Contributions to Discrete Mathematics

Volume 6, Number 1, Pages 98–141
ISSN 1715-0868

DUAL LINEAR SPACES GENERATED BY A NON-DESARGUESIAN CONFIGURATION

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Abstract. We describe a method to try to construct non-Desarguesian projective planes of a given finite order using a computer program. If a projective plane of order n exists, then it can be constructed from a dual linear space by a sequence of one-line extensions. This motivates a careful analysis of the failures of Desargues' Law in a dual linear space. Up to isomorphism, there are 105 dual linear spaces generated by a non-Desarguesian configuration with which to start the extension process. A further reduction allows us to limit the number of starting configurations to 15.

1. Introduction

The availability, speed and memory of modern computers have made feasible computational approaches to problems that only a few years ago would have seemed hopeless. This paper reports on our attempts on a problem that has gone from total impracticality to the barest margins of feasibility: the search for non-Desarguesian projective planes of either prime order or non-prime-power order.

A dual linear space is a partial projective plane that contains the intersection of every pair of its lines. Every dual linear space can be extended to a projective plane, usually infinite, by a sequence of one line extensions. Moreover, one may describe necessary conditions for the sequence of one-line extensions to terminate after finitely many steps with a finite projective plane of a given order. Our program attempts to construct a finite projective plane from a dual linear space generated by a non-Desarguesian configuration by a sequence of one-line extensions. An exhaustive search is not practical, so some random choices are built into the search. Since about the year 2000, we have been running various versions of the program in (so far unsuccessful) attempts to build non-Desarguesian planes of orders 11, 12, 13 and 15.

Received by the editors December 8, 2009, and in revised form February 8, 2011.
2000 *Mathematics Subject Classification.* 51E14, 51A35.
Key words and phrases. Finite non-Desarguesian projective plane, dual linear space.
The first author was supported in part by a grant from the U.S. Civilian Research & Development Foundation.

Our algorithm begins with a dual linear space that fails Desargues' Law. Thus an important step in limiting the search is to classify the dual linear spaces generated by a non-Desarguesian configuration, as these are the possible inputs for the extension algorithm. Our main theoretical result is that there are 105 non-isomorphic initial dual linear spaces generated by the basic non-Desarguesian configuration. Moreover, it turns out that for these 105 types of failures, the existence of one type in a projective plane forces the existence of another type. This allows us to reduce the number of starting configurations to 15.

The authors gratefully acknowledge the contributions of Cherise Ho to an early version of this search.

2. Partial Projective Planes and Semiplanes

In this section we describe the types of partial projective planes that are used in our program to try to construct non-Desarguesian planes. These are various geometric structures S given by a triple $S = \langle P, L, \leq \rangle$ consisting of a set P of elements called *points*, a set L of distinguished subsets of points, called *lines*, and a binary relation $\leq$ contained in $P \times L$. The relation $p \leq \ell$ is interpreted to mean that the point p is on the line ℓ.

A *configuration* is a triple $C = \langle P, L, \leq \rangle$ with the property that no two distinct lines contain two distinct points in common. Thus in a configuration two distinct points determine at most one line, and two distinct lines intersect in at most one point. We use $p \vee q$ to denote the line through points p and q, if there is one. Similarly, $k \wedge \ell$ denotes the point on both lines k and ℓ, if there is one.

A *linear space* is a structure $S = \langle P, L, \leq \rangle$ satisfying the following axioms.

(L1) Any two distinct points of S belong to exactly one line of S.
(L2) Any line of S contains at least two points of S.

We define a *dual linear space*, or *semiplane*, to be a triple $S' = (P', L', \leq')$ satisfying the dual statements of (L1) and (L2). Thus in a dual linear space, every pair of lines intersect in exactly one point, and every point lies on at least two lines.

A *projective plane* is then a structure $\Pi = \langle P, L, \leq \rangle$ satisfying these axioms.

(PP1) Any two distinct points lie on exactly one line.
(PP2) Any two distinct lines intersect in exactly one point.
(PP3) There exist four points, no three of which are on a line.

Thus a projective plane is a linear space which is also a dual linear space. We will be concerned in this paper with describing how to extend a dual linear space to a projective plane by adding lines. The decision to start with a dual linear space rather than a linear space is of course arbitrary. We think of a dual linear space as the meet semilattice generated by a set of lines in a projective plane (hence the term semiplane). This is justified by the theorem of Marshall Hall that any configuration can be extended to a projective plane [7]. However, the plane given by Hall's construction is always infinite, and we would like to extend a finite configuration to a finite projective plane. It is not known whether this can always be done. Even more, we would like to prescribe the order of the projective plane in advance.

For any point p in a finite configuration, we let $r(p)$ denote the number of lines on p. Similarly, we let $r(\ell)$ denote the number of points on the line ℓ. We shall refer to $r(p)$ and $r(\ell)$ as the *rank* of the respective point p and line ℓ.

It is useful to observe that we can count the total number of point-line incidences $p \leq \ell$ in two different ways.

Lemma 1. *If $S = \langle P, L, \leq \rangle$ is a finite configuration, then*

$$\sum_{l \in L} r(l) = \sum_{p \in P} r(p).$$

A linear space with n points, where $n \geq 3$, that has one line containing $n-1$ points and $n-1$ lines containing two points, is called a *near pencil*.

The following result of de Bruijn and Erdős [4] is the Fundamental Theorem of finite linear spaces. For the proof of the theorem, see [2].

Theorem 2. *Let $S = \langle P, L, \leq \rangle$ be a finite linear space with at least two points. Then $|L| \geq |P|$. Moreover, equality holds if and only if S is a projective plane or a near-pencil.*

Since we will be working in the dual situation, we note that in a finite dual linear space, we have $|P| \geq |L|$.

We say that a configuration $S' = (P', L', \leq')$ is an *extension* of $S = \langle P, L, \leq \rangle$ if $P \subseteq P'$, $L \subseteq L'$, and the relation $\leq$ is the restriction of $\leq'$ to $P \times L$. This is denoted by $S \sqsubseteq S'$.

3. A Necessary Condition for Extendibility to a Plane of Order n

One can describe various combinatorial properties that are necessary for a semiplane to be extendible to a projective plane of order n. Clearly we want the semiplane to have at most $n^2 + n + 1$ points (in which case it will have at most that many lines), and for each point and line to have rank at most $n + 1$. Beyond these conditions, we have found one test to be particularly useful.

Given the putative order n and a semiplane $\Pi = \langle P_0, L_0, \leq_0 \rangle$, define

$$\rho_n(\Pi) = \sum_{\ell \in L_0} r_\Pi(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n+1).$$

Theorem 3. *Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane that can be extended to a projective plane $\Sigma = \langle P, L, \leq \rangle$ of order n. Then*

$$\rho_n(\Pi) = |\{p \in P : p \nleq \ell \textit{ for all } \ell \in L_0\}|.$$

Hence Π *can be extended to a projective plane of order* n *only if* $\rho_n(\Pi) \geq 0$.

Proof. The lines of L_0 contain $|P_0| + \sum_{\ell \in L_0}(n+1-r_\Pi(\ell))$ points in Σ. So there are $n^2+n+1+\sum_{\ell \in L_0} r_\Pi(\ell) - |P_0| - |L_0|(n+1) = \rho_n(\Pi)$ points in P that are on no line of L_0. □

There are various other necessary conditions for a semiplane to be extendible to a projective plane of order n. In practice, however, we have found that semiplanes that fail those conditions also have $\rho_n(\Pi)$ negative. Therefore we have removed those tests from our programs, and the condition $\rho_n(\Pi) \geq 0$ remains our main test for extendibility until a better candidate comes along.

4. One Line Extensions

Let $\Sigma = \langle P, L, \leq \rangle$ be a projective plane. Observe that there is a natural correspondence between subsets of L and semiplanes Π contained in Σ. This is given by the map $K \mapsto \Pi_K$, where K is a subset of L and $\Pi_K = \langle Q, K, \leq^* \rangle$ is a semiplane obtained by taking $Q = \{k \wedge \ell : k, \ell \in K \text{ and } k \neq \ell\}$ and $\leq^*$ the restriction of $\leq$ to $Q \times K$.

Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane, and let $\Sigma = \langle P, L, \leq \rangle$ be a semiplane properly extending Π. Let $k \in L - L_0$, and let $M = \{p \in P_0 : p \leq k\}$. Note that $s \vee t$ is undefined in Π for all pairs s, $t \in M$. Let $D = \{\ell \in L_0 : \ell \wedge k \notin P_0\}$. For each $\ell \in D$, let $q_\ell = \ell \wedge k$; note that $\ell \neq \ell'$ implies $q_\ell \neq q_{\ell'}$. Define an extension $\Pi' = \langle P_1, L_1, \leq_1 \rangle$ of Π as follows:

(1) $P_1 = P_0 \cup \{q_\ell : \ell \in D\}$,
(2) $L_1 = L_0 \cup \{k\}$, and
(3) $\leq_1$ is the order induced by Σ, which consists precisely of the relations holding in Π and the new relations $m \leq_1 k$ for all $m \in M$, and $q_\ell \leq_1 \ell$ and $q_\ell \leq_1 k$ for each $\ell \in D$.

Then Π' is a semiplane with $\Pi \sqsubseteq \Pi' \sqsubseteq \Sigma$. Moreover, $s \vee t$ is defined in Π' for all s, $t \in M$, *viz.*, $s \vee t = k$.

Now let us describe this construction without reference to the plane Σ. Let $\Pi = \langle P_0, L_0, \leq_0 \rangle$ be a semiplane that is not a projective plane, and let $M \subseteq P_0$ be a set of points whose pairwise join is undefined in Π. (We allow $|M| = 0$ or 1, which can happen in the case of a near pencil, but the important case is when $|M| \geq 2$.) Let k be a symbol for a new line. Let $D = \{\ell \in L_0 : p \not\leq_0 \ell \text{ for all } p \in M\}$. For each line $\ell \in D$, introduce a new point q_ℓ. Define $\Pi' = \langle P_1, L_1, \leq_1 \rangle$ in the following way:

(1) $P_1 = P_0 \cup \{q_\ell : \ell \in D\}$,
(2) $L_1 = L_0 \cup \{k\}$, and
(3) $\leq_1$ consists of $\leq_0$, the relations $m \leq_1 k$ for all $m \in M$, and for each $\ell \in D$ the relations $q_\ell \leq_1 \ell$ and $q_\ell \leq_1 k$.

We will denote the extension Π' obtained thusly by $\alpha_M(\Pi)$. We refer to the construction as a *one line extension*, or the one-line extension of Π determined by M.

Theorem 4. *If Π is a semiplane and M is a set of points whose pairwise join is undefined in Π, then $\alpha_M(\Pi)$ is a semiplane with $\Pi \sqsubseteq \alpha_M(\Pi)$. Moreover, let Σ be a semiplane with $\Pi \sqsubseteq \Sigma$, where $\Pi = \langle P_0, L_0, \leq_0 \rangle$ and $\Sigma = \langle P, L, \leq \rangle$. Let $k \in L - L_0$, and let $M = \{p \in P_0 : p \leq k\}$. Then the subsemiplane $\Pi_{(L_0 \cup \{k\})}$ of Σ is isomorphic to $\alpha_M(\Pi)$.*

Let Π and Σ be two semiplanes. We say that Σ *properly extends* Π if $\Pi \sqsubseteq \Sigma$ and Σ has at least one line not in Π.

Corollary 5. *Let Π be a semiplane. Every semiplane properly extending Π can be obtained as the union of a (possibly infinite) sequence of one line extensions starting with Π. Every projective plane that is a minimal extension of Π (i.e., does not have a proper subplane containing Π) can be obtained as the union of a (possibly infinite) sequence of one line extensions with $|M| \geq 2$.*

The construction of $\alpha_M(\Pi)$ depends, of course, on the choice of M. For example, by always taking $|M| = 2$ we can obtain a sequence whose union is the projective plane freely generated by Π, as constructed by M. Hall [7].

Now let us look at some changes in parameters affected by a one line extension $\Pi' = \alpha_M(\Pi)$, with the notation as above.

(1) $|P_1| = |P_0| + |D|$.
(2) $|L_1| = |L_0| + 1$.

Since we have to consider both Π and Π', define the *rank* $r_\Pi(p)$ of a point $p \in P_0$ by $r_\Pi(p) = |\{\ell \in L_0 : p \leq_0 \ell\}|$, and define $r_{\Pi'}(p)$, $r_\Pi(\ell)$, $r_{\Pi'}(\ell)$ similarly.

(3) For $p \in P_0$,

$$r_{\Pi'}(p) = \begin{cases} r_\Pi(p) + 1 & \text{if } p \in M \\ r_\Pi(p) & \text{if } p \notin M \end{cases}$$

and $r_{\Pi'}(q_\ell) = 2$ for $\ell \in D$.

(4) For $\ell \in L_0$,

$$r_{\Pi'}(\ell) = \begin{cases} r_\Pi(\ell) + 1 & \text{if } \ell \in D \\ r_\Pi(\ell) & \text{if } \ell \notin D \end{cases}$$

and $r_{\Pi'}(k) = |M| + |D|$.

(5) The new joins that are defined in Π' are

$$\begin{aligned} s \vee t &= k && \text{for distinct } s, t \in M, \\ q_\ell \vee t &= k && \text{for } \ell \in D \text{ and } t \in M, \\ q_\ell \vee q_m &= k && \text{for distinct } \ell, m \in D, \text{ and} \\ q_\ell \vee u &= \ell && \text{for } u \leq \ell \in D. \end{aligned}$$

Finally, given $\Pi \sqsubseteq \Sigma$, we would like to know when there is a sequence of one line extensions from Π to Σ with $|M| \geq 2$ for every step. This cannot be done, for example, if Π is a subplane of the projective plane Σ. We need that at each step the new line should intersect the existing lines in at least two existing points, i.e., there are at least two existing points whose join is undefined. The following result gives a sufficient condition for this to occur.

Theorem 6. *Let* $\Pi \sqsubseteq \Sigma$*, where* Π *is a semiplane and* Σ *is a projective plane of order* n*. If* Π *is not a near pencil or a (sub)plane, then there is a sequence of one line extensions from* Π *to* Σ *with* $|M| \geq 2$ *for every step.*

After a one line extension $\Pi' = \alpha_M(\Pi)$, adding a new line k, we have

$$\begin{aligned}\Delta\rho_n = \rho_n(\Pi') - \rho_n(\Pi) &= r_{\Pi'}(k) + \Big(\sum_{\ell \in L_0} r_{\Pi'}(\ell) - r_{\Pi}(\ell)\Big) - |D| - (n+1)\\ &= r_{\Pi'}(k) + |D| - |D| - (n+1)\\ &= r_{\Pi'}(k) - (n+1).\end{aligned}$$

Thus $\Delta\rho_n$ is nonpositive, i.e., ρ_n is always decreasing, so long as we keep $r_{\Pi'}(k) = |M| + |D|$ at most $n+1$. Moreover, if $\rho_n(\Pi) = 0$ then every one-line extension that could possibly lead to a projective plane of order n must have its new line satisfying $r_{\Pi'}(k) = n+1$. So if the semiplane Π is not a projective plane, and $\rho_n(\Pi) = 0$, and $|M| + |D| \neq n+1$ for every choice of M, then Π cannot be extended to a projective plane of order n.

