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# CLAW-FREENESS, 3-HOMOGENEOUS SUBSETS OF A GRAPH AND A RECONSTRUCTION PROBLEM

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Abstract. We describe Forb$\{K_{1,3}, \overline{K_{1,3}}\}$, the class of graphs $G$ such that $G$ and its complement $\overline{G}$ are claw-free. With few exceptions, it is made up of graphs whose connected components consist of cycles of length at least 4, paths, and of the complements of these graphs. Considering the hypergraph $\mathcal{H}^{(3)}(G)$ made of the 3-element subsets of the vertex set of a graph $G$ on which $G$ induces a clique or an independent subset, we deduce from above a description of the Boolean sum $G\dot{+}G'$ of two graphs $G$ and $G'$ giving the same hypergraph. We indicate the role of this latter description in a reconstruction problem of graphs up to complementation.

## 1. Results and motivation

Our notations and terminology mostly follow [3]. The graphs we consider in this paper are undirected, simple and have no loops, that is, a *graph* is a pair $G := (V, \mathcal{E})$, where $\mathcal{E}$ is a subset of $[V]^2$, the set of 2-element subsets of $V$. Elements of $V$ are the *vertices* of $G$ and elements of $\mathcal{E}$ its *edges*. We denote by $V(G)$ the vertex set of $G$ and by $E(G)$ its edge set. We look at members of $[V]^2$ as unordered pairs of distinct vertices. If $A$ is a subset of $V$, the pair $G_{\restriction A} := (A, \mathcal{E} \cap [A]^2)$ is the *graph induced by $G$ on $A$*. The *complement* of $G$ is the simple graph $\overline{G}$ whose vertex set is $V$ and whose edges are the unordered pairs of nonadjacent and distinct vertices of $G$, that is $\overline{G} = (V, \overline{\mathcal{E}})$, where $\overline{\mathcal{E}} = [V]^2 \setminus \mathcal{E}$. We denote by $K_3$ the complete graph on 3 vertices and by $K_{1,3}$ the graph made of a vertex linked to a $\overline{K_3}$. The graph $K_{1,3}$ is called a *claw*, the graph $\overline{K_{1,3}}$ a *co-claw*.

In [4], Brandstädt and Mahfud give a structural characterization of graphs with no claw and no co-claw; they deduce several algorithmic consequences

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(relying on bounded clique width). We will give a more precise characterization of such graphs.

We denote by $A_6$ the graph on 6 vertices made of a $K_3$ bounded by three $K_3$ (cf. Figure 1) and by $C_n$ the $n$-element cycle, $n \geq 4$. We denote by $P_9$ the Paley graph on 9 vertices (cf. Figure 1). Note that $P_9$ is isomorphic to its complement $\overline{P_9}$, to the line-graph of $K_{3,3}$ and also to $K_3 \square K_3$, the cartesian product of $K_3$ by itself (see [3, p. 30] if needed for a definition of the *cartesian product of graphs*, and see [15, p. 176] and [3, p. 28] for a definition and basic properties of *Paley graphs*).

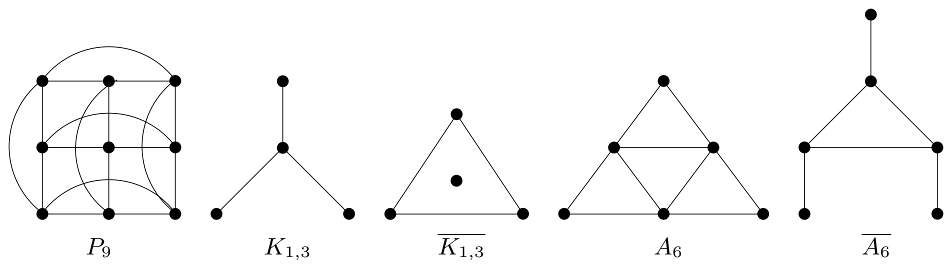


FIGURE 1.

Given a set $\mathcal{F}$ of graphs, we denote by $\mathrm{Forb}\,\mathcal{F}$ the class of graphs $G$ such that no member of $\mathcal{F}$ is isomorphic to an induced subgraph of $G$. Members of $\mathrm{Forb}\{K_3\}$ and $\mathrm{Forb}\{K_{1,3}\}$ are called, resp., *triangle-free* and *claw-free* graphs.

