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SIGNED STAR k-DOMATIC NUMBER OF A GRAPH

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Abstract. Let G be a simple graph without isolated vertices with vertex set $V(G)$ and edge set $E(G)$ and let k be a positive integer. A function $f : E(G) \to \{-1, 1\}$ is said to be a signed star k-dominating function on G if $\sum_{e\in E(v)} f(e) \geq k$ for every vertex v of G, where $E(v) = \{uv \in E(G) \mid u \in N(v)\}$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed star k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq 1$ for each $e \in E(G)$, is called a signed star k-dominating family (of functions) on G. The maximum number of functions in a signed star k-dominating family on G is the signed star k-domatic number of G, denoted by $d_{kSS}(G)$.

In this paper we study the properties of the signed star k-domatic number $d_{kSS}(G)$. In particular, we determine the signed star k-domatic number of some classes of graphs. Some of our results extend these one given by Atapour et al. [1] for the signed star domatic number.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [8] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. For every nonempty subset E' of $E(G)$, the subgraph $G[E']$ induced by E' is the graph whose vertex set consists of those vertices of G incident with at least one edge of E' and whose edge set is E'.

Two edges e_1 and e_2 of G are called *adjacent* if they are distinct and have a common vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \to \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e\in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges incident with the vertex v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e\in E_G(v)} f(e)$.

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Let k be a positive integer. A function $f : E(G) \to \{-1, 1\}$ is called a *signed star k-dominating function* (SSkDF) on G, if $f(v) \geq k$ for every vertex v of G. The *signed star k-domination number* of a graph G is $\gamma_{kSS}(G) = \min\{\sum_{e\in E} f(e) \mid f \text{ is a SSkDF on } G\}$. The signed star k-dominating function f on G with $f(E(G)) = \gamma_{kSS}(G)$ is called a $\gamma_{kSS}(G)$-*function*. As assuming $\delta(G) \geq k$ is clearly necessary, we will always assume that when we discuss $\gamma_{kSS}(G)$ all graphs involved satisfy $\delta(G) \geq k$. The signed star k-domination number, introduced by Xu and Li in [11], has been studied by several authors (see for instance [2, 7]). The signed star 1-domination number is the usual signed star domination number which has been introduced by Xu in [9] and has been studied by several authors (see for instance [5, 6, 10]).

A set $\{f_1, f_2, \ldots, f_d\}$ of signed star k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq 1$ for each $e \in E(G)$, is called a *signed star k-dominating family* (of functions) on G. The maximum number of functions in a signed star k-dominating family on G is the *signed star k-domatic number* of G, denoted by $d_{kSS}(G)$. The signed star k-domatic number is well-defined and $d_{kSS}(G) \geq 1$ for all graphs G with $\delta(G) \geq k$, since the set consisting of any one SSkD function forms a SSkD family on G. A d_{kSS}-*family* of a graph G is a SSkD family containing $d_{kSS}(D)$ SSkD functions. The signed star 1-domatic number $d_{1SS}(G)$ is the usual signed star domatic number $d_{SS}(G)$ which was introduced by Atapour et al. in [1].

Our purpose in this paper is to initiate the study of signed star k-domatic number in graphs. We first study basic properties and bounds for the signed star k-domatic number of a graph, some of which are analogous to those of the signed star domatic number $d_{SS}(G)$ in [1]. In addition, we determine the signed star k-domatic number of some classes of graphs.

Observation 1.1. *Let G be a graph of order $n \geq 3$ and size m. If $k \in \{n-2, n-1\}$ and $\delta(k) \geq k$, then $\gamma_{kSS}(G) = m$ and hence $d_{kSS}(G) = 1$.*

Observation 1.2. *Let G be a graph of size m with $\delta(G) \geq k$. Then $\gamma_{kSS}(G) = m$ if and only if each edge $e \in E(G)$ has an endpoint u such that* $\deg(u) = k$ *or* $\deg(u) = k + 1$.

Proof. If each edge $e \in E(G)$ has an endpoint u such that $\deg(u) = k$ or $\deg(u) = k + 1$, then trivially $\gamma_{kSS}(G) = m$.

Conversely, assume that $\gamma_{kSS}(G) = m$. Suppose to the contrary that there exists an edge $e = uv \in E(G)$ such that $\min\{\deg(u), \deg(v)\} \geq k + 2$. Define $f : E(G) \to \{-1, 1\}$ by $f(e) = -1$ and $f(e') = 1$ for $e' \in E(G) \setminus \{e\}$. Obviously, f is a signed star k-dominating function of G with weight less than m, a contradiction. This completes the proof. □

2. Basic properties of the signed star k-domatic number

In this section we study basic properties of $d_{kSS}(G)$.