In practice, we will be trying to extend a relatively small initial semiplane Π_0 to a projective plane Σ of order n by a sequence of one line extensions. We may start with $\rho_n(\Pi_0)$ relatively large, but soon enough we will reach extensions $\Pi_j \sqsupseteq \Pi_0$ with $\rho_n(\Pi_j) = 0$. Thus in fact *most* of the one line extensions in the sequence must have $\Delta\rho_n = 0$.

Note. The function ρ_n in the condition $\rho_n(\Pi) \geq 0$ depends on n, and for small values it may be decreasing in n. Thus a semiplane Π may be extendible to a plane of order $m < n$ even though $\rho_n(\Pi) < 0$, and indeed this does occur. Let A_4 be the affine plane of order 4. Clearly A_4 can be extended to the projective plane of order 4, but $\rho_5(A_4) = -25$.

5. The Algorithm

We can now outline the algorithm underlying our program to construct non-Desarguesian projective planes of a given finite order n.

1: Read the input of the order of the projective plane and the initial semiplane.

2: Find all choices of M that keep
  (a) $\rho_n \geq 0$,
  (b) $r(k) \leq n+1$, where k is the new line,
  (c) $r(\ell) \leq n+1$ for all lines ℓ in the semiplane,
  (d) $r(p) \leq n+1$ for all points p in the semiplane, and
  (e) the total number of points is no more than n^2+n+1.

If the number of choices is zero, then go to Step 3; otherwise, pick a choice and go to Step 4.

3: Remove the last line added, and adjust all the parameters. If possible, pick the next choice in line and go to Step 4. If there are no more choices, then we have the following two cases:
  3a: If we are not back to the original configuration, repeat Step 3.
  3b: Otherwise, all the choices have been tested, and there is no projective plane. Output the conclusion that the semiplane cannot be extended to a projective plane of order n, and stop the program.

4: Add the line, and adjust all the parameters. If the number of lines is $n^2 + n + 1$, then we have obtained a projective plane. Output the plane and stop the program; otherwise go to Step 2.

In principle, this is a finite algorithm. However, even for relatively small orders, the number of choices for M at each stage can be large. Hence, it is not practical to run the program straight through as described.

With a Desargues configuration as the initial input, this program does produce the Desarguesian projective planes of order 3, 4, 5 and 7. When we tried to extend a semiplane containing a non-Desarguesian configuration to a plane of order 9, there are simply too many choices for M at each stage. But with a little help from us, which means that we directed the computer to make certain choices for a few steps, the program was able to produce a Hall plane of that order. For orders 11 and higher, there are a lot more choices for M, making an exhaustive search out of the question. Thus we resorted to making random choices for the first twenty or so lines to be added. In the spirit of genetic algorithms, those sequences that initially produced semiplanes with a relatively large number of lines (depending on the prescribed order) were then subjected to longer test runs.

The results of our attempts to construct non-Desarguesian projective planes of order 11, 12, 13 and 15 are described in the last section before the Appendix. (There is no plane of order 14 by the Bruck-Ryser theorem [3].)

6. Initial non-Desarguesian Configurations

Given a non-Desarguesian configuration, there are two basic ways to extend it to a semiplane:

(1) Make lines go through existing points; or
(2) Make pairwise nonconcurrent lines meet at new points.

Figure 1 illustrates the basic non-Desarguesian configuration, that is, two triangles are centrally perspective from a point but not axially perspective from a line, and establishes our notation.

Based on the non-Desarguesian configuration of Figure 1, Table 1 lists the meets of all the lines. A question mark indicates that the meet is undefined. Table 2 gives the details of which lines pass through which point. By using these two tables, we can easily check whether method one or two for our extension would work for certain lines. For instance, from Table 2 we know

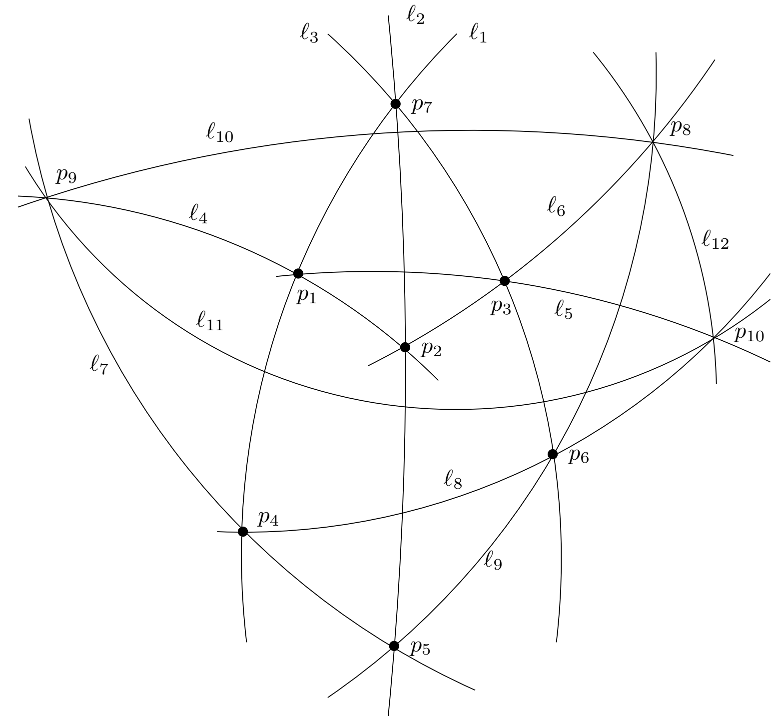

FIGURE 1. Non-Desarguesian configuration.

TABLE 1. Intersection of lines.

$\wedge$	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}
l_1		p_7	p_7	p_1	p_1	?	p_4	p_4	?	?	?	?
l_2	p_7		p_7	p_2	?	p_2	p_5	?	p_5	?	?	?
l_3	p_7	p_7		?	p_3	p_3	?	p_6	p_6	?	?	?
l_4	p_1	p_2	?		p_1	p_2	p_9	?	?	p_9	p_9	?
l_5	p_1	?	p_3	p_1		p_3	?	p_{10}	?	?	p_{10}	p_{10}
l_6	?	p_2	p_3	p_2	p_3		?	?	p_8	p_8	?	p_8
l_7	p_4	p_5	?	p_9	?	?		p_4	p_5	p_9	p_9	?
l_8	p_4	?	p_6	?	p_{10}	?	p_4		p_6	?	p_{10}	p_{10}
l_9	?	p_5	p_6	?	?	p_8	p_5	p_6		p_8	?	p_8
l_{10}	?	?	?	p_9	?	p_8	p_9	?	p_8		p_9	p_8
l_{11}	?	?	?	p_9	p_{10}	?	p_9	p_{10}	?	p_9		p_{10}
l_{12}	?	?	?	?	p_{10}	p_8	?	p_{10}	p_8	p_8	p_{10}	

TABLE 2. Lines through each point.

Point	Lines
p_1	l_1 l_4 l_5
p_2	l_2 l_4 l_6
p_3	l_3 l_5 l_6
p_4	l_1 l_7 l_8
p_5	l_2 l_7 l_9

Point	Lines
p_6	l_3 l_8 l_9
p_7	l_1 l_2 l_3
p_8	l_6 l_9 l_{10} l_{12}
p_9	l_4 l_7 l_{10} l_{11}
p_{10}	l_5 l_8 l_{11} l_{12}

that p_1 is on l_1, l_4 and l_5. So we can only make a line, which does not intersect any of those lines, pass through p_1. In other words, we need to see question marks under those lines on some row of Table 1. Now if we check Table 1 carefully, we know that l_9 would be our only choice. We would like to note here that no more than three lines can meet at a new point because any set of four lines contains at least one intersecting pair. The following is a complete list of all possible ways to extend the non-Desarguesian configuration of Figure 1. When method one is applied, we group them according to the rank of the points, those with rank 4 and those with rank 3. We further separate the latter ones into three smaller groups depending on whether they are on the first triangle p_1, p_2, p_3 or the second triangle p_4, p_5, p_6 or neither.

I.) Make a line go through a point of rank 4.
- 1.) $l_1 \longrightarrow p_8$
- 2.) $l_2 \longrightarrow p_{10}$
- 3.) $l_3 \longrightarrow p_9$

II.) Make a line go through a point of rank 3.
- 1.) Points on the second triangle.
  - a.) $l_4 \longrightarrow p_6$
  - b.) $l_5 \longrightarrow p_5$
  - c.) $l_6 \longrightarrow p_4$
- 2.) Points on the first triangle.
  - a.) $l_7 \longrightarrow p_3$
  - b.) $l_8 \longrightarrow p_2$
  - c.) $l_9 \longrightarrow p_1$
- 3.) Point on neither of the triangles.
  - a.) $l_{10} \longrightarrow p_7$
  - b.) $l_{11} \longrightarrow p_7$
  - c.) $l_{12} \longrightarrow p_7$

III.) Make three pairwise nonconcurrent lines meet at a new point.
- 1.) l_1, l_6, l_{11}
- 2.) l_1, l_9, l_{11}
- 3.) l_2, l_5, l_{10}
- 4.) l_2, l_8, l_{10}

5.) l_3, l_4, l_{12}
6.) l_3, l_7, l_{12}

IV.) Make two nonconcurrent lines meet at a new point.

Since in a projective plane any two points lie on exactly one line, and any two lines intersect in exactly one point, some methods from group I, II and III may not be used simultaneously for our extension. For instance, method I(1) cannot go with method II(1)(c), for otherwise $p_8 = l_1 \wedge l_6 = p_4$, a contradiction. Note that method IV can be used at any time of the extension, and it is the only method which can be used repeatedly. The following list shows how meets will be defined if a method is applied.

I(1): $l_1 \wedge l_6 = l_1 \wedge l_9 = l_1 \wedge l_{10} = l_1 \wedge l_{12} = p_8$
I(2): $l_2 \wedge l_5 = l_2 \wedge l_8 = l_2 \wedge l_{11} = l_2 \wedge l_{12} = p_{10}$
I(3): $l_3 \wedge l_4 = l_3 \wedge l_7 = l_3 \wedge l_{10} = l_3 \wedge l_{11} = p_9$
II(1)(a): $l_3 \wedge l_4 = l_4 \wedge l_8 = l_4 \wedge l_9 = p_6$
II(1)(b): $l_2 \wedge l_5 = l_5 \wedge l_7 = l_5 \wedge l_9 = p_5$
II(1)(c): $l_1 \wedge l_6 = l_6 \wedge l_7 = l_6 \wedge l_8 = p_4$
II(2)(a): $l_3 \wedge l_7 = l_5 \wedge l_7 = l_6 \wedge l_7 = p_3$
II(2)(b): $l_2 \wedge l_8 = l_4 \wedge l_8 = l_6 \wedge l_8 = p_2$
II(2)(c): $l_1 \wedge l_9 = l_4 \wedge l_9 = l_5 \wedge l_9 = p_1$
II(3)(a): $l_1 \wedge l_{10} = l_2 \wedge l_{10} = l_3 \wedge l_{10} = p_7$
II(3)(b): $l_1 \wedge l_{11} = l_2 \wedge l_{11} = l_3 \wedge l_{11} = p_7$
II(3)(c): $l_1 \wedge l_{12} = l_2 \wedge l_{12} = l_3 \wedge l_{12} = p_7$
III(1): $l_1 \wedge l_6 \wedge l_{11} =$ Some new point
III(2): $l_1 \wedge l_9 \wedge l_{11} =$ Some new point
III(3): $l_2 \wedge l_5 \wedge l_{10} =$ Some new point
III(4): $l_2 \wedge l_8 \wedge l_{10} =$ Some new point
III(5): $l_3 \wedge l_4 \wedge l_{12} =$ Some new point
III(6): $l_3 \wedge l_7 \wedge l_{12} =$ Some new point

Using the list above, we obtain the table of inconsistency which gives a complete list of all the conflicts among the methods.

Lemma 7. *Altogether at most six applications of methods from groups I, II(1), II(2), II(3) and III can be made.*

Proof. If we are to choose seven or more methods from the five groups, not necessarily from all five, then we are forced to take at least two methods from at least one of the five groups. Note that no more than one method may be chosen from group II(3). Also note that only up to three methods may be taken from group III at a time.

Case I:

If we take any two methods from group I, then by Table 3, we know that we may possibly pick the remaining one from the same group, one method from group II(1), one method from group II(2), and two methods from group III. But the two methods from group III don't go together. Therefore, we may take no more than six methods in this case.

Table 3. Table of inconsistency.

I(1)	II(1)(c) II(2)(c) II(3)(a) II(3)(c) III(1) III(2)
I(2)	II(1)(b) II(2)(b) II(3)(b) II(3)(c) III(3) III(4)
I(3)	II(1)(a) II(2)(a) II(3)(a) II(3)(b) III(5) III(6)
II(1)(a)	I(3) II(2)(b) II(2)(c) III(5)
II(1)(b)	I(2) II(2)(a) II(2)(c) III(3)
II(1)(c)	I(1) II(2)(a) II(2)(b) III(1)
II(2)(a)	I(3) II(1)(b) II(1)(c) III(6)
II(2)(b)	I(2) II(1)(a) II(1)(c) III(4)
II(2)(c)	I(1) II(1)(a) II(1)(b) III(2)
II(3)(a)	I(1) I(3) II(3)(b) II(3)(c) III(3) III(4)
II(3)(b)	I(2) I(3) II(3)(a) II(3)(c) III(1) III(2)
II(3)(c)	I(1) I(2) II(3)(a) II(3)(b) III(5) III(6)
III(1)	I(1) II(1)(c) II(3)(b) III(2)
III(2)	I(1) II(2)(c) II(3)(b) III(1)
III(3)	I(2) II(1)(b) II(3)(a) III(4)
III(4)	I(2) II(2)(b) II(3)(a) III(3)
III(5)	I(3) II(1)(a) II(3)(c) III(6)
III(6)	I(3) II(2)(a) II(3)(c) III(5)

For example, if we take I(1) and I(2), then Table 3 suggests that I(3), II(1)(a), II(2)(a), III(5) and III(6) are the possible choices to add. But III(5) and III(6) are inconsistent, hence we may only take up to six methods for this case.