The main result of this note asserts:

**Theorem 1.1.** *The class* $\mathrm{Forb}\{K_{1,3}, \overline{K_{1,3}}\}$ *consists of* $A_6$*; of the induced subgraphs of* $P_9$*; of graphs whose connected components are cycles of length at least* 4 *or paths; and of the complements of these graphs.*

As an immediate consequence of Theorem 1.1, note that the graphs $A_6$ and $\overline{A_6}$ are the only members of $\mathrm{Forb}\{K_{1,3}, \overline{K_{1,3}}\}$ which contain a $K_3$ and a $\overline{K_3}$ with no vertex in common. Note also that $A_6$ and $\overline{A_6}$ are very important graphs for the study of how maximal cliques and stable sets overlap in general graphs. See the main theorem of [7], and also [8]. A list of all self-complementary line-graphs is given in [9, p. 31]. Apart from $C_5$, they are all induced subgraphs of $P_9$.

From Theorem 1.1 we obtain a characterization of the Boolean sum of two graphs having the same 3-homogeneous subsets. For that, we say that a subset of vertices of a graph $G$ is *homogeneous** if it is a clique or an independent set. Let $\mathcal{H}^{(3)}(G)$ be the hypergraph having the same vertices as $G$ and whose hyperedges are the 3-element homogeneous subsets of $G$. Given two graphs $G$ and $G'$ on the same vertex set $V$, we recall that the

*Note that the word homogeneous is used with this meaning in Ramsey theory; in other areas of graph theory it has other meanings, several in fact.

*Boolean sum* $G \dot{+} G'$ of $G$ and $G'$ is the graph on $V$ whose edges are unordered pairs $e$ of distinct vertices such that $e \in E(G)$ if and only if $e \notin E(G')$. Note that $E(G \dot{+} G')$ is the symmetric difference $E(G) \Delta E(G')$ of $E(G)$ and $E(G')$. The graph $G \dot{+} G'$ is also called the *symmetric difference* of $G$ and $G'$ and denoted by $G \Delta G'$ in [3]. Given a graph $U$ with vertex set $V$, the *edge-graph* of $U$ is the graph $S(U)$ whose vertices are the edges $u$ of $U$ and whose edges are unordered pairs $uv$ such that $u = xy$, $v = xz$ for three distinct elements $x, y, z \in V$ such that $yz$ is not an edge of $U$. Note that the edge-graph $S(U)$ is a spanning subgraph of $L(U)$, the *line-graph* of $U$, not to be confused with it.

Claw-free graphs and triangle-free graphs are related by means of the edge-graph construction. Indeed, as it is immediate to see, for every graph $U$, we have

$$U \in \mathrm{Forb}\{K_{1,3}\} \iff S(U) \in \mathrm{Forb}\{K_3\} \tag{1.1}$$

Our characterization is as follows.

**Theorem 1.2.** *Let $U$ be a graph. The following properties are equivalent.*

*(1) There are two graphs $G$ and $G'$ having the same 3-element homogeneous subsets such that $U := G \dot{+} G'$;*
*(2) $S(U)$ and $S(\overline{U})$ are bipartite;*
*(3) Either*
  - *(i) $U$ is an induced subgraph of $P_9$, or*
  - *(ii) the connected components of $U$, or of its complement $\overline{U}$, are cycles of even length or paths.*

As a consequence, if the graph $U$ satisfying Property (1) is disconnected, then $U$ contains no 3-element cycle, moreover, if $U$ contains no 3-element cycle then each connected component of $U$ is a cycle of even length, or a path, in particular $U$ is bipartite.

The implication (2) $\Rightarrow$ (3) in Theorem 1.2 follows immediately from Theorem 1.1. Indeed, suppose that Property (2) holds, that is $S(U)$ and $S(\overline{U})$ are bipartite, then from (1.1) and from the fact that $S(A_6)$ and $S(C_n)$, $n \geq 4$, are respectively isomorphic to $C_9$ and to $C_n$, we have:

$$U \in \mathrm{Forb}\{K_{1,3}, \overline{K_{1,3}}, A_6, \overline{A_6}, C_{2n+1}, \overline{C_{2n+1}} : n \geq 2\}.$$

From Theorem 1.1, Property (3) holds. The other implications, obtained by more straigthforward arguments, are given in Subsection 2.3.

This leaves open the following question.

**Problem 1.3.** *Which pairs of graphs $G$ and $G'$ with the same 3-element homogeneous subsets have a given Boolean sum $U := G \dot{+} G'$?*

A partial answer, motivated by the reconstruction problem discussed below, is given in [5]. We mention that two graphs $G$ and $G'$ as above are determined by the graphs induced on the connected components of $U := G \dot{+} G'$ and on a system of distinct representatives of these connected components ([5, Proposition 10]).