Theorem 2.1. *Let G be a graph of size m with $\delta(G) \geq k$, signed star k-domination number $\gamma_{kSS}(G)$ and signed star k-domatic number $d_{kSS}(G)$. Then*

$$\gamma_{kSS}(G) \cdot d_{kSS}(G) \leq m.$$

Moreover, if we have $\gamma_{kSS}(G) \cdot d_{kSS}(G) = m$, then for each d_{kSS}-family $\{f_1, f_2, \ldots, f_d\}$ of G, each function f_i is a γ_{kSS}-function and $\sum_{i=1}^{d} f_i(e) = 1$ for all $e \in E(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed star k-dominating family on G such that $d = d_{kSS}(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{kSS}(G) &= \sum_{i=1}^{d} \gamma_{kSS}(G) \leq \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) \\ &= \sum_{e \in E(G)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E(G)} 1 = m \end{aligned}$$

as desired.

If $\gamma_{kSS}(G) \cdot d_{kSS}(G) = m$, then the two inequalities occurring in the proof become equalities. Hence for the d_{kSS}-family $\{f_1, f_2, \ldots, f_d\}$ of G and for each i, $\sum_{e \in E(G)} f_i(e) = \gamma_{kSS}(G)$, thus each function f_i is a γ_{kSS}-function, and $\sum_{i=1}^{d} f_i(e) = 1$ for all $e \in E(G)$. □

Corollary 2.2. *If G is a graph of size m and $\delta(G) \geq k$, then*

$$\gamma_{kSS}(G) + d_{kSS}(G) \leq m + 1 .$$

Proof. By Theorem 2.1,

$$\gamma_{kSS}(G) + d_{kSS}(G) \leq d_{kSS}(G) + \frac{m}{d_{kSS}(G)} . \tag{2.1}$$

Using the fact that the function $g(x) = x + m/x$ is decreasing for $1 \leq x \leq \sqrt{m}$ and increasing for $\sqrt{m} \leq x \leq m$, this inequality leads to the desired bound immediately. □

Corollary 2.3. *Let G be a graph of size m and $\delta(G) \geq k$. If $2 \leq \gamma_{kSS}(G) \leq m - 1$, then*

$$\gamma_{kSS}(G) + d_{kSS}(G) \leq m.$$

Proof. Theorem 2.1 implies that

$$\gamma_{kSS}(G) + d_{kSS}(G) \leq \gamma_{kSS}(G) + \frac{m}{\gamma_{kSS}(G)} . \tag{2.2}$$

If we define $x = \gamma_{kSS}(G)$ and $g(x) = x + m/x$ for $x > 0$, then because $2 \leq \gamma_{kSS}(G) \leq m - 1$, we have to determine the maximum of the function

g in the interval $I : 2 \leq x \leq m-1$. It is easy to see that

$$\begin{aligned}\max_{x\in I}\{g(x)\} &= \max\{g(2), g(m-1)\} \\ &= \max\left\{2+\frac{m}{2}, m-1+\frac{m}{m-1}\right\} \\ &= m-1+\frac{m}{m-1} < m+1,\end{aligned}$$

and we obtain $\gamma_{kSS}(G) + d_{kSS}(G) \leq m$. This completes the proof. □

Corollary 2.4. *Let $k \geq 1$ be an integer, and let G be a graph of size m and $\delta(G) \geq k$. If $\min\{\gamma_{kSS}(G), d_{kSS}(G)\} \geq 2$, then*

$$\gamma_{kSS}(G) + d_{kSS}(G) \leq \frac{m}{2} + 2\ .$$

Proof. Since $\min\{\gamma_{kSS}(G), d_{kSS}(G)\} \geq 2$, it follows by Theorem 2.1 that $2 \leq d_{kSS}(G) \leq m/2$. By (2.1) and the fact that the maximum of $g(x) = x + m/x$ on the interval $2 \leq x \leq m/2$ is $g(2) = g(m/2)$, we see that

$$\gamma_{kSS}(G) + d_{kSS}(G) \leq d_{kSS}(G) + \frac{m}{d_{kSS}(G)} \leq \frac{m}{2} + 2\ .$$

□

Observation 1.1 demonstrates that Corollary 2.4 is no longer true in the case that $\min\{\gamma_{kSS}(G), d_{kSS}(G)\} = 1$.