Case II:

If we take any two methods from group II(1), then we may possibly take the remaining one from the same group, one method from group I, one method from group II(3), and three methods from group III.

a.) If we take the one from group I, then we may take up to two from group III with or without taking one from group II(3).
b.) If we take the remaining one from the same group, then we can either take up to three methods from group III, or take one from group II(3) together with no more than two from group III.
c.) If we do not take the possible one from group I and the remaining one from the same group, then we can take up to a total of three from group II(3) and III.

In any case, we may only take a maximum of six methods.

For example, suppose we take II(1)(a) and II(1)(b). Then I(1), II(1)(c), any one from group II(3), III(1), III(2), III(4), and III(6) are the possible choices from which we may pick. If we choose I(1), then we cannot pick II(1)(c), II(3)(a), II(3)(c), III(1) and III(2). So now we are left with II(3)(b), III(4) and III(6) as our possible choices. Table 3 suggests that II(3)(b) has no conflicts with III(4) or III(6). Therefore we may pick all three of them, which gives us a total of six methods. If we choose II(1)(c), which is the remaining method from the same group, then I(1) and III(1) are eliminated from our list. Since II(3)(a) does not go with III(4), II(3)(b) does not go with III(2) and II(3)(c) does not go with III(6), we can either take III(2), III(4) and III(6) all at once, or we can take only two of those and one from group II(3). Either case gives a total of six methods. Suppose we ignore both I(1) and II(1)(c). Since no more than three can be chosen from group III, again we are left with only six choices.

Case III:

If we take any two methods from group II(2), then a similar argument as in Case II will lead to the same conclusion.

Case IV:

If we take any two from group III, then only one method from group I, four methods in total from group II(1) and II(2), one method from group II(3), and two methods from group III could possibly be added. But the two methods from group III are inconsistent, and none of them is consistent with either the one from group I or the one from group II(3). We may choose up to three of the four methods from group II(1) and II(2) at a time, but if we choose the one method from group I, then only up to two of those methods are possible. Therefore only up to six methods may be taken at a time in this case.

For example, suppose we take III(1) and III(3), then I(3), II(1)(a), II(2)(a), II(2)(b), II(2)(c), II(3)(c), III(5) and III(6) are our possible choices to add. Notice that III(5) and III(6) are inconsistent, and both of them are inconsistent with I(3) and II(3)(c). Also notice that we may choose II(2)(a), II(2)(b) and II(2)(c) out of the four methods from group II(1) and II(2), but if we choose I(3), then we may only take II(2)(b) and II(2)(c). Therefore we may take only up to four methods from the list which implies a maximum of six methods for this case.

□

Theorem 8. *There are* 875 *ways to extend the non-Desarguesian configuration of Figure* 1 *to a semiplane without adding a new line.*

The proof of this proposition is a straightforward case analysis using our table of inconsistency and the above lemma. Rather than list all the cases at this time, let us consider the automorphisms of the non-Desarguesian configuration and then list them by isomorphism classes.

Theorem 9. *Every automorphism of the non-Desarguesian configuration of Figure* 1 *fixes* p_7*.*

Proof. Since p_7 has rank 3, it could possibly be mapped only to one of the points from p_1 to p_7. Note that every point on the lines passing through p_7 has rank 3, but there always exists a point of rank 4 on at least one of the lines passing through any of the points from p_1 to p_6. Therefore, any automorphism of the non-Desarguesian configuration of Figure 1 must map p_7 to p_7. □

Corollary 10. *There are 12 automorphisms of the non-Desarguesian configuration of Figure* 1*.*

Proof. Since any automorphism will fix p_7, we may only map l_1, l_2, and l_3 into themselves. This implies that we may map p_1 to itself, p_2, p_3, p_4, p_5, or p_6. And under each of the cases mentioned, there are two choices to map p_2 to. Therefore we obtain 12 automorphisms. Table 4 gives all the details. □

Among the 875 extensions, some of them may be isomorphic. Table 5 shows how the methods change when automorphisms are applied.

Theorem 11. *The non-Desarguesian configuration of Figure* 1 *can be extended to 105 semiplanes up to isomorphism without adding a new line.*

When we apply the automorphisms to the semiplanes listed in Theorem 8, we obtain a list of all 105 isomorphism classes, which is given in the Appendix. In this list, we provide two additional properties about the semiplanes in each isomorphism class, which are number of points in the semiplanes and the ρ value which is computed with $n = 12$. In summary, one semiplane, with 37 points, forms one class by itself, 6 semiplanes with 35 points form one class, 9 semiplanes with 34 points form 2 classes, 15 semiplanes with 33 points form 3 classes, 42 semiplanes with 32 points form 4 classes, 47 semiplanes with 31 points form 7 classes, 75 semiplanes with 30 points form 9 classes, 111 semiplanes with 29 points form 12 classes, 95 semiplanes with 28 points form 11 classes, 126 semiplanes with 27 points form 15 classes, 120 semiplanes with 26 points form 12 classes, 79 semiplanes with 25 points form 9 classes, 66 semiplanes with 24 points form 9 classes, 45 semiplanes with 23 points form 5 classes, 26 semiplanes with 22 points form 3 classes, 6 semiplanes with 21 points form one class and 6 semiplanes with 20 points form one class. In the list, the parenthetical number(s) following a semiplane indicate(s) the automorphism(s) which will map the first semiplane in the class to that semiplane. Note that semiplanes with the same number of points may not have the same ρ value. Also note that a semiplane with fewer points does not necessarily have a smaller ρ value than a semiplane with more points.

TABLE 4. Automorphisms.

$\star$	p_7	p_1	p_2	p_3	p_4	p_5	p_6	p_8	p_9	p_{10}	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}
1	p_7	p_1	p_2	p_3	p_4	p_5	p_6	p_8	p_9	p_{10}	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}
2	p_7	p_1	p_3	p_2	p_4	p_6	p_5	p_8	p_{10}	p_9	l_1	l_3	l_2	l_5	l_4	l_6	l_8	l_7	l_9	l_{12}	l_{11}	l_{10}
3	p_7	p_2	p_1	p_3	p_5	p_4	p_6	p_{10}	p_9	p_8	l_2	l_1	l_3	l_4	l_6	l_5	l_7	l_9	l_8	l_{11}	l_{10}	l_{12}
4	p_7	p_2	p_3	p_1	p_5	p_6	p_4	p_{10}	p_8	p_9	l_2	l_3	l_1	l_6	l_4	l_5	l_9	l_7	l_8	l_{12}	l_{10}	l_{11}
5	p_7	p_3	p_1	p_2	p_6	p_4	p_5	p_9	p_{10}	p_8	l_3	l_1	l_2	l_5	l_6	l_4	l_8	l_9	l_7	l_{11}	l_{12}	l_{10}
6	p_7	p_3	p_2	p_1	p_6	p_5	p_4	p_9	p_8	p_{10}	l_3	l_2	l_1	l_6	l_5	l_4	l_9	l_8	l_7	l_{10}	l_{12}	l_{11}
7	p_7	p_4	p_5	p_6	p_1	p_2	p_3	p_8	p_9	p_{10}	l_1	l_2	l_3	l_7	l_8	l_9	l_4	l_5	l_6	l_{10}	l_{11}	l_{12}
8	p_7	p_4	p_6	p_5	p_1	p_3	p_2	p_8	p_{10}	p_9	l_1	l_3	l_2	l_8	l_7	l_9	l_5	l_4	l_6	l_{12}	l_{11}	l_{10}
9	p_7	p_5	p_4	p_6	p_2	p_1	p_3	p_{10}	p_9	p_8	l_2	l_1	l_3	l_7	l_9	l_8	l_4	l_6	l_5	l_{11}	l_{10}	l_{12}
10	p_7	p_5	p_6	p_4	p_2	p_3	p_1	p_{10}	p_8	p_9	l_2	l_3	l_1	l_9	l_7	l_8	l_6	l_4	l_5	l_{12}	l_{10}	l_{11}
11	p_7	p_6	p_4	p_5	p_3	p_1	p_2	p_9	p_{10}	p_8	l_3	l_1	l_2	l_8	l_9	l_7	l_5	l_6	l_4	l_{11}	l_{12}	l_{10}
12	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_9	p_8	p_{10}	l_3	l_2	l_1	l_9	l_8	l_7	l_6	l_5	l_4	l_{10}	l_{12}	l_{11}

Table 5. Changes in methods corresponding to automorphisms.

	1	2	3	4	5	6	7	8	9	10	11	12
I(1)	I(1)	I(1)	I(2)	I(2)	I(3)	I(3)	I(1)	I(1)	I(2)	I(2)	I(3)	I(3)
I(2)	I(2)	I(3)	I(1)	I(3)	I(1)	I(2)	I(2)	I(3)	I(1)	I(3)	I(1)	I(2)
I(3)	I(3)	I(2)	I(3)	I(1)	I(2)	I(1)	I(3)	I(2)	I(3)	I(1)	I(2)	I(1)
II(1)(a)	II(1)(a)	II(1)(b)	II(1)(a)	II(1)(c)	II(1)(b)	II(1)(c)	II(2)(a)	II(2)(b)	II(2)(a)	II(2)(c)	II(2)(b)	II(2)(c)
II(1)(b)	II(1)(b)	II(1)(a)	II(1)(c)	II(1)(a)	II(1)(c)	II(1)(b)	II(2)(b)	II(2)(a)	II(2)(c)	II(2)(a)	II(2)(c)	II(2)(b)
II(1)(c)	II(1)(c)	II(1)(c)	II(1)(b)	II(1)(b)	II(1)(a)	II(1)(a)	II(2)(c)	II(2)(c)	II(2)(b)	II(2)(b)	II(2)(a)	II(2)(a)
II(2)(a)	II(2)(a)	II(2)(b)	II(2)(a)	II(2)(c)	II(2)(b)	II(2)(c)	II(1)(a)	II(1)(b)	II(1)(a)	II(1)(c)	II(1)(b)	II(1)(c)
II(2)(b)	II(2)(b)	II(2)(a)	II(2)(c)	II(2)(a)	II(2)(c)	II(2)(b)	II(1)(b)	II(1)(a)	II(1)(c)	II(1)(a)	II(1)(c)	II(1)(b)
II(2)(c)	II(2)(c)	II(2)(c)	II(2)(b)	II(2)(b)	II(2)(a)	II(2)(a)	II(1)(c)	II(1)(c)	II(1)(b)	II(1)(b)	II(1)(a)	II(1)(a)
II(3)(a)	II(3)(a)	II(3)(c)	II(3)(b)	II(3)(c)	II(3)(b)	II(3)(a)	II(3)(a)	II(3)(c)	II(3)(b)	II(3)(c)	II(3)(b)	II(3)(a)
II(3)(b)	II(3)(b)	II(3)(b)	II(3)(a)	II(3)(a)	II(3)(c)	II(3)(c)	II(3)(b)	II(3)(b)	II(3)(a)	II(3)(a)	II(3)(c)	II(3)(c)
II(3)(c)	II(3)(c)	II(3)(a)	II(3)(c)	II(3)(b)	II(3)(a)	II(3)(b)	II(3)(c)	II(3)(a)	II(3)(c)	II(3)(b)	II(3)(a)	II(3)(b)
III(1)	III(1)	III(1)	III(3)	III(3)	III(5)	III(5)	III(2)	III(2)	III(4)	III(4)	III(6)	III(6)
III(2)	III(2)	III(2)	III(4)	III(4)	III(6)	III(6)	III(1)	III(1)	III(3)	III(3)	III(5)	III(5)
III(3)	III(3)	III(5)	III(1)	III(5)	III(1)	III(3)	III(4)	III(6)	III(2)	III(6)	III(2)	III(4)
III(4)	III(4)	III(6)	III(2)	III(6)	III(2)	III(4)	III(3)	III(5)	III(1)	III(5)	III(1)	III(3)
III(5)	III(5)	III(3)	III(5)	III(1)	III(3)	III(1)	III(6)	III(4)	III(6)	III(2)	III(4)	III(2)
III(6)	III(6)	III(4)	III(6)	III(2)	III(4)	III(2)	III(5)	III(3)	III(5)	III(1)	III(3)	III(1)

7. Further reductions

We have seen that every non-Desarguesian plane contains one of 105 types of initial non-Desarguesian semiplanes. But in fact the non-Desarguesian configuration is never unique, and we can use this to reduce the number of initial semiplanes to consider.

Theorem 12. *Every non-Desarguesian plane contains a configuration in one of the following fifteen classes: 1, 2, 6, 11, 12, 20, 24, 25, 32, 33, 46, 63, 64, 73, 74.*

To prove this, we must study in some detail how one non-Desarguesian configuration gives rise to another.

A *partial non-Desarguesian configuration* is a semiplane that satisfies the conditions of Figure 1, except that only one of the three lines ℓ_{10}, ℓ_{11}, ℓ_{12} need be defined.

Lemma 13. *A partial non-Desarguesian configuration in a projective plane can be extended to a non-Desarguesian configuration.*

Lemma 14. *In a semiplane generated by our standard non-Desarguesian configuration, the following points and lines form a partial non-Desarguesian configuration.*

$$\begin{aligned} x_1 &= p_2 & m_1 &= \ell_4 \\ x_2 &= p_3 & m_2 &= \ell_5 \\ x_3 &= p_7 & m_3 &= \ell_1 \\ x_4 &= p_9 & m_4 &= \ell_6 \\ x_5 &= p_{10} & m_5 &= \ell_2 \\ x_6 &= p_4 & m_6 &= \ell_3 \\ x_7 &= p_1 & m_7 &= \ell_{11} \\ x_8 &= p_6 & m_8 &= \ell_7 \\ x_9 &= \ell_6 \wedge \ell_{11} & m_9 &= \ell_8 \\ x_{10} &= p_5 & m_{12} &= \ell_9 \end{aligned}$$

Lemma 15. *The partial non-Desarguesian configuration $\langle \mathbf{x}, \mathbf{m} \rangle$ satisfies the second condition in the following list if and only if the non-Desarguesian configuration $\langle \mathbf{p}, \ell \rangle$ satisfies the first condition.*

$$\begin{aligned} \mathit{I(1)} &\longrightarrow \mathit{III(5)} \\ \mathit{I(2)} &\longrightarrow \mathit{II(1)(b)} \\ \mathit{I(3)} &\longrightarrow \mathit{II(1)(c)} \\ \mathit{II(1)(a)} &\longrightarrow \mathit{I(1)} \\ \mathit{II(1)(b)} &\longrightarrow \mathit{I(2)} \\ \mathit{II(1)(c)} &\longrightarrow \mathit{II(1)(a)} \end{aligned}$$

II(2)(a) $\longrightarrow$ *II(2)(b)*
II(2)(b) $\longrightarrow$ *II(2)(c)*
II(2)(c) $\longrightarrow$ *II(3)(c)*
II(3)(b) $\longrightarrow$ *II(2)(a)*
III(1) $\longrightarrow$ *I(3)*
III(2) $\longrightarrow$ *III(6)*

We can write conditions for the remaining properties, which are II(3)(a), II(3)(c), III(3), III(4), III(5) and III(6) on the left, and II(3)(a), II(3)(c), III(1), III(2), III(3) and III(4) on the right, but they are not as direct as those in the lemma.