A quite natural problem, related to the study of Ramsey numbers for triples, is this question:

**Problem 1.4.** *Which hypergraphs are of the form $\mathcal{H}^{(3)}(G)$?*

An asymptotic lower bound of the size of $\mathcal{H}^{(3)}(G)$ in terms of $|V(G)|$ was established by A. W. Goodman [10].

The motivation for Theorem 1.2 (and thus Theorem 1.1) originates in a reconstruction problem on graphs that we present now. Considering two graphs $G$ and $G'$ on the same set $V$ of vertices, we say that $G$ and $G'$ are *isomorphic up to complementation* if $G'$ is isomorphic to $G$ or to the complement $\overline{G}$ of $G$. Let $k$ be a non-negative integer, we say that $G$ and $G'$ are *$k$-hypomorphic up to complementation* if for every $k$-element subset $K$ of $V$, the graphs $G_{\restriction K}$ and $G'_{\restriction K}$ induced by $G$ and $G'$ on $K$ are isomorphic up to complementation. Finally, we say that $G$ is *$k$-reconstructible up to complementation* if every graph $G'$ which is $k$-hypomorphic to $G$ up to complementation is in fact isomorphic to $G$ up to complementation. The following problem emerged from a question of P. Ille [13]:

**Problem 1.5.** *For which pairs $(k, v)$ of integers, $k < v$, every graph $G$ on $v$ vertices is $k$-reconstructible up to complementation?*

It is immediate to see that if the conclusion of the problem above is positive for some $k, v$, then $v$ is distinct from 3 and 4 and, with a little bit of thought, that if $v \geq 5$ then $k \geq 4$ (see Proposition 4.1 of [6]). With J. Dammak, G. Lopez [5, 6] we proved that the conclusion is positive if

(i) $4 \leq k \leq v - 3$ or
(ii) $4 \leq k = v - 2$ and $v \equiv 2 \pmod 4$.

We do not know if in (ii) the condition $v \equiv 2 \pmod 4$ can be dropped. For $4 \leq k = v - 1$, we checked that the conclusion holds if $v = 5$ and noticed that for larger values of $v$ it could be negative or extremely hard to obtain, indeed, a positive conclusion would imply that Ulam's reconstruction conjecture holds (see Proposition 19 of [5]).

The reason for which Theorem 1.2 plays a role in that matter relies on properties of incidence matrices. Given non-negative integers $t$, $k$, let $W_{tk}$ be the $\binom{v}{2}$ by $\binom{v}{k}$ incidence matrix of 0's and 1's, the rows of which are indexed by $t$-element subsets $T$ of $V$, the columns are indexed by the $k$-element subsets $K$ of $V$, and where the entry $W_{tk}(T, K)$ is 1 if $T \subseteq K$ and is 0 otherwise.

Let $U := G \dot{+} G'$ and $M_U$ be the column vector associated to the graph $U$. The matrix product ${}^{T}W_{2k}M_U$ where the computation is made in the two elements field $\mathbb{Z}/2\mathbb{Z}$ is 0 if and only if the number of edges of $G_{\restriction K}$ and $G'_{\restriction K}$ have the same parity for all $K$'s, a condition satisfied if $G$ and $G'$ are $k$-hypomorphic up to complementation and $k \equiv 0 \pmod 4$ or $k \equiv 1 \pmod 4$. According to R. M. Wilson [16], the dimension (over $\mathbb{Z}/2\mathbb{Z}$) of the kernel of ${}^{T}W_{2k}$ is 1 if $2 \leq k \leq v - 2$ and $k \equiv 0 \pmod 4$ that is $M_U$ is the constant matrix 0 or 1, and thus $G'$ is equal to $G$ or to $\overline{G}$. If $k \equiv 1 \pmod 4$, the dimension of the kernel of ${}^TW_{2k}$ is $v$ and this kernel consists of (the column matrices of) complete bipartite graphs and their complements [6]. If we add the fact that $G$ and $G'$ have the same 3-homogeneous subsets then, according to Theorem 1.2, $U$ is (claw, co-claw)-free. If $v \geq 5$, it follows readily that $U$ is either the empty graph or the complete graph. Hence $G'$ is equal to $G$ or to $\overline{G}$. If $3 \leq k \leq v-3$, it turns out that *two graphs $G$ and $G'$ which are $k$-hypomorphic up to complementation are 3-hypomorphic up to complementation, which amounts to the fact that $G$ and $G'$ have the same 3-homogeneous subsets*, thus in the case $k \equiv 1 \pmod 4$, $G$ and $G'$ are equal up to complementation. Indeed, a famous Gottlieb-Kantor theorem on incidence matrices ([11, 14]) asserts that the matrix $W_{tk}$ has full row rank over the field of rational numbers provided that $t \leq \min\{k, v-k\}$, from which follows the our next proposition.