Theorem 2.5. *Let G be a graph with $\delta(G) \geq k$ and let $v \in V(G)$. Then*

$$d_{kSS}(G) \leq \begin{cases} \dfrac{\deg(v)}{k} & \textit{if } \deg(v) \equiv k \pmod 2\ , \\ \dfrac{\deg(v)}{k+1} & \textit{if } \deg(v) \equiv k+1 \pmod 2\ . \end{cases}$$

Moreover, if the equality holds, then for each function f_i of a SSkD family $\{f_1, f_2, \ldots, f_d\}$ and for every $e \in E(v)$,

$$\sum_{e\in E(v)} f_i(e) = \begin{cases} k & \textit{if } \deg(v) \equiv k \pmod 2\ , \\ k+1 & \textit{if } \deg(v) \equiv k+1 \pmod 2\ , \end{cases}$$

and $\sum_{i=1}^{d} f_i(e) = 1$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SSkD family of G such that $d = d_{kSS}(G)$. If $\deg(v) \equiv k \pmod 2$, then

$$d = \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \frac{1}{k} \sum_{e\in E(v)} f_i(e) = \frac{1}{k} \sum_{e\in E(v)} \sum_{i=1}^{d} f_i(e) \leq \frac{1}{k} \sum_{e\in E(v)} 1 = \frac{\deg(v)}{k}\ .$$

Similarly, if $\deg(v) \equiv k+1 \pmod 2$, then

$$d = \sum_{i=1}^{d} 1 \le \sum_{i=1}^{d} \frac{1}{k+1} \sum_{e \in E(v)} f_i(e)$$
$$= \frac{1}{k+1} \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) \le \frac{1}{k+1} \sum_{e \in E(v)} 1 = \frac{\deg(v)}{k+1} .$$

If $d_{kSS}(G) = \deg(v)/k$ when $\deg(v) \equiv k \pmod 2$ or $d_{kSS}(G) = \deg(v)/(k+1)$ when $\deg(v) \equiv k+1 \pmod 2$, then the two inequalities occurring in the proof of each corresponding case become equalities, which gives the properties given in the statement. □

Corollary 2.6. *Let G be a graph and $1 \le k \le \delta(G)$. Then*

$$d_{kSS}(G) \le \begin{cases} \dfrac{\delta(G)}{k} & \text{if } \delta(G) \equiv k \pmod 2 , \\ \dfrac{\delta(G)}{k+1} & \text{if } \delta(G) \equiv k+1 \pmod 2 . \end{cases}$$

Theorem 2.7. *The signed star k-domatic number is an odd integer.*

Proof. Let G be an arbitrary graph, and assume that $d = d_{kSS}(G)$ is even. Let $\{f_1, f_2, \ldots, f_d\}$ be the corresponding signed star k-dominating family on G. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \le 1$. On the left-hand side of this inequality, a sum of an even number of odd summands occurs. Therefore it is an even number, and we obtain $\sum_{i=1}^{d} f_i(e) \le 0$ for each $e \in E(G)$. This forces

$$kd = \sum_{i=1}^{d} k \le \sum_{i=1}^{d} \sum_{e \in E(v)} f_i(e) = \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) \le 0 ,$$

which is a contradiction. □

An immediate consequence of Theorems 2.5, 2.7 and Corollary 2.6 is the following result.

Corollary 2.8. *Let G be a graph with $\delta(G) \ge k$. If $\delta(G) < 3k$ or if G has a vertex v of degree $\deg(v) = 3k+1$, then $d_{kSS}(G) = 1$.*

Proof. If $\delta(G) < 3k$, then Corollary 2.6 implies that

$$d_{kSS}(G) \le \frac{\delta(G)}{k} < \frac{3k}{k} = 3.$$

Applying Theorem 2.7, we deduce that $d_{kSS}(G) \le 1$ and thus $d_{kSS}(G) = 1$. If G has a vertex v of degree $\deg(v) = 3k+1$, then $\deg(v) \equiv k+1 \pmod 2$ and thus it follows from Theorem 2.5 that

$$d_{kSS}(G) \le \frac{\deg(v)}{k+1} = \frac{3k+1}{k+1} < 3.$$

Again Theorem 2.7 leads to the desired result $d_{kSS}(G) = 1$. □

Corollary 2.9. *Let G be a graph of size m. Then $\gamma_{kSS}(G)+d_{kSS}(G) = m+1$ if and only if each edge $e \in E(G)$ has an endpoint u such that $\deg(u) = k$ or $\deg(u) = k+1$.*

Proof. If each edge $e \in E(G)$ has an endpoint u such that $\deg(u) = k$ or $\deg(u) = k+1$, then $\gamma_{kSS}(G) = m$ by Observation 1.2. Hence $d_{kSS}(G) = 1$ and the result follows.