Now let us partition the classes into two types, denoted $\mathcal{S}$ and $\mathcal{T}$. A class belongs to $\mathcal{S}$ if it has one of the following three properties.

(A) The class has no I(x), II(1)(x), II(2)(x), II(3)(x) nor III(x). This applies to class 1 only.
(B) The class contains one I(x) and no II(1)(y) nor II(2)(z). This applies to classes 2, 11, 12, 32, 33, 46, 73, 74.
(C) The class contains two or more I(x)'s. This applies to classes 6, 20, 24, 25, 63, 64.

The type $\mathcal{T}$ then contains all the remaining classes.

It remains to check only that the 90 classes in $\mathcal{T}$ each have a representative that transforms into a configuration satisfying one of (B) or (C). Some classes, *viz.* 4, 13, 41, 42, 47, 48, 82, 85, 86 and 99, require two or more applications of Lemma 15 and automorphisms. This verification is straightforward, if slightly tedious, and we only give details on the hardest cases.

Consider any class containing I(1) and II(1)(a), or equivalently by automorphism (4), I(2) and II(1)(c). Applying Lemma 15 twice to the latter yields I(1) and I(2), which is type (C).

Either one application of the lemma, or the preceding argument, together suffice to cover all but classes 4, 13, 47 and 48.

Class 4(b) has II(3)(b), IV. The lemma transforms this into II(2)(a) and possibly others from among II(3)(a), II(3)(b), III(1), III(2), III(3), III(4). If perchance II(3)(b) is not included, then automorphism (7) transforms this into II(1)(a) and others, still not including II(3)(b). Applying Lemma 15 again yields I(1), and possibly I(3), but no II(1)(x) nor II(2)(y). Thus we get either type (B) or (C). If, on the other hand, II(3)(b) is included at the second stage, then automorphism (9) transforms II(2)(a), II(3)(b) into II(1)(a), II(3)(a), which is inconsistent with II(3)(b), and possibly some III(x)'s. Another application of the lemma then gives (B) or (C).

Class 13(c) is easier. For II(1)(c), II(2)(c) transforms to II(1)(a), II(3)(c), which is inconsistent with II(3)(b). So another application of the lemma yields I(1) and possibly others, that may include I(3) but no II(1)(x), II(2)(y).

Class 47(e) has II(1)(c), II(2)(c) and II(3)(a). This transforms into II(1)(a), II(3)(c) and perhaps others, but not II(3)(b). Another application yields I(1) and maybe others, but no II(1)(x), II(2)(y).

Class 48(a) has II(1)(a), II(2)(a) and II(3)(b). This transforms into I(1), II(2)(b) and perhaps others. If there is no II(3)(b), then automorphism (8) takes this to I(1), II(1)(a) and preserves the lack of a II(3)(b). Then applying Lemma 15 yields I(1) and others, but no II(1)(x), II(2)(y). If, on the other hand, we have I(1), II(2)(b) and II(3)(b), then automorphism (3) changes this to I(2), II(2)(c) and II(3)(a). Applying the lemma gives II(1)(b), II(3)(c) and maybe others, conceivably including I(1). One more application yields I(1) and perhaps others, but no II(1)(x), II(2)(y).

Thus every non-Desarguesian projective plane contains a configuration in one of the 15 classes of Theorem 12.

8. Results of Computations

We have been running various versions of programs based on the one-line extension algorithm to construct projective planes since about the year 2000, usually on several computers simultaneously. Most of the time, the parameters were set to build a plane of order 12, but we have also spent a significant amount of computational time on attempts to construct non-Desarguesian planes of orders 11, 13 and 15.

So far we have done several hundred thousand hours of computer search. In all cases, ρ_n decreases to 0 in just a few steps. At the beginning of our computer search, the initial semiplane was chosen at random from the 105 possibilities. Arbitrarily, we concentrated on types 41, 42, 50, 86, 98 and 105. Theorem 12, which reduces the search to fifteen initial semiplanes, was not found until 2008. As luck would have it, none of these fifteen classes was among those extensively tested during the first nine years!

From all the trials that we have done so far on extending non-Desarguesian semiplanes to a plane of order 12, more than a hundred times we were able to extend the semiplanes from the initial 12 lines to 50 lines, but in only one case so far were we able to obtain 51 lines. In our less extensive investigations of order 11, we have obtained semiplanes with 46 lines; for order 13, semiplanes with 54 lines; for order 15, semiplanes with 57 lines.

In the construction of all the lower order planes, we observed that once the number of lines exceeded roughly one third of the total required, the extension to a projective plane was completed immediately with only one choice for the extension in all remaining steps. For example, this occurred after a semiplane with 29 lines (out of 91) was found while constructing the Hall plane of order 9. While we do not understand the exact nature of this phenomenon (and hope to work on it in the future), it may be that we are not necessarily as far from a plane of order 12 as it may appear.

We also checked that there are no embeddings of one semiplane into another among the 105 isomorphism types. All of them can be extended to a

Hall plane of order 9, except five of them, namely semiplanes 89, 90, 95, 96 and 98. These five types cannot be extended to the dual of that Hall plane either, together with two additional types, semiplanes 93 and 99.

For other results on projective plane of order 12, see Hall and Wilkinson [5] and Suetake [8].

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A. Isomorphism types of non-Desarguesian Semiplanes

Class	Semiplane Isomorphism Types	
1	Semiplane with 37 points and $\rho_{12} = 51$	
	(a) IV	
2	Semiplanes with 33 points and $\rho_{12} = 48$	
	(a) I(1), IV	(c) I(3), IV (5,6,11,12)
	(b) I(2), IV (3,4,9,10)	
3	Semiplanes with 34 points and $\rho_{12} = 49$	
	(a) II(1)(a), IV	(d) II(2)(a), IV (7,9)
	(b) II(1)(b), IV (2,5)	(e) II(2)(b), IV (8,11)
	(c) II(1)(c), IV (4,6)	(f) II(2)(c), IV (10,12)
4	Semiplanes with 34 points and $\rho_{12} = 49$	
	(a) II(3)(a), IV	(c) II(3)(c), IV (2,4,8,10)
	(b) II(3)(b), IV (3,5,9,11)	
5	Semiplanes with 35 points and $\rho_{12} = 50$	
	(a) III(1), IV	(d) III(4), IV (9,10)
	(b) III(2), IV (7,8)	(e) III(5), IV (5,6)
	(c) III(3), IV (3,4)	(f) III(6), IV (11,12)
6	Semiplanes with 29 points and $\rho_{12} = 45$	
	(a) I(1), I(2), IV	(c) I(2), I(3), IV (4,6,10,12)
	(b) I(1), I(3), IV (2,5,8,11)	
7	Semiplanes with 31 points and $\rho_{12} = 47$	
	(a) II(1)(a), II(1)(b), IV	(d) II(2)(a), II(2)(b), IV (7,8)
	(b) II(1)(a), II(1)(c), IV (3,4)	(e) II(2)(a), II(2)(c), IV (9,10)
	(c) II(1)(b), II(1)(c), IV (5,6)	(f) II(2)(b), II(2)(c), IV (11,12)

Class	Semiplane Isomorphism Types	
8	Semiplanes with 33 points and $\rho_{12} = 49$	
	(a) III(1), III(3), IV	(d) III(2), III(6), IV (8,11)
	(b) III(1), III(5), IV (2,5)	(e) III(3), III(5), IV (4,6)
	(c) III(2), III(4), IV (7,9)	(f) III(4), III(6), IV (10,12)
9	Semiplanes with 33 points and $\rho_{12} = 49$	
	(a) III(1), III(4), IV	(d) III(2), III(5), IV (5,8)
	(b) III(1), III(6), IV (2,11)	(e) III(3), III(6), IV (4,12)
	(c) III(2), III(3), IV (3,7)	(f) III(4), III(5), IV (6,10)
10	Semiplanes with 30 points and $\rho_{12} = 46$	
	(a) I(1), II(1)(a), IV	(g) I(2), II(2)(a), IV (9)
	(b) I(1), II(1)(b), IV (2)	(h) I(2), II(2)(c), IV (10)
	(c) I(1), II(2)(a), IV (7)	(i) I(3), II(1)(b), IV (5)
	(d) I(1), II(2)(b), IV (8)	(j) I(3), II(1)(c), IV (6)
	(e) I(2), II(1)(a), IV (3)	(k) I(3), II(2)(b), IV (11)
	(f) I(2), II(1)(c), IV (4)	(l) I(3), II(2)(c), IV (12)
11	Semiplanes with 30 points and $\rho_{12} = 46$	
	(a) I(1), II(3)(b), IV	(c) I(3), II(3)(c), IV (5,6,11,12)
	(b) I(2), II(3)(a), IV (3,4,9,10)	
12	Semiplanes with 31 points and $\rho_{12} = 47$	
	(a) I(1), III(3), IV	(g) I(2), III(5), IV (4)
	(b) I(1), III(4), IV (7)	(h) I(2), III(6), IV (10)
	(c) I(1), III(5), IV (2)	(i) I(3), III(1), IV (5)
	(d) I(1), III(6), IV (8)	(j) I(3), III(2), IV (11)
	(e) I(2), III(1), IV (3)	(k) I(3), III(3), IV (6)

Class	Semiplane Isomorphism Types			
12 (cont.)	(f)	I(2), III(2), IV (9)	(l)	I(3), III(4), IV (12)
13	Semiplanes with 31 points and $\rho_{12} = 47$			
	(a)	II(1)(a), II(2)(a), IV	(c)	II(1)(c), II(2)(c), IV (4,6,10,12)
	(b)	II(1)(b), II(2)(b), IV (2,5,8,11)		
14	Semiplanes with 31 points and $\rho_{12} = 47$			
	(a)	II(1)(a), II(3)(a), IV	(g)	II(2)(a), II(3)(a), IV (7)
	(b)	II(1)(a), II(3)(b), IV (3)	(h)	II(2)(a), II(3)(b), IV (9)
	(c)	II(1)(b), II(3)(b), IV (5)	(i)	II(2)(b), II(3)(b), IV (11)
	(d)	II(1)(b), II(3)(c), IV (2)	(j)	II(2)(b), II(3)(c), IV (8)
	(e)	II(1)(c), II(3)(a), IV (6)	(k)	II(2)(c), II(3)(a), IV (12)
	(f)	II(1)(c), II(3)(c), IV (4)	(l)	II(2)(c), II(3)(c), IV (10)
15	Semiplanes with 31 points and $\rho_{12} = 47$			
	(a)	II(1)(a), II(3)(c), IV	(d)	II(2)(a), II(3)(c), IV (7,9)
	(b)	II(1)(b), II(3)(a), IV (2,5)	(e)	II(2)(b), II(3)(a), IV (8,11)
	(c)	II(1)(c), II(3)(b), IV (4,6)	(f)	II(2)(c), II(3)(b), IV (10,12)
16	Semiplanes with 32 points and $\rho_{12} = 48$			
	(a)	II(1)(a), III(1), IV	(g)	II(2)(a), III(2), IV (7)
	(b)	II(1)(a), III(3), IV (3)	(h)	II(2)(a), III(4), IV (9)
	(c)	II(1)(b), III(1), IV (2)	(i)	II(2)(b), III(2), IV (8)
	(d)	II(1)(b), III(5), IV (5)	(j)	II(2)(b), III(6), IV (11)
	(e)	II(1)(c), III(3), IV (4)	(k)	II(2)(c), III(4), IV (10)
	(f)	II(1)(c), III(5), IV (6)	(l)	II(2)(c), III(6), IV (12)

Class	Semiplane Isomorphism Types			
17	Semiplanes with 32 points and $\rho_{12} = 48$			
	(a)	II(1)(a), III(2), IV	(g)	II(2)(a), III(1), IV (7)
	(b)	II(1)(a), III(4), IV (3)	(h)	II(2)(a), III(3), IV (9)
	(c)	II(1)(b), III(2), IV (2)	(i)	II(2)(b), III(1), IV (8)
	(d)	II(1)(b), III(6), IV (5)	(j)	II(2)(b), III(5), IV (11)
	(e)	II(1)(c), III(4), IV (4)	(k)	II(2)(c), III(3), IV (10)
	(f)	II(1)(c), III(6), IV (6)	(l)	II(2)(c), III(5), IV (12)
18	Semiplanes with 32 points and $\rho_{12} = 48$			
	(a)	II(1)(a), III(6), IV	(d)	II(2)(a), III(5), IV (7,9)
	(b)	II(1)(b), III(4), IV (2,5)	(e)	II(2)(b), III(3), IV (8,11)
	(c)	II(1)(c), III(2), IV (4,6)	(f)	II(2)(c), III(1), IV (10,12)
19	Semiplanes with 32 points and $\rho_{12} = 48$			
	(a)	II(3)(a), III(1), IV	(g)	II(3)(b), III(5), IV (5)
	(b)	II(3)(a), III(2), IV (7)	(h)	II(3)(b), III(6), IV (11)
	(c)	II(3)(a), III(5), IV (6)	(i)	II(3)(c), III(1), IV (2)
	(d)	II(3)(a), III(6), IV (12)	(j)	II(3)(c), III(2), IV (8)
	(e)	II(3)(b), III(3), IV (3)	(k)	II(3)(c), III(3), IV (4)
	(f)	II(3)(b), III(4), IV (9)	(l)	II(3)(c), III(4), IV (10)
20	Semiplane with 25 points and $\rho_{12} = 42$			
	(a)	I(1), I(2), I(3), IV		
21	Semiplanes with 28 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(1)(b), II(1)(c), IV	(b)	II(2)(a), II(2)(b), II(2)(c), IV (7,8,9,10,11,12)