**Proposition 1.6.** *(*[6, Proposition 2.4]*) Let $t \leq min(k, v-k)$ and $G$ and $G'$ be two graphs on the same set $V$ of $v$ vertices. If $G$ and $G'$ are $k$-hypomorphic up to complementation then they are $t$-hypomorphic up to complementation.*

Up to now, Wilson theorem has not been applied successfully to the cases $k \equiv 2 \pmod 4$ and $k \equiv 3 \pmod 4$. Instead, efforts concentrated on the structure of pairs of $k$-hypomorphic graphs $G$ and $G'$ with the same 3-homogeneous subsets. The form of their Boolean sum as given in Theorem 1.2(3) was the first step of a description. With that in hands, it was shown in [5] that the additional hypothesis that $G$ and $G'$ are $k$-hypomorphic to complementation for some $k$, $4 \leq k \leq v-2$, was enough to ensure that $G$ and $G'$ are isomorphic up to complementation.

## 2. Proofs

Let $U$ be a graph. For an unordered pair $e := xy$ of distinct vertices, we set $U(e) = 1$ if $e \in E(U)$ and $U(e) = 0$ otherwise. Let $x \in V(U)$; we denote by $N_U(x)$ and $d_U(x)$ the *neighborhood* and the *degree* of $x$ (that is $N_U(x) := \{y \in V(U) : xy \in E(U)\}$ and $d_U(x) := |N_U(x)|$). For $X \subseteq V(U)$, we set $N_U(X) := (\cup_{x\in X} N_U(x)) \setminus X$.

### 2.1. **Proof of Theorem 1.1.**

*Proof.* Trivially, the graphs described in Theorem 1.1 belong to Forb$\{K_{1,3}, \overline{K_{1,3}}\}$. We prove the converse.

The *diamond* is the graph on four vertices with five edges. We say that *a graph $G$ contains a graph $H$* when $G$ has an induced subgraph isomorphic to $H$.

**Theorem 2.1.** *(Harary and Holzmann* [12]*) A graph $G$ is the line-graph of a triangle-free graph if and only if $G$ contains no claw and no diamond.*

*Proof.* Since [12] is very difficult to find, we include a short proof. Checking that a line-graph of a triangle-free graph contains no claw and no diamond is a routine matter. Conversely, let $G$ be graph with no claw and no diamond. A theorem of Beineke [1] states that there exists a list $\mathcal{L}$ of nine graphs such any graph that does not contain a graph from $\mathcal{L}$ is a line-graph. One of the nine graphs is the claw and the eight remaining ones all contain a diamond. So, $G = L(R)$ for some graph $R$. Let $R'$ be the graph obtained from $R$ by replacing each connected component of $R$ isomorphic to a triangle by a claw. So, $L(R) = L(R') = G$. We claim that $R'$ is triangle-free. Else let $T$ be a triangle of $R'$. From the construction of $R'$, there is a vertex $v \notin T$ in the connected component of $R'$ that contains $T$. So we may choose $v$ with a neighbor in $T$. Now the edges of $T$ and one edge from $v$ to $T$ induce a diamond of $G$, a contradiction. □

Let $G$ be in the class Forb$\{K_{1,3}, \overline{K_{1,3}}\}$.

(1) *We may assume that $G$ and $\overline{G}$ are connected.*
Else, up to symmetry, $G$ is disconnected. If $G$ contains a vertex $v$ of degree at least 3, then $N_G(v)$ contains an edge (for otherwise there is a claw), so $G$ contains a triangle. This is a contradiction since by picking a vertex in another component we obtain a co-claw. So all vertices of $G$ are of degree at most 2. It follows that the components of $G$ are cycles (of length at least 4, or there is a co-claw) or paths, an outcome of the theorem. This proves our assumption.

(2) *We may assume that $G$ and $\overline{G}$ contain no induced path on six vertices.*
Else $G$ has an induced subgraph $H$ that is either a path on at least 6 vertices or a cycle on at least 7 vertices. Suppose $H$ maximal with respect to this property. If $G = H$ then we are done. Else, by (1), we pick a vertex $v$ in $G \backslash H$ with at least one neighbor in $H$. From the maximality of $H$, $v$ has a neighbor $p_i$ in the interior of some $P_6 = p_1p_2p_3p_4p_5p_6$ of $H$. Up to symmetry we assume that v has a neighbor $p_i$ where $i \in \{2, 3\}$. So $N_G(v) \cap \{p_1, p_2, p_3, p_4\}$ contains an edge $e$ for otherwise $\{p_i, p_{i-1}, p_{i+1}, v\}$ induces a claw. If $e = p_1p_2$ then $v$ must be adjacent to $p_4, p_5, p_6$ for otherwise there is a co-claw; so $\{v, p_1, p_4, p_6\}$ induces a claw. If $e = p_2p_3$ then $v$ must be adjacent to $p_5, p_6$ for otherwise there is a co-claw, so from the symmetry between $\{p_1, p_2\}$ and $\{p_5, p_6\}$ we may rely on the previous case. If $e = p_3p_4$ then $v$ must be adjacent to $p_1, p_6$ for otherwise there is a co-claw; so $\{v, p_1, p_3, p_6\}$ induces a claw. In all cases there is a contradiction. This proves our second assumption.

