbf{p}'$  is  $(d, \mathbf{r})$ -nice and  $\mathbf{q}'$  is an expansion of  $\mathbf{p}'$ . Then it follows from Corollary 5.3 that

$$(5.4) \quad \left. \frac{\partial^k}{\partial s^k} (V_{d+2k}(\mathbf{q}', \mathbf{r}(s)) - V_{d+2k}(\mathbf{p}', \mathbf{r}(s))) \right|_{s=0} \geq 0.$$

By choosing  $\mathbf{p}'$  and  $\mathbf{q}'$  arbitrarily close to  $\mathbf{p}$  and  $\mathbf{q}$  respectively, we get the following inequality as a limiting case of (5.4).

$$\left. \frac{\partial^k}{\partial s^k} (V_{d+2k}(\mathbf{q}, \mathbf{r}(s)) - V_{d+2k}(\mathbf{p}, \mathbf{r}(s))) \right|_{s=0} \geq 0.$$

Together with (5.3) this proves that

$$V_d(\mathbf{q}, \mathbf{r}) \geq V_d(\mathbf{p}, \mathbf{r}).$$

□

#### ACKNOWLEDGMENTS

I would like to thank Robert Connelly for many fruitful discussions on the subject of the paper and for his constant support as one of my Ph.D. supervisors. I would also like to thank the anonymous referee for providing a very detailed review and for pointing out the inaccuracies in the first version of the text.

#### REFERENCES

1. Ralph Alexander, *Lipschitzian mappings and total mean curvature of polyhedral surfaces. I*, Trans. Amer. Math. Soc. **288** (1985), no. 2, 661–678.
2. Károly Bezdek and Robert Connelly, *Pushing disks apart—the Kneser-Poulsen conjecture in the plane*, J. Reine Angew. Math. **553** (2002), 221–236.
3. Vasilis Capovleas and János Pach, *On the perimeter of a point set in the plane*, Discrete and computational geometry (New Brunswick, NJ, 1989/1990), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 6, Amer. Math. Soc., Providence, RI, 1991, pp. 67–76.
4. B. Csikós, *On the volume of the union of balls*, Discrete Comput. Geom. **20** (1998), no. 4, 449–461.
5. Balázs Csikós, *A Schläfli-type formula for polytopes with curved faces and its application to the Kneser-Poulsen conjecture*, Monatsh. Math. **147** (2006), no. 4, 273–292.
6. M. D. Kirschbraun, *Über die zusammenziehende und Lipschitzsche Transformationen*, Fundamenta Mathematicae **22** (1934), 77–108.
7. Martin Kneser, *Einige Bemerkungen über das Minkowskische Flächenmass*, Arch. Math. (Basel) **6** (1955), 382–390.
8. Ebbe Thue Poulsen, *Problem 10*, Math. Scand. **2** (1954), 346.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, ROOM 6290, 40 ST.  
 GEORGE STREET, TORONTO, ONTARIO, CANADA M5S 2E4  
*E-mail address:* igors.gorbovickis@utoronto.ca