Class	Semiplane Isomorphism Types			
22	Semiplanes with 31 points and $\rho_{12} = 48$			
	(a)	III(1), III(3), III(5), IV	(b)	III(2), III(4), III(6), IV (7–12)
23	Semiplanes with 31 points and $\rho_{12} = 48$			
	(a)	III(1), III(3), III(6), IV	(d)	III(2), III(3), III(5), IV (4,6)
	(b)	III(1), III(4), III(5), IV (2,5)	(e)	III(2), III(3), III(6), IV (8,11)
	(c)	III(1), III(4), III(6), IV (10,12)	(f)	III(2), III(4), III(5), IV (7,9)
24	Semiplanes with 26 points and $\rho_{12} = 43$			
	(a)	I(1), I(2), II(1)(a), IV	(d)	I(1), I(3), II(2)(b), IV (8,11)
	(b)	I(1), I(2), II(2)(a), IV (7,9)	(e)	I(2), I(3), II(1)(c), IV (4,6)
	(c)	I(1), I(3), II(1)(b), IV (2,5)	(f)	I(2), I(3), II(2)(c), IV (10,12)
25	Semiplanes with 27 points and $\rho_{12} = 44$			
	(a)	I(1), I(2), III(5), IV	(d)	I(1), I(3), III(4), IV (8,11)
	(b)	I(1), I(2), III(6), IV (7,9)	(e)	I(2), I(3), III(1), IV (4,6)
	(c)	I(1), I(3), III(3), IV (2,5)	(f)	I(2), I(3), III(2), IV (10,12)
26	Semiplanes with 28 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(1)(b), II(3)(a), IV	(g)	II(2)(a), II(2)(b), II(3)(a), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(c), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(c), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(b), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(b), IV (9)
	(d)	II(1)(a), II(1)(c), II(3)(c), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(c), IV (10)
	(e)	II(1)(b), II(1)(c), II(3)(a), IV (6)	(k)	II(2)(b), II(2)(c), II(3)(a), IV (12)
	(f)	II(1)(b), II(1)(c), II(3)(b), IV (5)	(l)	II(2)(b), II(2)(c), II(3)(b), IV (11)
27	Semiplanes with 28 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(1)(b), II(3)(b), IV	(d)	II(2)(a), II(2)(b), II(3)(b), IV (7,8)

Class	Semiplane Isomorphism Types			
27	(b)	II(1)(a), II(1)(c), II(3)(a), IV (3,4)	(e)	II(2)(a), II(2)(c), II(3)(a), IV (9,10)
(cont.)	(c)	II(1)(b), II(1)(c), II(3)(c), IV (5,6)	(f)	II(2)(b), II(2)(c), II(3)(c), IV (11,12)
28	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(1)(b), III(1), IV	(d)	II(2)(a), II(2)(b), III(2), IV (7,8)
	(b)	II(1)(a), II(1)(c), III(3), IV (3,4)	(e)	II(2)(a), II(2)(c), III(4), IV (9,10)
	(c)	II(1)(b), II(1)(c), III(5), IV (5,6)	(f)	II(2)(b), II(2)(c), III(6), IV (11,12)
29	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(1)(b), III(2), IV	(d)	II(2)(a), II(2)(b), III(1), IV (7,8)
	(b)	II(1)(a), II(1)(c), III(4), IV (3,4)	(e)	II(2)(a), II(2)(c), III(3), IV (9,10)
	(c)	II(1)(b), II(1)(c), III(6), IV (5,6)	(f)	II(2)(b), II(2)(c), III(5), IV (11,12)
30	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(1)(b), III(4), IV	(g)	II(2)(a), II(2)(b), III(3), IV (7)
	(b)	II(1)(a), II(1)(b), III(6), IV (2)	(h)	II(2)(a), II(2)(b), III(5), IV (8)
	(c)	II(1)(a), II(1)(c), III(2), IV (3)	(i)	II(2)(a), II(2)(c), III(1), IV (9)
	(d)	II(1)(a), II(1)(c), III(6), IV (4)	(j)	II(2)(a), II(2)(c), III(5), IV (10)
	(e)	II(1)(b), II(1)(c), III(2), IV (5)	(k)	II(2)(b), II(2)(c), III(1), IV (11)
	(f)	II(1)(b), II(1)(c), III(4), IV (6)	(l)	II(2)(b), II(2)(c), III(3), IV (12)
31	Semiplanes with 27 points and $\rho_{12} = 44$			
	(a)	I(1), II(1)(a), II(1)(b), IV	(d)	I(2), II(2)(a), II(2)(c), IV (9,10)
	(b)	I(1), II(2)(a), II(2)(b), IV (7,8)	(e)	I(3), II(1)(b), II(1)(c), IV (5,6)
	(c)	I(2), II(1)(a), II(1)(c), IV (3,4)	(f)	I(3), II(2)(b), II(2)(c), IV (11,12)
32	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	I(1), III(3), III(5), IV	(d)	I(2), III(2), III(6), IV (9,10)

Class	Semiplane Isomorphism Types			
32	(b)	I(1), III(4), III(6), IV (7,8)	(e)	I(3), III(1), III(3), IV (5,6)
(cont.)	(c)	I(2), III(1), III(5), IV (3,4)	(f)	I(3), III(2), III(4), IV (11,12)
33	Semiplanes with 29 and $\rho_{12} = 46$			
	(a)	I(1), III(3), III(6), IV	(d)	I(2), III(2), III(5), IV (4,9)
	(b)	I(1), III(5), III(4), IV (2,7)	(e)	I(3), III(1), III(4), IV (5,12)
	(c)	I(2), III(1), III(6), IV (3,10)	(f)	I(3), III(2), III(3), IV (6,11)
34	Semiplanes with 30 points and $\rho_{12} = 45$			
	(a)	II(1)(a), III(1), III(3), IV	(d)	II(2)(a), III(2), III(4), IV (7,9)
	(b)	II(1)(b), III(1), III(5), IV (2,5)	(e)	II(2)(b), III(2), III(6), IV (8,11)
	(c)	II(1)(c), III(3), III(5), IV (4,6)	(f)	II(2)(c), III(4), III(6), IV (10,12)
35	Semiplanes with 30 points and $\rho_{12} = 47$			
	(a)	II(1)(a), III(1), III(4), IV	(g)	II(2)(a), III(1), III(4), IV (9)
	(b)	II(1)(a), III(2), III(3), IV (3)	(h)	II(2)(a), III(2), III(3), IV (7)
	(c)	II(1)(b), III(1), III(6), IV (2)	(i)	II(2)(b), III(1), III(6), IV (11)
	(d)	II(1)(b), III(2), III(5), IV (5)	(j)	II(2)(b), III(2), III(5), IV (8)
	(e)	II(1)(c), III(3), III(6), IV (4)	(k)	II(2)(c), III(3), III(6), IV (12)
	(f)	II(1)(c), III(4), III(5), IV (6)	(l)	II(2)(c), III(4), III(5), IV (10)
36	Semiplanes with 30 points and $\rho_{12} = 47$			
	(a)	II(1)(a), III(1), III(6), IV	(g)	II(2)(a), III(2), III(5), IV (7)
	(b)	II(1)(a), III(3), III(6), IV (3)	(h)	II(2)(a), III(4), III(5), IV (9)
	(c)	II(1)(b), III(1), III(4), IV (2)	(i)	II(2)(b), III(2), III(3), IV (8)
	(d)	II(1)(b), III(4), III(5), IV (5)	(j)	II(2)(b), III(3), III(6), IV (11)
	(e)	II(1)(c), III(2), III(3), IV (4)	(k)	II(2)(c), III(1), III(4), IV (10)

Class		Semiplane Isomorphism Types		
36 (cont.)	(f)	II(1)(c), III(2), III(5), IV (6)	(l)	II(2)(c), III(1), III(6), IV (12)
37		Semiplanes with 30 points and $\rho_{12} = 47$		
	(a)	II(1)(a), III(2), III(4), IV	(d)	II(2)(a), III(1), III(3), IV (7,9)
	(b)	II(1)(b), III(2), III(6), IV (2,5)	(e)	II(2)(b), III(1), III(5), IV (8,11)
	(c)	II(1)(c), III(4), III(6), IV (4,6)	(f)	II(2)(c), III(3), III(5), IV (10,12)
38		Semiplanes with 30 points and $\rho_{12} = 47$		
	(a)	II(1)(a), III(2), III(6), IV	(g)	II(2)(a), III(1), III(5), IV (7)
	(b)	II(1)(a), III(4), III(6), IV (3)	(h)	II(2)(a), III(3), III(5), IV (9)
	(c)	II(1)(b), III(2), III(4), IV (2)	(i)	II(2)(b), III(1), III(3), IV (8)
	(d)	II(1)(b), III(4), III(6), IV (5)	(j)	II(2)(b), III(3), III(5), IV (11)
	(e)	II(1)(c), III(2), III(4), IV (4)	(k)	II(2)(c), III(1), III(3), IV (10)
	(f)	II(1)(c), III(2), III(6), IV (6)	(l)	II(2)(c), III(1), III(5), IV (12)
39		Semiplanes with 30 points and $\rho_{12} = 47$		
	(a)	II(3)(a), III(1), III(5), IV	(d)	II(3)(b), III(4), III(6), IV (9,11)
	(b)	II(3)(a), III(2), III(6), IV (7,12)	(e)	II(3)(c), III(1), III(3), IV (2,4)
	(c)	II(3)(b), III(3), III(5), IV (3,5)	(f)	II(3)(c), III(2), III(4), IV (8,10)
40		Semiplanes with 30 points and $\rho_{12} = 47$		
	(a)	II(3)(a), III(1), III(6), IV	(d)	II(3)(b), III(4), III(5), IV (5,9)
	(b)	II(3)(a), III(2), III(5), IV (6,7)	(e)	II(3)(c), III(1), III(4), IV (2,10)
	(c)	II(3)(b), III(3), III(6), IV (3,11)	(f)	II(3)(c), III(2), III(3), IV (4,8)
41		Semiplanes with 27 points and $\rho_{12} = 44$		
	(a)	I(1), II(1)(a), II(2)(a), IV	(d)	I(2), II(1)(c), II(2)(c), IV (4,10)

Class	Semiplane Isomorphism Types			
41	(b)	I(1), II(1)(b), II(2)(b), IV (2,8)	(e)	I(3), II(1)(b), II(2)(b), IV (5,11)
(cont.)	(c)	I(2), II(1)(a), II(2)(a), IV (3,9)	(f)	I(3), II(1)(c), II(2)(c), IV (6,12)
42	Semiplanes with 27 points and $\rho_{12} = 44$			
	(a)	I(1), II(1)(a), II(3)(b), IV	(g)	I(2), II(2)(a), II(3)(a), IV (9)
	(b)	I(1), II(1)(b), II(3)(b), IV (2)	(h)	I(2), II(2)(c), II(3)(a), IV (10)
	(c)	I(1), II(2)(a), II(3)(b), IV (7)	(i)	I(3), II(1)(b), II(3)(c), IV (5)
	(d)	I(1), II(2)(b), II(3)(b), IV (8)	(j)	I(3), II(1)(c), II(3)(c), IV (6)
	(e)	I(2), II(1)(a), II(3)(a), IV (3)	(k)	I(3), II(2)(b), II(3)(c), IV (11)
	(f)	I(2), II(1)(c), II(3)(a), IV (4)	(l)	I(3), II(2)(c), II(3)(c), IV (12)
43	Semiplanes with 28 points and $\rho_{12} = 45$			
	(a)	I(1), II(1)(a), III(3), IV	(g)	I(2), II(2)(a), III(2), IV (9)
	(b)	I(1), II(1)(b), III(5), IV (2)	(h)	I(2), II(2)(c), III(6), IV (10)
	(c)	I(1), II(2)(a), III(4), IV (7)	(i)	I(3), II(1)(b), III(1), IV (5)
	(d)	I(1), II(2)(b), III(6), IV (8)	(j)	I(3), II(1)(c), III(3), IV (6)
	(e)	I(2), II(1)(a), III(1), IV (3)	(k)	I(3), II(2)(b), III(2), IV (11)
	(f)	I(2), II(1)(c), III(5), IV (4)	(l)	I(3), II(2)(c), III(4), IV (12)
44	Semiplanes with 28 points and $\rho_{12} = 45$			
	(a)	I(1), II(1)(a), III(4), IV	(g)	I(2), II(2)(a), III(1), IV (9)
	(b)	I(1), II(1)(b), III(6), IV (2)	(h)	I(2), II(2)(c), III(5), IV (10)
	(c)	I(1), II(2)(a), III(3), IV (7)	(i)	I(3), II(1)(b), III(2), IV (5)
	(d)	I(1), II(2)(b), III(5), IV (8)	(j)	I(3), II(1)(c), III(4), IV (6)
	(e)	I(2), II(1)(a), III(2), IV (3)	(k)	I(3), II(2)(b), III(1), IV (11)
	(f)	I(2), II(1)(c), III(6), IV (4)	(l)	I(3), II(2)(c), III(3), IV (12)

Class	Semiplane Isomorphism Types	
45	Semiplanes with 28 points and $\rho_{12} = 45$	
	(a) I(1), II(1)(a), III(6), IV	(g) I(2), II(2)(a), III(5), IV (9)
	(b) I(1), II(1)(b), III(4), IV (2)	(h) I(2), II(2)(c), III(1), IV (10)
	(c) I(1), II(2)(a), III(5), IV (7)	(i) I(3), II(1)(b), III(4), IV (5)
	(d) I(1), II(2)(b), III(3), IV (8)	(j) I(3), II(1)(c), III(2), IV (6)
	(e) I(2), II(1)(a), III(6), IV (3)	(k) I(3), II(2)(b), III(3), IV (11)
	(f) I(2), II(1)(c), III(2), IV (4)	(l) I(3), II(2)(c), III(1), IV (12)
46	Semiplanes with 28 points and $\rho_{12} = 45$	
	(a) I(1), II(3)(b), III(3), IV	(g) I(2), II(3)(a), III(5), IV (4)
	(b) I(1), II(3)(b), III(4), IV (7)	(h) I(2), II(3)(a), III(6), IV (10)
	(c) I(1), II(3)(b), III(5), IV (2)	(i) I(3), II(3)(c), III(1), IV (5)
	(d) I(1), II(3)(b), III(6), IV (8)	(j) I(3), II(3)(c), III(2), IV (11)
	(e) I(2), II(3)(a), III(1), IV (3)	(k) I(3), II(3)(c), III(3), IV (6)
	(f) I(2), II(3)(a), III(2), IV (9)	(l) I(3), II(3)(c), III(4), IV (12)
47	Semiplanes with 28 points and $\rho_{12} = 45$	
	(a) II(1)(a), II(2)(a), II(3)(a), IV	(d) II(1)(b), II(2)(b), II(3)(c), IV (2,8)
	(b) II(1)(a), II(2)(a), II(3)(b), IV (3,9)	(e) II(1)(c), II(2)(c), II(3)(a), IV (6,12)
	(c) II(1)(b), II(2)(b), II(3)(b), IV (5,11)	(f) II(1)(c), II(2)(c), II(3)(c), IV (4,10)
48	Semiplanes with 28 points and $\rho_{12} = 45$	
	(a) II(1)(a), II(2)(a), II(3)(c), IV	(c) II(1)(c), II(2)(c), II(3)(b), IV (4,6,10,12)
	(b) II(1)(b), II(2)(b), II(3)(a), IV (2,5,8,11)	
49	Semiplanes with 29 points and $\rho_{12} = 46$	
	(a) II(1)(a), II(2)(a), III(1), IV	(g) II(1)(b), II(2)(b), III(5), IV (5)