(3) *We may assume that $G$ and $\overline{G}$ contain no $A_6$.*
Suppose that $G$ contains $\overline{A_6}$. Then, let $aa', bb', cc'$ be three disjoint edges of $G$ such that the only edges between them are $ab, bc, ca$. If $V(G) = \{a, a', b, b', c, c'\}$, an outcome of the theorem is satisfied, so let $v$ be a seventh vertex of $G$. We may assume that $av \in E(G)$ (else there is a co-claw). If $a'v \in E(G)$ then $vb', vc' \in E(G)$ (else there is a co-claw) so $\{v, a', b', c'\}$ is a claw. Hence $a'v \notin E(G)$. We have $vb \in E(G)$

(or $\{a, a', v, b\}$ is a claw) and similarly $vc \in E(G)$. So $\{a', v, b, c\}$ is a co-claw. This proves our third assumption.

(4) *Finally, we may assume that $G$ and $\overline{G}$ contain no diamond.*
Suppose for a contradiction that $\overline{G}$ contains a diamond. Then, $G$ contains a co-diamond, that is four vertices $a, b, c, d$ that induce only one edge, say $ab$. By (1), there is a path $P$ from $\{c, d\}$ to some vertex $w$ that has a neighbor in $\{a, b\}$. We choose such a path $P$ minimal and we assume up to symmetry that the path is from $c$.

If $w$ is adjacent to both $a, b$ then $\{a, b, w, d\}$ induces a co-claw unless $w$ is adjacent to $d$, similarly $w$ is adjacent to $c$, so $\{w, a, c, d\}$ induces a claw. Hence $w$ is adjacent to exactly one of $a, b$, say to a. So, $P' = cPwab$ is an induced path and for convenience we rename its vertices $p_1, \dots, p_k$. If $d$ has a neighbor in $P'$ then, from the minimality of $P'$, this neighbor must be $p_2$. So, $\{p_2, p_1, p_3, d\}$ induces a claw. Hence, $d$ has no neighbor in $P'$.

By (1), there is a path $Q$ from $d$ to some vertex $v$ that has a neighbor in $P'$. We choose $Q$ minimal with respect to this property. From the paragraph above, $v \neq d$. Let $p_i$ (resp. $p_j$) be the neighor of $v$ in $P'$ with minimum (resp. maximum) index. If $i = j = 1$ then $dQvp_1Pwp_{k-1}p_k$ is a path on at least 6 vertices a contradiction to (2). So, if $i = j$ then $i \neq 1$ and symmetrically, $i \neq k$, so $\{p_{i-1}, p_i, p_{i+1}, v\}$ is a claw. Hence $i \neq j$. If $j > i+1$ then $\{v, v', p_i, p_j\}$, where $v'$ is the neighbor of $v$ along $Q$, is a claw. So, $j = i+1$. So $vp_ip_j$ is a triangle. Hence $P' = p_1p_2p_3p_4$, $Q = dv$ and $i = 2$, for otherwise there is a co-claw. Hence, $P' \cup Q$ form an induced $\overline{A_6}$ of $G$, a contradiction to (3). This proves our final assumption.

Now $G$ is connected and contains no claw and no diamond. So, by Theorem 2.1, $G$ is the line-graph of some connected triangle-free graph $R$. Symmetrically, $\overline{G}$ is also a line-graph. These graphs are studied in [2].

If $R$ contains a vertex $v$ of degree at least 4 then all edges of $R$ must be incident with $v$, for else an edge $e$ non-incident with $v$ together with three edges of $R$ incident with $v$ and non-adjacent to $e$ form a co-claw in $G$. So all vertices of $R$ have degree at most 3 since otherwise, $G$ is a clique, a contradiction to (1). We may assume that $R$ has a vertex $a$ of degree 3 for otherwise $G$ is a path or a cycle. Let $b, b', b''$ be the neighbors of $a$. Since $a$ has degree 3, all edges of $R$ must be incident with $b, b'$ or $b''$ for otherwise $G$ contains a co-claw.