Class	Semiplane Isomorphism Types			
49	(b)	II(1)(a), II(2)(a), III(2), IV (7)	(h)	II(1)(b), II(2)(b), III(6), IV (11)
(cont.)	(c)	II(1)(a), II(2)(a), III(3), IV (3)	(i)	II(1)(c), II(2)(c), III(3), IV (4)
	(d)	II(1)(a), II(2)(a), III(4), IV (9)	(j)	II(1)(c), II(2)(c), III(4), IV (10)
	(e)	II(1)(b), II(2)(b), III(1), IV (2)	(k)	II(1)(c), II(2)(c), III(5), IV (6)
	(f)	II(1)(b), II(2)(b), III(2), IV (8)	(l)	II(1)(c), II(2)(c), III(6), IV (12)
50	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(3)(a), III(1), IV	(g)	II(2)(a), II(3)(a), III(2), IV (7)
	(b)	II(1)(a), II(3)(b), III(3), IV (3)	(h)	II(2)(a), II(3)(b), III(4), IV (9)
	(c)	II(1)(b), II(3)(b), III(5), IV (5)	(i)	II(2)(b), II(3)(b), III(6), IV (11)
	(d)	II(1)(b), II(3)(c), III(1), IV (2)	(j)	II(2)(b), II(3)(c), III(2), IV (8)
	(e)	II(1)(c), II(3)(a), III(5), IV (6)	(k)	II(2)(c), II(3)(a), III(6), IV (12)
	(f)	II(1)(c), II(3)(c), III(3), IV (4)	(l)	II(2)(c), II(3)(c), III(4), IV (10)
51	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(3)(a), III(2), IV	(g)	II(2)(a), II(3)(a), III(1), IV (7)
	(b)	II(1)(a), II(3)(b), III(4), IV (3)	(h)	II(2)(a), II(3)(b), III(3), IV (9)
	(c)	II(1)(b), II(3)(b), III(6), IV (5)	(i)	II(2)(b), II(3)(b), III(5), IV (11)
	(d)	II(1)(b), II(3)(c), III(2), IV (2)	(j)	II(2)(b), II(3)(c), III(1), IV (8)
	(e)	II(1)(c), II(3)(a), III(6), IV (6)	(k)	II(2)(c), II(3)(a), III(5), IV (12)
	(f)	II(1)(c), II(3)(c), III(4), IV (4)	(l)	II(2)(c), II(3)(c), III(3), IV (10)
52	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(3)(a), III(6), IV	(g)	II(2)(a), II(3)(a), III(5), IV (7)
	(b)	II(1)(a), II(3)(b), III(6), IV (3)	(h)	II(2)(a), II(3)(b), III(5), IV (9)
	(c)	II(1)(b), II(3)(b), III(4), IV (5)	(i)	II(2)(b), II(3)(b), III(3), IV (11)

Class	Semiplane Isomorphism Types			
52	(d)	II(1)(b), II(3)(c), III(4), IV (2)	(j)	II(2)(b), II(3)(c), III(3), IV (8)
(cont.)	(e)	II(1)(c), II(3)(a), III(2), IV (6)	(k)	II(2)(c), II(3)(a), III(1), IV (12)
	(f)	II(1)(c), II(3)(c), III(2), IV (4)	(l)	II(2)(c), II(3)(c), III(1), IV (10)
53	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(3)(c), III(1), IV	(g)	II(2)(a), II(3)(c), III(2), IV (7)
	(b)	II(1)(a), II(3)(c), III(3), IV (3)	(h)	II(2)(a), II(3)(c), III(4), IV (9)
	(c)	II(1)(b), II(3)(a), III(1), IV (2)	(i)	II(2)(b), II(3)(a), III(2), IV (8)
	(d)	II(1)(b), II(3)(a), III(5), IV (5)	(j)	II(2)(b), II(3)(a), III(6), IV (11)
	(e)	II(1)(c), II(3)(b), III(3), IV (4)	(k)	II(2)(c), II(3)(b), III(4), IV (10)
	(f)	II(1)(c), II(3)(b), III(5), IV (6)	(l)	II(2)(c), II(3)(b), III(6), IV (12)
54	Semiplanes with 29 points and $\rho_{12} = 46$			
	(a)	II(1)(a), II(3)(c), III(2), IV	(g)	II(2)(a), II(3)(c), III(1), IV (7)
	(b)	II(1)(a), II(3)(c), III(4), IV (3)	(h)	II(2)(a), II(3)(c), III(3), IV (9)
	(c)	II(1)(b), II(3)(a), III(2), IV (2)	(i)	II(2)(b), II(3)(a), III(1), IV (8)
	(d)	II(1)(b), II(3)(a), III(6), IV (5)	(j)	II(2)(b), II(3)(a), III(5), IV (11)
	(e)	II(1)(c), II(3)(b), III(4), IV (4)	(k)	II(2)(c), II(3)(b), III(3), IV (10)
	(f)	II(1)(c), II(3)(b), III(6), IV (6)	(l)	II(2)(c), II(3)(b), III(5), IV (12)
55	Semiplanes with 25 points and $\rho_{12} = 47$			
	(a)	II(1)(a), II(1)(b), II(1)(c), II(3)(a), IV	(d)	II(2)(a), II(2)(b), II(2)(c), II(3)(a), IV (7,12)
	(b)	II(1)(a), II(1)(b), II(1)(c), II(3)(b), IV (3,5)	(e)	II(2)(a), II(2)(b), II(2)(c), II(3)(b), IV (9,11)
	(c)	II(1)(a), II(1)(b), II(1)(c), II(3)(c), IV (2,4)	(f)	II(2)(a), II(2)(b), II(2)(c), II(3)(c), IV (8,10)
56	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), II(1)(c), III(2), IV	(d)	II(2)(a), II(2)(b), II(2)(c), III(1), IV (7,8)

Class	Semiplane Isomorphism Types			
56	(b)	II(1)(a), II(1)(b), II(1)(c), III(4), IV (3,4)	(e)	II(2)(a), II(2)(b), II(2)(c), III(3), IV (9,10)
(cont.)	(c)	II(1)(a), II(1)(b), II(1)(c), III(6), IV (5,6)	(f)	II(2)(a), II(2)(b), II(2)(c), III(5), IV (11,12)
57	Semiplanes with 28 points and $\rho_{12} = 46$			
	(a)	II(1)(a), III(1), III(3), III(6), IV	(d)	II(2)(a), III(2), III(4), III(5), IV (7,9)
	(b)	II(1)(b), III(1), III(4), III(5), IV (2,5)	(e)	II(2)(b), III(2), III(3), III(6), IV (8,11)
	(c)	II(1)(c), III(2), III(3), III(5), IV (4,6)	(f)	II(2)(c), III(1), III(4), III(6), IV (10,12)
58	Semiplanes with 28 points and $\rho_{12} = 46$			
	(a)	II(1)(a), III(1), III(4), III(6), IV	(g)	II(2)(a), III(1), III(4), III(5), IV (9)
	(b)	II(1)(a), III(2), III(3), III(6), IV (3)	(h)	II(2)(a), III(2), III(3), III(5), IV (7)
	(c)	II(1)(b), III(1), III(4), III(6), IV (2)	(i)	II(2)(b), III(1), III(3), III(6), IV (11)
	(d)	II(1)(b), III(2), III(4), III(5), IV (5)	(j)	II(2)(b), III(2), III(3), III(5), IV (8)
	(e)	II(1)(c), III(2), III(3), III(6), IV (4)	(k)	II(2)(c), III(1), III(3), III(6), IV (12)
	(f)	II(1)(c), III(2), III(4), III(5), IV (6)	(l)	II(2)(c), III(1), III(4), III(5), IV (10)
59	Semiplanes with 28 points and $\rho_{12} = 46$			
	(a)	II(1)(a), III(2), III(4), III(6), IV	(d)	II(2)(a), III(1), III(3), III(5), IV (7,9)
	(b)	II(1)(b), III(2), III(4), III(6), IV (2,5)	(e)	II(2)(b), III(1), III(3), III(5), IV (8,11)
	(c)	II(1)(c), III(2), III(4), III(6), IV (4,6)	(f)	II(2)(c), III(1), III(3), III(5), IV (10,12)
60	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(1)(b), III(1), III(4), IV	(g)	II(2)(a), II(2)(b), III(2), III(3), IV (7)
	(b)	II(1)(a), II(1)(b), III(1), III(6), IV (2)	(h)	II(2)(a), II(2)(b), III(2), III(5), IV (8)
	(c)	II(1)(a), II(1)(c), III(2), III(3), IV (3)	(i)	II(2)(a), II(2)(c), III(1), III(4), IV (9)
	(d)	II(1)(a), II(1)(c), III(3), III(6), IV (4)	(j)	II(2)(a), II(2)(c), III(4), III(5), IV (10)
	(e)	II(1)(b), II(1)(c), III(2), III(5), IV (5)	(k)	II(2)(b), II(2)(c), III(1), III(6), IV (11)
	(f)	II(1)(b), II(1)(c), III(4), III(5), IV (6)	(l)	II(2)(b), II(2)(c), III(3), III(6), IV (12)

Class	Semiplane Isomorphism Types			
61	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(1)(b), III(2), III(4), IV	(g)	II(2)(a), II(2)(b), III(1), III(3), IV (7)
	(b)	II(1)(a), II(1)(b), III(2), III(6), IV (2)	(h)	II(2)(a), II(2)(b), III(1), III(5), IV (8)
	(c)	II(1)(a), II(1)(c), III(2), III(4), IV (3)	(i)	II(2)(a), II(2)(c), III(1), III(3), IV (9)
	(d)	II(1)(a), II(1)(c), III(4), III(6), IV (4)	(j)	II(2)(a), II(2)(c), III(3), III(5), IV (10)
	(e)	II(1)(b), II(1)(c), III(2), III(6), IV (5)	(k)	II(2)(b), II(2)(c), III(1), III(5), IV (11)
	(f)	II(1)(b), II(1)(c), III(4), III(6), IV (6)	(l)	II(2)(b), II(2)(c), III(3), III(5), IV (12)
62	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(1)(b), III(4), III(6), IV	(d)	II(2)(a), II(2)(b), III(3), III(5), IV (7,8)
	(b)	II(1)(a), II(1)(c), III(2), III(6), IV (3,4)	(e)	II(2)(a), II(2)(c), III(1), III(5), IV (9,10)
	(c)	II(1)(b), II(1)(c), III(2), III(4), IV (5,6)	(f)	II(2)(b), II(2)(c), III(1), III(3), IV (11,12)
63	Semiplanes with 23 points and $\rho_{12} = 41$			
	(a)	I(1), I(2), II(1)(a), II(2)(a), IV	(c)	I(2), I(3), II(1)(c), II(2)(c), IV (4,6,10,12)
	(b)	I(1), I(3), II(1)(b), II(2)(b), IV (2,5,8,11)		
64	Semiplanes with 24 points and $\rho_{12} = 42$			
	(a)	I(1), I(2), II(1)(a), III(6), IV	(d)	I(1), I(2), II(2)(a), III(5), IV (7,9)
	(b)	I(1), I(3), II(1)(b), III(4), IV (2,5)	(e)	I(1), I(3), II(2)(b), III(3), IV (8,11)
	(c)	I(2), I(3), II(1)(c), III(2), IV (4,6)	(f)	I(2), I(3), II(2)(c), III(1), IV (10,12)
65	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), II(3)(a), III(1), IV	(g)	II(2)(a), II(2)(b), II(3)(a), III(2), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(c), III(1), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(c), III(2), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(b), III(3), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(b), III(4), IV (9)
	(d)	II(1)(a), II(1)(c), II(3)(c), III(3), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(c), III(4), IV (10)

Class	Semiplane Isomorphism Types			
65	(e)	II(1)(b), II(1)(c), II(3)(a), III(5), IV (6)	(k)	II(2)(b), II(2)(c), II(3)(a), III(6), IV (12)
(cont.)	(f)	II(1)(b), II(1)(c), II(3)(b), III(5), IV (5)	(l)	II(2)(b), II(2)(c), II(3)(b), III(6), IV (11)
66	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), II(3)(a), III(2), IV	(g)	II(2)(a), II(2)(b), II(3)(a), III(1), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(c), III(2), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(c), III(1), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(b), III(4), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(b), III(3), IV (9)
	(d)	II(1)(a), II(1)(c), II(3)(c), III(4), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(c), III(3), IV (10)
	(e)	II(1)(b), II(1)(c), II(3)(a), III(6), IV (6)	(k)	II(2)(b), II(2)(c), II(3)(a), III(5), IV (12)
	(f)	II(1)(b), II(1)(c), II(3)(b), III(6), IV (5)	(l)	II(2)(b), II(2)(c), II(3)(b), III(5), IV (11)
67	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), II(3)(a), III(6), IV	(g)	II(2)(a), II(2)(b), II(3)(a), III(5), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(c), III(4), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(c), III(3), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(b), III(6), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(b), III(5), IV (9)
	(d)	II(1)(a), II(1)(c), II(3)(c), III(2), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(c), III(1), IV (10)
	(e)	II(1)(b), II(1)(c), II(3)(a), III(2), IV (6)	(k)	II(2)(b), II(2)(c), II(3)(a), III(1), IV (12)
	(f)	II(1)(b), II(1)(c), II(3)(b), III(4), IV (5)	(l)	II(2)(b), II(2)(c), II(3)(b), III(3), IV (11)
68	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), II(3)(b), III(4), IV	(g)	II(2)(a), II(2)(b), II(3)(b), III(3), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(b), III(6), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(b), III(5), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(a), III(2), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(a), III(1), IV (9)
	(d)	II(1)(a), II(1)(c), II(3)(a), III(6), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(a), III(5), IV (10)
	(e)	II(1)(b), II(1)(c), II(3)(c), III(2), IV (5)	(k)	II(2)(b), II(2)(c), II(3)(c), III(1), IV (11)
	(f)	II(1)(b), II(1)(c), II(3)(c), III(4), IV (6)	(l)	II(2)(b), II(2)(c), II(3)(c), III(3), IV (12)