If one of $b, b', b''$, say $b$, is of degree 3, then $N_R(b) = \{a, a', a''\}$ and all edges of $R$ are incident with one of $a, a', a''$ (or there is a co-claw). So $R$ is a subgraph of $K_{3,3}$. So, since $P_9 = L(K_{3,3})$, $G = L(R)$ is an induced subgraph of $P_9$, an outcome of the theorem. Hence we assume that $b, b', b''$ are of degree at most 2. If $|N_R(\{b, b', b''\}) \setminus \{a\}| \geq 3$, then $R$ contains the pairwise non-adjacent edges $bc, b'c', b''c''$ say, and the edges $ab, ab', ab'', bc, b'c', b''c''$

are vertices of $G$ that induce an $\overline{A_6}$, a contradiction to (3). So,

$$|N_R(\{b, b', b''\}) \setminus \{a\}| \leq 2$$

which means again that $R$ is a subgraph of $K_{3,3}$. □

2.2. **Ingredients for the proof of Theorem 1.2.** The proof of the equivalence between Properties (1) and (2) of Theorem 1.2 relies on the following lemma.

**Lemma 2.2.** *Let $G$ and $G'$ be two graphs on the same vertex set $V$ and let $U := G\dot{+}G'$. Then, the following properties are equivalent.*

*(a) $G$ and $G'$ have the same 3-element homogeneous subsets;*
*(b) $U(xy) = U(xz) \neq U(yz) \Longrightarrow G(xy) \neq G(xz)$ for all distinct elements $x, y, z$ of $V$.*
*(c) The sets $A_1 := E(U) \cap E(G)$ and $A_2 := E(U) \setminus E(G)$ divide $V(S(U))$ into two independent sets and also the sets $B_1 := E(\overline{U}) \cap E(G)$ and $B_2 := E(\overline{U}) \setminus E(G)$ divide $V(S(\overline{U}))$ into two independent sets.*

*Proof.* Observe first that Property (b) is equivalent to the conjunction of the following properties:

($b_U$) If $uv$ is an edge of $S(U)$ then $u \in E(G)$ iff $v \notin E(G)$, and
($b_{\overline{U}}$) If $uv$ is an edge of $S(\overline{U})$ then $u \in E(G)$ iff $v \notin E(G)$.

$(a) \Longrightarrow (b)$.

Let us show $(a) \Longrightarrow (b_U)$. Let $uv \in E(S(U))$, then $u, v \in E(U)$. By contradiction, we may suppose that $u, v \in E(G)$ (the other case implies $u, v \in E(G')$ thus is similar). Since $u$ and $v$ are edges of $U = G\dot{+}G'$ then $u, v \notin E(G')$. Let $w := yz$ such that $u = xy$, $v = xz$. Then $w \notin E(U)$ and thus $w \in E(G)$ iff $w \in E(G')$.

If $w \in E(G)$, $\{x, y, z\}$ is a homogeneous subset of $G$. Since $G$ and $G'$ have the same 3-element homogeneous subsets, $\{x, y, z\}$ is a homogeneous subset of $G'$. Hence, since $u, v \notin E(G')$, $w = yz \notin E(G')$, thus $w \notin E(G)$, a contradiction.

If $w \notin E(G)$, then $w \notin E(G')$; since $u, v \notin E(G')$ it follows that $\{x, y, z\}$ is a homogeneous subset of $G'$. Consequently $\{x, y, z\}$ is a homogeneous subset of $G$. Since $u, v \in E(G)$, then $w \in E(G)$, a contradiction.

The implication $(a) \Longrightarrow (b_{\overline{U}})$ is similar.

$(b) \Longrightarrow (a)$.

Let $T$ be a $K_3$ of $G$. Suppose that $T$ is not a homogeneous subset of $G'$ then we may suppose $T = \{u, v, w\}$ with $u, v \in E(G')$ and $w \notin E(G')$ or $u, v \in E(\overline{G'})$ and $w \notin E(\overline{G'})$. In the first case $uv \in E(S(\overline{U}))$, which contradicts Property $(b_{\overline{U}})$, in the second case $uv \in E(S(U))$, which contradicts Property $(b_U)$.

$(b) \Longrightarrow (c)$.

First $V(S(U)) = E(U) = A_1 \cup A_2$ and $V(S(\overline{U})) = E(\overline{U}) = B_1 \cup B_2$. Let $u, v$ be two distinct elements of $A_1$ (respectively $A_2$). Then $u, v \in E(G)$ (respectively $u, v \notin E(G)$). From $(b_U)$ we have $uv \notin E(S(U))$. Then $A_1$ and $A_2$ are independent sets of $V(S(U))$. The proof that $B_1$ and $B_2$ are independent sets of $V(S(\overline{U}))$ is similar.