Class	Semiplane Isomorphism Types			
69	Semiplanes with 24 points and $\rho_{12} = 42$			
	(a)	I(1), II(1)(a), II(1)(b), II(3)(b), IV	(d)	I(1), II(2)(a), II(2)(b), II(3)(b), IV (7,8)
	(b)	I(2), II(1)(a), II(1)(c), II(3)(a), IV (3,4)	(e)	I(2), II(2)(a), II(2)(c), II(3)(a), IV (9,10)
	(c)	I(3), II(1)(b), II(1)(c), II(3)(c), IV (5,6)	(f)	I(3), II(2)(b), II(2)(c), II(3)(c), IV (11,12)
70	Semiplanes with 25 points and $\rho_{12} = 43$			
	(a)	I(1), II(1)(a), II(1)(b), III(4), IV	(g)	I(1), II(2)(a), II(2)(b), III(3), IV (7)
	(b)	I(1), II(1)(a), II(1)(b), III(6), IV (2)	(h)	I(1), II(2)(a), II(2)(b), III(5), IV (8)
	(c)	I(2), II(1)(a), II(1)(c), III(2), IV (3)	(i)	I(2), II(2)(a), II(2)(c), III(1), IV (9)
	(d)	I(2), II(1)(a), II(1)(c), III(6), IV (4)	(j)	I(2), II(2)(a), II(2)(c), III(5), IV (10)
	(e)	I(3), II(1)(b), II(1)(c), III(2), IV (5)	(k)	I(3), II(2)(b), II(2)(c), III(1), IV (11)
	(f)	I(3), II(1)(b), II(1)(c), III(4), IV (6)	(l)	I(3), II(2)(b), II(2)(c), III(3), IV (12)
71	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	I(1), II(1)(a), III(3), III(6), IV	(g)	I(1), II(2)(a), III(4), III(5), IV (7)
	(b)	I(1), II(1)(b), III(4), III(5), IV (2)	(h)	I(1), II(2)(b), III(3), III(6), IV (8)
	(c)	I(2), II(1)(a), III(1), III(6), IV (3)	(i)	I(2), II(2)(a), III(2), III(5), IV (9)
	(d)	I(2), II(1)(c), III(2), III(5), IV (4)	(j)	I(2), II(2)(c), III(1), III(6), IV (10)
	(e)	I(3), II(1)(b), III(1), III(4), IV (5)	(k)	I(3), II(2)(b), III(2), III(3), IV (11)
	(f)	I(3), II(1)(c), III(2), III(3), IV (6)	(l)	I(3), II(2)(c), III(1), III(4), IV (12)
72	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	I(1), II(1)(a), III(4), III(6), IV	(g)	I(1), II(2)(a), III(3), III(5), IV (7)
	(b)	I(1), II(1)(b), III(4), III(6), IV (2)	(h)	I(1), II(2)(b), III(3), III(5), IV (8)
	(c)	I(2), II(1)(a), III(2), III(6), IV (3)	(i)	I(2), II(2)(a), III(1), III(5), IV (9)
	(d)	I(2), II(1)(c), III(2), III(6), IV (4)	(j)	I(2), II(2)(c), III(1), III(5), IV (10)

Class	Semiplane Isomorphism Types			
72	(e)	I(3), II(1)(b), III(2), III(4), IV (5)	(k)	I(3), II(2)(b), III(1), III(3), IV (11)
(cont.)	(f)	I(3), II(1)(c), III(2), III(4), IV (6)	(l)	I(3), II(2)(c), III(1), III(3), IV (12)
73	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	I(1), II(3)(b), III(3), III(5), IV	(d)	I(2), II(3)(a), III(2), III(6), IV (9,10)
	(b)	I(1), II(3)(b), III(4), III(6), IV (7,8)	(e)	I(3), II(3)(c), III(1), III(3), IV (5,6)
	(c)	I(2), II(3)(a), III(1), III(5), IV (3,4)	(f)	I(3), II(3)(c), III(2), III(4), IV (11,12)
74	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	I(1), II(3)(b), III(3), III(6), IV	(d)	I(2), II(3)(a), III(2), III(5), IV (4,9)
	(b)	I(1), II(3)(b), III(4), III(5), IV (2,7)	(e)	I(3), II(3)(c), III(1), III(4), IV (5,12)
	(c)	I(2), II(3)(a), III(1), III(6), IV (3,10)	(f)	I(3), II(3)(c), III(2), III(3), IV (6,11)
75	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(2)(a), III(1), III(3), IV	(d)	II(1)(b), II(2)(b), III(2), III(6), IV (8,11)
	(b)	II(1)(a), II(2)(a), III(2), III(4), IV (7,9)	(e)	II(1)(c), II(2)(c), III(3), III(5), IV (4,6)
	(c)	II(1)(b), II(2)(b), III(1), III(5), IV (2,5)	(f)	II(1)(c), II(2)(c), III(4), III(6), IV (10,12)
76	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(2)(a), III(1), III(4), IV	(d)	II(1)(b), II(2)(b), III(2), III(5), IV (5,8)
	(b)	II(1)(a), II(2)(a), III(2), III(3), IV (3,7)	(e)	II(1)(c), II(2)(c), III(3), III(6), IV (4,12)
	(c)	II(1)(b), II(2)(b), III(1), III(6), IV (2,11)	(f)	II(1)(c), II(2)(c), III(4), III(5), IV (6,10)
77	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(3)(a), III(1), III(6), IV	(g)	II(2)(a), II(3)(a), III(2), III(5), IV (7)
	(b)	II(1)(a), II(3)(b), III(3), III(6), IV (3)	(h)	II(2)(a), II(3)(b), III(4), III(5), IV (9)
	(c)	II(1)(b), II(3)(b), III(4), III(5), IV (5)	(i)	II(2)(b), II(3)(b), III(3), III(6), IV (11)
	(d)	II(1)(b), II(3)(c), III(1), III(4), IV (2)	(j)	II(2)(b), II(3)(c), III(2), III(3), IV (8)

Class	Semiplane Isomorphism Types			
77	(e)	II(1)(c), II(3)(a), III(2), III(5), IV (6)	(k)	II(2)(c), II(3)(a), III(1), III(6), IV (12)
(cont.)	(f)	II(1)(c), II(3)(c), III(2), III(3), IV (4)	(l)	II(2)(c), II(3)(c), III(1), III(4), IV (10)
78	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(3)(a), III(2), III(6), IV	(g)	II(2)(a), II(3)(a), III(1), III(5), IV (7)
	(b)	II(1)(a), II(3)(b), III(4), III(6), IV (3)	(h)	II(2)(a), II(3)(b), III(3), III(5), IV (9)
	(c)	II(1)(b), II(3)(b), III(4), III(6), IV (5)	(i)	II(2)(b), II(3)(b), III(3), III(5), IV (11)
	(d)	II(1)(b), II(3)(c), III(2), III(4), IV (2)	(j)	II(2)(b), II(3)(c), III(1), III(3), IV (8)
	(e)	II(1)(c), II(3)(a), III(2), III(6), IV (6)	(k)	II(2)(c), II(3)(a), III(1), III(5), IV (12)
	(f)	II(1)(c), II(3)(c), III(2), III(4), IV (4)	(l)	II(2)(c), II(3)(c), III(1), III(3), IV (10)
79	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(3)(c), III(1), III(3), IV	(d)	II(2)(a), II(3)(c), III(2), III(4), IV (7,9)
	(b)	II(1)(b), II(3)(a), III(1), III(5), IV (2,5)	(e)	II(2)(b), II(3)(a), III(2), III(6), IV (8,11)
	(c)	II(1)(c), II(3)(b), III(3), III(5), IV (4,6)	(f)	II(2)(c), II(3)(b), III(4), III(6), IV (10,12)
80	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(3)(c), III(1), III(4), IV	(g)	II(2)(a), II(3)(c), III(1), III(4), IV (9)
	(b)	II(1)(a), II(3)(c), III(2), III(3), IV (3)	(h)	II(2)(a), II(3)(c), III(2), III(3), IV (7)
	(c)	II(1)(b), II(3)(a), III(1), III(6), IV (2)	(i)	II(2)(b), II(3)(a), III(1), III(6), IV (11)
	(d)	II(1)(b), II(3)(a), III(2), III(5), IV (5)	(j)	II(2)(b), II(3)(a), III(2), III(5), IV (8)
	(e)	II(1)(c), II(3)(b), III(3), III(6), IV (4)	(k)	II(2)(c), II(3)(b), III(3), III(6), IV (12)
	(f)	II(1)(c), II(3)(b), III(4), III(5), IV (6)	(l)	II(2)(c), II(3)(b), III(4), III(5), IV (10)
81	Semiplanes with 27 points and $\rho_{12} = 45$			
	(a)	II(1)(a), II(3)(c), III(2), III(4), IV	(d)	II(2)(a), II(3)(c), III(1), III(3), IV (7,9)
	(b)	II(1)(b), II(3)(a), III(2), III(6), IV (2,5)	(e)	II(2)(b), II(3)(a), III(1), III(5), IV (8,11)
	(c)	II(1)(c), II(3)(b), III(4), III(6), IV (4,6)	(f)	II(2)(c), II(3)(b), III(3), III(5), IV (10,12)

Class	Semiplane Isomorphism Types			
82	Semiplanes with 24 points and $\rho_{12} = 42$			
	(a)	I(1), II(1)(a), II(2)(a), II(3)(b), IV	(d)	I(2), II(1)(c), II(2)(c), II(3)(a), IV (4,10)
	(b)	I(1), II(1)(b), II(2)(b), II(3)(b), IV (2,8)	(e)	I(3), II(1)(b), II(2)(b), II(3)(c), IV (5,11)
	(c)	I(2), II(1)(a), II(2)(a), II(3)(a), IV (3,9)	(f)	I(3), II(1)(c), II(2)(c), II(3)(c), IV (6,12)
83	Semiplanes with 25 points and $\rho_{12} = 43$			
	(a)	I(1), II(1)(a), II(2)(a), III(3), IV	(g)	I(2), II(1)(c), II(2)(c), III(5), IV (4)
	(b)	I(1), II(1)(a), II(2)(a), III(4), IV (7)	(h)	I(2), II(1)(c), II(2)(c), III(6), IV (10)
	(c)	I(1), II(1)(b), II(2)(b), III(5), IV (2)	(i)	I(3), II(1)(b), II(2)(b), III(1), IV (5)
	(d)	I(1), II(1)(b), II(2)(b), III(6), IV (8)	(j)	I(3), II(1)(b), II(2)(b), III(2), IV (11)
	(e)	I(2), II(1)(a), II(2)(a), III(1), IV (3)	(k)	I(3), II(1)(c), II(2)(c), III(3), IV (6)
	(f)	I(2), II(1)(a), II(2)(a), III(2), IV (9)	(l)	I(3), II(1)(c), II(2)(c), III(4), IV (12)
84	Semiplanes with 25 points and $\rho_{12} = 43$			
	(a)	I(1), II(1)(a), II(3)(b), III(3), IV	(g)	I(2), II(2)(a), II(3)(a), III(2), IV (9)
	(b)	I(1), II(1)(b), II(3)(b), III(5), IV (2)	(h)	I(2), II(2)(c), II(3)(a), III(6), IV (10)
	(c)	I(1), II(2)(a), II(3)(b), III(4), IV (7)	(i)	I(3), II(1)(b), II(3)(c), III(1), IV (5)
	(d)	I(1), II(2)(b), II(3)(b), III(6), IV (8)	(j)	I(3), II(1)(c), II(3)(c), III(3), IV (6)
	(e)	I(2), II(1)(a), II(3)(a), III(1), IV (3)	(k)	I(3), II(2)(b), II(3)(c), III(2), IV (11)
	(f)	I(2), II(1)(c), II(3)(a), III(5), IV (4)	(l)	I(3), II(2)(c), II(3)(c), III(4), IV (12)
85	Semiplanes with 25 points and $\rho_{12} = 43$			
	(a)	I(1), II(1)(a), II(3)(b), III(4), IV	(g)	I(2), II(2)(a), II(3)(a), III(1), IV (9)
	(b)	I(1), II(1)(b), II(3)(b), III(6), IV (2)	(h)	I(2), II(2)(c), II(3)(a), III(5), IV (10)
	(c)	I(1), II(2)(a), II(3)(b), III(3), IV (7)	(i)	I(3), II(1)(b), II(3)(c), III(2), IV (5)
	(d)	I(1), II(2)(b), II(3)(b), III(5), IV (8)	(j)	I(3), II(1)(c), II(3)(c), III(4), IV (6)

Class	Semiplane Isomorphism Types			
85	(e)	I(2), II(1)(a), II(3)(a), III(2), IV (3)	(k)	I(3), II(2)(b), II(3)(c), III(1), IV (11)
(cont.)	(f)	I(2), II(1)(c), II(3)(a), III(6), IV (4)	(l)	I(3), II(2)(c), II(3)(c), III(3), IV (12)
86	Semiplanes with 25 points and $\rho_{12} = 43$			
	(a)	I(1), II(1)(a), II(3)(b), III(6), IV	(g)	I(2), II(2)(a), II(3)(a), III(5), IV (9)
	(b)	I(1), II(1)(b), II(3)(b), III(4), IV (2)	(h)	I(2), II(2)(c), II(3)(a), III(1), IV (10)
	(c)	I(1), II(2)(a), II(3)(b), III(5), IV (7)	(i)	I(3), II(1)(b), II(3)(c), III(4), IV (5)
	(d)	I(1), II(2)(b), II(3)(b), III(3), IV (8)	(j)	I(3), II(1)(c), II(3)(c), III(2), IV (6)
	(e)	I(2), II(1)(a), II(3)(a), III(6), IV (3)	(k)	I(3), II(2)(b), II(3)(c), III(3), IV (11)
	(f)	I(2), II(1)(c), II(3)(a), III(2), IV (4)	(l)	I(3), II(2)(c), II(3)(c), III(1), IV (12)
87	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(2)(a), II(3)(a), III(1), IV	(g)	II(1)(b), II(2)(b), II(3)(c), III(1), IV (2)
	(b)	II(1)(a), II(2)(a), II(3)(a), III(2), IV (7)	(h)	II(1)(b), II(2)(b), II(3)(c), III(2), IV (8)
	(c)	II(1)(a), II(2)(a), II(3)(b), III(3), IV (3)	(i)	II(1)(c), II(2)(c), II(3)(a), III(5), IV (6)
	(d)	II(1)(a), II(2)(a), II(3)(b), III(4), IV (9)	(j)	II(1)(c), II(2)(c), II(3)(a), III(6), IV (12)
	(e)	II(1)(b), II(2)(b), II(3)(b), III(5), IV (5)	(k)	II(1)(c), II(2)(c), II(3)(c), III(3), IV (4)
	(f)	II(1)(b), II(2)(b), II(3)(b), III(6), IV (11)	(l)	II(1)(c), II(2)(c), II(3)(c), III(4), IV (10)
88	Semiplanes with 26 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(2)(a), II(3)(c), III(1), IV	(g)	II(1)(b), II(2)(b), II(3)(a), III(5), IV (5)
	(b)	II(1)(a), II(2)(a), II(3)(c), III(2), IV (7)	(h)	II(1)(b), II(2)(b), II(3)(a), III(6), IV (11)
	(c)	II(1)(a), II(2)(a), II(3)(c), III(3), IV (3)	(i)	II(1)(c), II(2)(c), II(3)(b), III(3), IV (4)
	(d)	II(1)(a), II(2)(a), II(3)(c), III(4), IV (9)	(j)	II(1)(c), II(2)(c), II(3)(b), III(4), IV (10)
	(e)	II(1)(b), II(2)(b), II(3)(a), III(1), IV (2)	(k)	II(1)(c), II(2)(c), II(3)(b), III(5), IV (6)
	(f)	II(1)(b), II(2)(b), II(3)(a), III(2), IV (8)	(l)	II(1)(c), II(2)(c), II(3)(b), III(6), IV (12)

Class	Semiplane Isomorphism Types			
89	Semiplanes with 24 points and $\rho_{12} = 43$			
	(a)	II(1)(a), II(1)(b), II(1)(c), III(2), III(4), IV	(d)	II(2)(a), II(2)(b), II(2)(c), III(1), III(3), IV (7,9)
	(b)	II(1)(a), II(1)(b), II(1)(c), III(2), III(6), IV (2,5)	(e)	II(2)(a), II(2)(b), II(2)(c), III(1), III(5), IV (8,11)
	(c)	II(1)(a), II(1)(b), II(1)(c), III(4), III(6), IV (4,6)	(f)	II(2)(a), II(2)(b), II(2)(c), III(3), III(5), IV (10,12)
90	Semiplanes with 25 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), III(1), III(4), III(6), IV	(d)	II(2)(a), II(2)(b), III(2), III(3), III(5), IV (7,8)
	(b)	II(1)(a), II(1)(c), III(2), III(3), III(6), IV (3,4)	(e)	II(2)(a), II(2)(c), III(1), III(4), III(5), IV (9,10)
	(c)	II(1)(b), II(1)(c), III(2), III(4), III(5), IV (5,6)	(f)	II(2)(b), II(2)(c), III(1), III(3), III(6), IV (11,12)
91	Semiplanes with 25 points and $\rho_{12} = 44$			
	(a)	II(1)(a), II(1)(b), III(2), III(4), III(6), IV	(d)	II(2)(a), II(2)(b), III(1), III(3), III(5), IV (7,8)
	(b)	II(1)(a), II(1)(c), III(2), III(4), III(6), IV (3,4)	(e)	II(2)(a), II(2)(c), III(1), III(3), III(5), IV (9,10)
	(c)	II(1)(b), II(1)(c), III(2), III(4), III(6), IV (5,6)	(f)	II(2)(b), II(2)(c), III(1), III(3), III(5), IV (11,12)
92	Semiplanes with 23 points and $\rho_{12} = 42$			
	(a)	II(1)(a), II(1)(b), II(1)(c), II(3)(a), III(2), IV	(g)	II(2)(a), II(2)(b), II(2)(c), II(3)(a), III(1), IV (7)
	(b)	II(1)(a), II(1)(b), II(1)(c), II(3)(a), III(6), IV (6)	(h)	II(2)(a), II(2)(b), II(2)(c), II(3)(a), III(5), IV (12)
	(c)	II(1)(a), II(1)(b), II(1)(c), II(3)(b), III(4), IV (3)	(i)	II(2)(a), II(2)(b), II(2)(c), II(3)(b), III(3), IV (9)
	(d)	II(1)(a), II(1)(b), II(1)(c), II(3)(b), III(6), IV (5)	(j)	II(2)(a), II(2)(b), II(2)(c), II(3)(b), III(5), IV (11)
	(e)	II(1)(a), II(1)(b), II(1)(c), II(3)(c), III(2), IV (2)	(k)	II(2)(a), II(2)(b), II(2)(c), II(3)(c), III(1), IV (8)
	(f)	II(1)(a), II(1)(b), II(1)(c), II(3)(c), III(4), IV (4)	(l)	II(2)(a), II(2)(b), II(2)(c), II(3)(c), III(3), IV (10)
93	Semiplanes with 24 points and $\rho_{12} = 43$			
	(a)	II(1)(a), II(1)(b), II(3)(a), III(1), III(6), IV	(g)	II(2)(a), II(2)(b), II(3)(a), III(2), III(5), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(c), III(1), III(4), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(c), III(2), III(3), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(b), III(3), III(6), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(b), III(4), III(5), IV (9)

Class	Semiplane Isomorphism Types			
93	(d)	II(1)(a), II(1)(c), II(3)(c), III(2), III(3), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(c), III(1), III(4), IV (10)
(cont.)	(e)	II(1)(b), II(1)(c), II(3)(a), III(2), III(5), IV (6)	(k)	II(2)(b), II(2)(c), II(3)(a), III(1), III(6), IV (12)
	(f)	II(1)(b), II(1)(c), II(3)(b), III(4), III(5), IV (5)	(l)	II(2)(b), II(2)(c), II(3)(b), III(3), III(6), IV (11)
94	Semiplanes with 24 points and $\rho_{12} = 43$			
	(a)	II(1)(a), II(1)(b), II(3)(a), III(2), III(6), IV	(g)	II(2)(a), II(2)(b), II(3)(a), III(1), III(5), IV (7)
	(b)	II(1)(a), II(1)(b), II(3)(c), III(2), III(4), IV (2)	(h)	II(2)(a), II(2)(b), II(3)(c), III(1), III(3), IV (8)
	(c)	II(1)(a), II(1)(c), II(3)(b), III(4), III(6), IV (3)	(i)	II(2)(a), II(2)(c), II(3)(b), III(3), III(5), IV (9)
	(d)	II(1)(a), II(1)(c), II(3)(c), III(2), III(4), IV (4)	(j)	II(2)(a), II(2)(c), II(3)(c), III(1), III(3), IV (10)
	(e)	II(1)(b), II(1)(c), II(3)(a), III(2), III(6), IV (6)	(k)	II(2)(b), II(2)(c), II(3)(a), III(1), III(5), IV (12)
	(f)	II(1)(b), II(1)(c), II(3)(b), III(4), III(6), IV (5)	(l)	II(2)(b), II(2)(c), II(3)(b), III(3), III(5), IV (11)
95	Semiplanes with 24 points and $\rho_{12} = 43$			
	(a)	II(1)(a), II(1)(b), II(3)(b), III(4), III(6), IV	(d)	II(2)(a), II(2)(b), II(3)(b), III(3), III(5), IV (7,8)
	(b)	II(1)(a), II(1)(c), II(3)(a), III(2), III(6), IV (3,4)	(e)	II(2)(a), II(2)(c), II(3)(a), III(1), III(5), IV (9,10)
	(c)	II(1)(b), II(1)(c), II(3)(c), III(2), III(4), IV (5,6)	(f)	II(2)(b), II(2)(c), II(3)(c), III(1), III(3), IV (11,12)
96	Semiplanes with 23 points and $\rho_{12} = 42$			
	(a)	I(1), II(1)(a), II(1)(b), III(4), III(6), IV	(d)	I(1), II(2)(a), II(2)(b), III(3), III(5), IV (7,8)
	(b)	I(2), II(1)(a), II(1)(c), III(2), III(6), IV (3,4)	(e)	I(2), II(2)(a), II(2)(c), III(1), III(5), IV (9,10)
	(c)	I(3), II(1)(b), II(1)(c), III(2), III(4), IV (5,6)	(f)	I(3), II(2)(b), II(2)(c), III(1), III(3), IV (11,12)
97	Semiplanes with 22 points and $\rho_{12} = 41$			
	(a)	I(1), II(1)(a), II(1)(b), II(3)(b), III(4), IV	(g)	I(1), II(2)(a), II(2)(b), II(3)(b), III(3), IV (7)
	(b)	I(1), II(1)(a), II(1)(b), II(3)(b), III(6), IV (2)	(h)	I(1), II(2)(a), II(2)(b), II(3)(b), III(5), IV (8)
	(c)	I(2), II(1)(a), II(1)(c), II(3)(a), III(2), IV (3)	(i)	I(2), II(2)(a), II(2)(c), II(3)(a), III(1), IV (9)
	(d)	I(2), II(1)(a), II(1)(c), II(3)(a), III(6), IV (4)	(j)	I(2), II(2)(a), II(2)(c), II(3)(a), III(5), IV (10)

Class	Semiplane Isomorphism Types			
97	(e)	I(3), II(1)(b), II(1)(c), II(3)(c), III(2), IV (5)	(k)	I(3), II(2)(b), II(2)(c), II(3)(c), III(1), IV (11)
(cont.)	(f)	I(3), II(1)(b), II(1)(c), II(3)(c), III(4), IV (6)	(l)	I(3), II(2)(b), II(2)(c), II(3)(c), III(3), IV (12)
98	Semiplanes with 23 points and $\rho_{12} = 42$			
	(a)	I(1), II(1)(a), II(3)(b), III(3), III(6), IV	(g)	I(2), II(2)(a), II(3)(a), III(2), III(5), IV (9)
	(b)	I(1), II(1)(b), II(3)(b), III(4), III(5), IV (2)	(h)	I(2), II(2)(c), II(3)(a), III(1), III(6), IV (10)
	(c)	I(1), II(2)(a), II(3)(b), III(4), III(5), IV (7)	(i)	I(3), II(1)(b), II(3)(c), III(1), III(4), IV (5)
	(d)	I(1), II(2)(b), II(3)(b), III(3), III(6), IV (8)	(j)	I(3), II(1)(c), II(3)(c), III(2), III(3), IV (6)
	(e)	I(2), II(1)(a), II(3)(a), III(1), III(6), IV (3)	(k)	I(3), II(2)(b), II(3)(c), III(2), III(3), IV (11)
	(f)	I(2), II(1)(c), II(3)(a), III(2), III(5), IV (4)	(l)	I(3), II(2)(c), II(3)(c), III(1), III(4), IV (12)
99	Semiplanes with 23 points and $\rho_{12} = 43$			
	(a)	I(1), II(1)(a), II(3)(b), III(4), III(6), IV	(g)	I(2), II(2)(a), II(3)(a), III(1), III(5), IV (9)
	(b)	I(1), II(1)(b), II(3)(b), III(4), III(6), IV (2)	(h)	I(2), II(2)(c), II(3)(a), III(1), III(5), IV (10)
	(c)	I(1), II(2)(a), II(3)(b), III(3), III(5), IV (7)	(i)	I(3), II(1)(b), II(3)(c), III(2), III(4), IV (5)
	(d)	I(1), II(2)(b), II(3)(b), III(3), III(5), IV (8)	(j)	I(3), II(1)(c), II(3)(c), III(2), III(4), IV (6)
	(e)	I(2), II(1)(a), II(3)(a), III(2), III(6), IV (3)	(k)	I(3), II(2)(b), II(3)(c), III(1), III(3), IV (11)
	(f)	I(2), II(1)(c), II(3)(a), III(2), III(6), IV (4)	(l)	I(3), II(2)(c), II(3)(c), III(1), III(3), IV (12)
100	Semiplanes with 24 points and $\rho_{12} = 43$			
	(a)	II(1)(a), II(2)(a), II(3)(c), III(1), III(3), IV	(d)	II(1)(b), II(2)(b), II(3)(a), III(2), III(6), IV (8,11)
	(b)	II(1)(a), II(2)(a), II(3)(c), III(2), III(4), IV (7,9)	(e)	II(1)(c), II(2)(c), II(3)(b), III(3), III(5), IV (4,6)
	(c)	II(1)(b), II(2)(b), II(3)(a), III(1), III(5), IV (2,5)	(f)	II(1)(c), II(2)(c), II(3)(b), III(4), III(6), IV (10,12)
101	Semiplanes with 24 points and $\rho_{12} = 43$			
	(a)	II(1)(a), II(2)(a), II(3)(c), III(1), III(4), IV	(d)	II(1)(b), II(2)(b), II(3)(a), III(2), III(5), IV (5,8)
	(b)	II(1)(a), II(2)(a), II(3)(c), III(2), III(3), IV (3,7)	(e)	II(1)(c), II(2)(c), II(3)(b), III(3), III(6), IV (4,12)

Class	Semiplane Isomorphism Types			
101 (cont.)	(c)	II(1)(b), II(2)(b), II(3)(a), III(1), III(6), IV (2,11)	(f)	II(1)(c), II(2)(c), II(3)(b), III(4), III(5), IV (6,10)
102	Semiplanes with 22 points and $\rho_{12} = 41$			
	(a)	I(1), II(1)(a), II(2)(a), II(3)(b), III(3), IV	(g)	I(2), II(1)(c), II(2)(c), II(3)(a), III(5), IV (4)
	(b)	I(1), II(1)(a), II(2)(a), II(3)(b), III(4), IV (7)	(h)	I(2), II(1)(c), II(2)(c), II(3)(a), III(6), IV (10)
	(c)	I(1), II(1)(b), II(2)(b), II(3)(b), III(5), IV (2)	(i)	I(3), II(1)(b), II(2)(b), II(3)(c), III(1), IV (5)
	(d)	I(1), II(1)(b), II(2)(b), II(3)(b), III(6), IV (8)	(j)	I(3), II(1)(b), II(2)(b), II(3)(c), III(2), IV (11)
	(e)	I(2), II(1)(a), II(2)(a), II(3)(a), III(1), IV (3)	(k)	I(3), II(1)(c), II(2)(c), II(3)(c), III(3), IV (6)
	(f)	I(2), II(1)(a), II(2)(a), II(3)(a), III(2), IV (9)	(l)	I(3), II(1)(c), II(2)(c), II(3)(c), III(4), IV (12)
103	Semiplanes with 22 points and $\rho_{12} = 42$			
	(a)	II(1)(a), II(1)(b), II(1)(c), III(2), III(4), III(6), IV	(b)	II(2)(a), II(2)(b), II(2)(c), III(1), III(3), III(5), IV (7–12)
104	Semiplanes with 21 points and $\rho_{12} = 41$			
	(a)	II(1)(a), II(1)(b), II(1)(c), II(3)(a), III(2), III(6), IV	(d)	II(2)(a), II(2)(b), II(2)(c), II(3)(a), III(1), III(5), IV (7,12)
	(b)	II(1)(a), II(1)(b), II(1)(c), II(3)(b), III(4), III(6), IV (3,5)	(e)	II(2)(a), II(2)(b), II(2)(c), II(3)(b), III(3), III(5), IV (9,11)
	(c)	II(1)(a), II(1)(b), II(1)(c), II(3)(c), III(2), III(4), IV (2,4)	(f)	II(2)(a), II(2)(b), II(2)(c), II(3)(c), III(1), III(3), IV (8,10)
105	Semiplanes with 20 points and $\rho_{12} = 40$			
	(a)	I(1), II(1)(a), II(1)(b), II(3)(b), III(4), III(6), IV	(d)	I(2), II(2)(a), II(2)(c), II(3)(a), III(1), III(5), IV (9,10)

Class	Semiplane Isomorphism Types			
105 (cont.)	(b)	I(1), II(2)(a), II(2)(b), II(3)(b), III(3), III(5), IV (7,8)	(e)	I(3), II(1)(b), II(1)(c), II(3)(c), III(2), III(4), IV (5,6)
	(c)	I(2), II(1)(a), II(1)(c), II(3)(a), III(2), III(6), IV (3,4)	(f)	I(3), II(2)(b), II(2)(c), II(3)(c), III(1), III(3), IV (11,12)