$(c) \Longrightarrow (b)$

This implication is trivial.

□

## 2.3. **Proof of Theorem 1.2.**

*Proof.*

$(1) \Longrightarrow (2)$

This implication follows directly from implication $(a) \Longrightarrow (c)$ of Lemma 2.2. Indeed, Property (c) implies trivially that $S(U)$ and $S(\overline{U})$ are bipartite.

$(2) \Longrightarrow (1)$.

Suppose that $S(U)$ and $S(\overline{U})$ are bipartite. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be respectively a partition of $V(S(U)) = E(U)$ and $V(S(\overline{U})) = E(\overline{U})$ into independent sets. Note that $A_i \cap B_j = \varnothing$, for $i, j \in \{1, 2\}$. Let $G, G'$ be two graphs with the same vertex set as $U$ such that $E(G) = A_1 \cup B_1$ and $E(G') = A_2 \cup B_1$. Clearly $E(G \dot{+} G') = A_1 \cup A_2 = E(U)$. Thus $U = G \dot{+} G'$. To conclude that Property (1) holds, it suffices to show that $G$ and $G'$ have the same 3-element homogeneous subsets, that is Property (a) of Lemma 2.2 holds. For that, note that $A_1 = E(U) \cap E(G)$, $A_2 = E(U) \setminus E(G)$, $B_1 = E(\overline{U}) \cap E(G)$ and $B_2 = E(\overline{U}) \setminus E(G)$ and thus Property (c) of Lemma 2.2 holds. It follows that Property (a) of this lemma holds.

$(2) \Longrightarrow (3)$

The proof of this implication was given in Section 1.

$(3) \Longrightarrow (2)$

For this converse implication, let $U$ be a graph satisfying Property (3). It is clear from Figure 1 that $S(P_9)$ is bipartite (vertical edges and horizontal edges form a partition). Since $\overline{P_9}$ is isomorphic to $P_9$, $S(\overline{P_9})$ is bipartite too. Thus, if $U$ is isomorphic to an induced subgraph of $P_9$, Property (2) holds. If not, we may suppose that the connected components of $U$ are cycles of even length or paths (otherwise, replace $U$ by $\overline{U}$). In this case, $S(U)$ is trivially bipartite. In order to prove that Property (2) holds, it suffices to prove that $S(\overline{U})$ is bipartite too. This is a direct consequence of the following claim.

**Claim 2.3.** *If $U$ is a bipartite graph, then $S(\overline{U})$ is bipartite too.*

*Proof.* If $c : V(U) \to \mathbb{Z}/2\mathbb{Z}$ is a colouring of $U$, set

$$c' : V(S(\overline{U})) \to \mathbb{Z}/2\mathbb{Z}$$

defined by

$$c'(\{x, y\}) := c(x) + c(y).$$

□

With this, the proof of Theorem 1.2 is complete. □

2.4. **A direct proof for (3) $\Longrightarrow$ (1) of Theorem 1.2.** In [6] we gave all possible decompositions of a graph $U$ satisfying (3) into a Boolean sum $G\dot{+}G'$ where $G$ and $G'$ have the same 3-element homogeneous sets.

When $U = P_9$, a decomposition $U = G\dot{+}G'$ can be given by a picture (see Figure 2). For the other cases, we introduce the following notation.

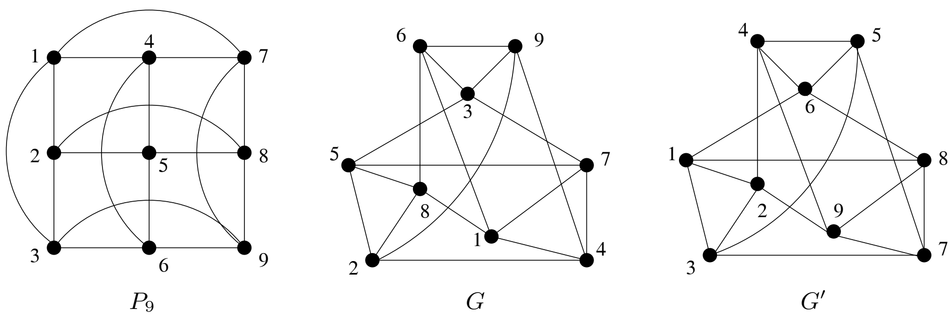


FIGURE 2.

Let $n \geq 2$. Let $X_n$ be an $n$-element set, $x_0, \ldots, x_{n-1}$ be an enumeration of $X_n$,

$$X_n^0 := \{x_i \in X_n : i \equiv 0 \pmod 2\} \text{ and}$$
$$X_n^1 := X_n \setminus X_n^0.$$

Set

$$R_n := [X_n^1]^2 \cup [X_n^2]^2,$$
$$S_n := \{\{x_{2i}, x_{2i+1}\} : 2i < n\}, \text{ and}$$
$$S_n' := \{\{x_{2i+1}, x_{2i+2}\} : 2i < n - 1\}.$$

Let $M_n$ and $M_n'$ be the graphs with vertex set $X_n$ and edge sets $E(M_n) := R_n \cup S_n$ and $E(M_n') := R_n \cup S_n'$, resp. Let

$$M_n'' := (X_n, R_n \cup S_n' \cup \{\{x_0, x_{n-1}\}\})$$

for $n$ even, $n \geq 4$. For $n \in \{6, 7\}$ we give a picture (see Figure 3). For convenience, we set $M_1 = M_1'$ the graph with one vertex and we put $V(M_1) := X_1^0 := \{x_0\}$. When $G$ is a graph of the form $M_n$, $M_n'$, or $M_n''$, with $n \geq 1$, we put $V^0(G) := X_n^0$ and $V^1(G) := X_n^1$.

When $U$ is a cycle of even size $2n$, a decomposition $U = G\dot{+}G'$ can be given by $G = M_{2n}$ and $G' = M_{2n}''$. When $U$ is a path of size $n$, a decomposition $U = G\dot{+}G'$ can be given by $G = M_n$ and $G' = M_n'$.

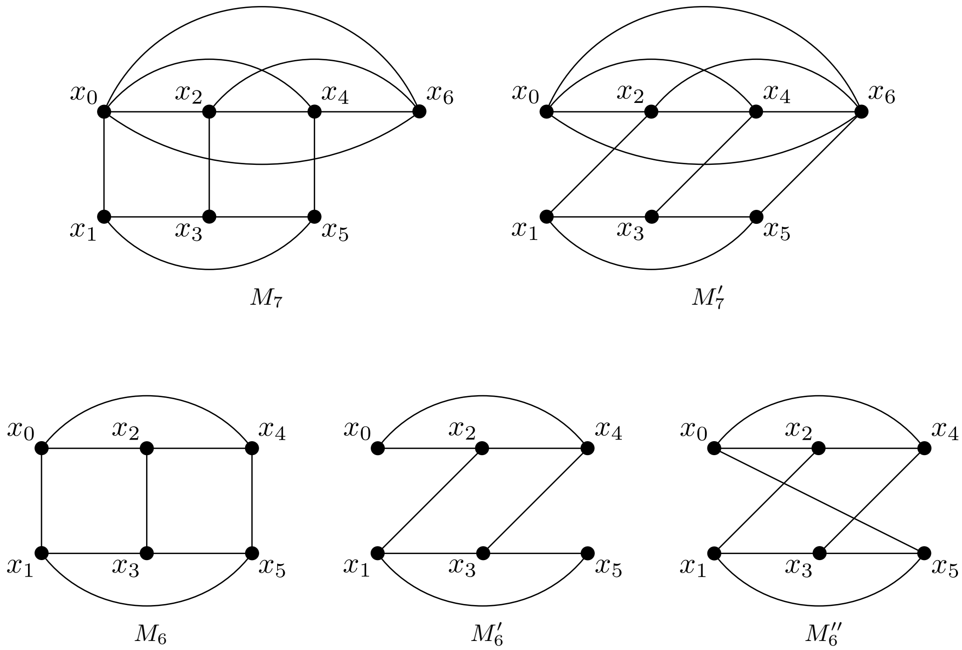


FIGURE 3.

When the connected components of $U$ are cycles of even length or paths, we define $G$ and $G'$ satisfying $U = G\dot{+}G'$ as follows: For each connected component $C$ of $U$, $(G_C, G'_C)$ is given by the previous step. For distinct connected components $C$ and $C'$ of $U$, $x \in C$, $x' \in C'$, $xx' \in E(G)$ (and $xx' \in E(G')$) if and only if $x \in V^0(G_C)$ and $x' \in V^0(G_{C'})$, or $x \in V^1(G_C)$ and $x' \in V^1(G_{C'})$.

When the connected components of $\overline{U}$ are cycles of even length or paths, from $\overline{U} = \overline{G}\dot{+}G'$, the previous step gives a pair $(\overline{G}, G')$, then a pair $(G, G')$.

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