Conversely, let $\gamma_{kSS}(G) + d_{kSS}(G) = m+1$. The result is obviously true for $m = 1, 2, 3$. Assume $m \geq 4$. By Corollary 2.4, we may assume that $\min\{\gamma_{kSS}(G), d_{kSS}(G)\} = 1$. If $\gamma_{kSS}(G) = 1$, then $d_{kSS}(G) = m$, which is a contradiction to Corollary 2.6. If $d_{kSS}(G) = 1$, then $\gamma_{kSS}(G) = m$ and the result follows by Observation 1.2. □

As an application of Corollary 2.6 and Theorem 2.7, we will prove the following Nordhaus-Gaddum type result.

Theorem 2.10. *For every graph G of order n with $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$,*

$$d_{kSS}(G) + d_{kSS}(\overline{G}) \leq \frac{n-1}{k}. \tag{2.3}$$

If $d_{kSS}(G) + d_{kSS}(\overline{G}) = (n-1)/k$, then G is regular, k and $\delta(G)$ are even and n is odd such that $n-1 \equiv 0 \pmod 4$.

Proof. Since $\delta(G) + \delta(\overline{G}) \leq n-1$, Corollary 2.6 leads to

$$d_{kSS}(G) + d_{kSS}(\overline{G}) \leq \frac{\delta(G)}{k} + \frac{\delta(\overline{G})}{k} \leq \frac{n-1}{k}\ .$$

If G is not regular, then $\delta(G) + \delta(\overline{G}) \leq n-2$ and hence we obtain the better bound $d_{kSS}(G) + d_{kSS}(\overline{G}) \leq (n-2)/k$. Thus assume now that G is $\delta(G)$-regular.

<u>Case 1:</u> Assume that k is odd. If $\delta(G)$ is even, then it follows from Corollary 2.6 that

$$\begin{aligned} d_{kSS}(G) + d_{kSS}(\overline{G}) &\leq \frac{\delta(G)}{k+1} + \frac{\delta(\overline{G})}{k} = \frac{\delta(G)}{k+1} + \frac{n-\delta(G)-1}{k} \\ &< \frac{\delta(G)}{k} + \frac{n-\delta(G)-1}{k} = \frac{n-1}{k}\ . \end{aligned}$$

If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Using Corollary 2.6, we find that

$$\begin{aligned} d_{kSS}(G) + d_{kSS}(\overline{G}) &\leq \frac{\delta(G)}{k} + \frac{\delta(\overline{G})}{k+1} = \frac{\delta(G)}{k} + \frac{n-\delta(G)-1}{k+1} \\ &< \frac{\delta(G)}{k} + \frac{n-\delta(G)-1}{k} = \frac{n-1}{k}\ . \end{aligned}$$

Combining these two bounds, we conclude that $d_{kSS}(G) + d_{kSS}(\overline{G}) < (n-1)/k$ when k is odd.

<u>Case 2:</u> Assume that k is even. If $\delta(G)$ is odd, then Corollary 2.6 implies $d_{kSS}(G)+d_{kSS}(\overline{G}) < (n-1)/k$ as above. If $\delta(G)$ is even and n is even, then

$\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the bound $d_{kSS}(G) + d_{kSS}(\overline{G}) < (n-1)/k$ as above.

Finally, assume that $\delta(G)$ is even and n is odd such that $n - 1 = 4p + 2$. If $d_{kSS}(G) + d_{kSS}(\overline{G}) = (n-1)/k$, then we observe that

$$d_{kSS}(G) = \frac{\delta(G)}{k} \quad \text{and} \quad d_{kSS}(\overline{G}) = \frac{\delta(\overline{G})}{k} \ .$$

According to Theorem 2.7, these two values are odd integers, say

$$d_{kSS}(G) = \frac{\delta(G)}{k} = 2s + 1 \quad \text{and} \quad d_{kSS}(\overline{G}) = \frac{\delta(\overline{G})}{k} = 2t + 1 \ .$$

If $k = 2i$, then we arrive at the contradiction

$$d_{kSS}(G) + d_{kSS}(\overline{G}) = \frac{\delta(G)}{k} + \frac{\delta(\overline{G})}{k} = 2(s + t + 1) = \frac{4p + 2}{2i} \ .$$

This contradiction completes the proof of Theorem 2.10. □

The following examples will demonstrate that $d_{kSS}(G) + d_{kSS}(\overline{G}) = (n-1)/k$ in Theorem 2.10 is possible when G is regular, k and $\delta(G)$ are even and n is odd such that $n - 1 \equiv 0 \pmod 4$.

Let $k \geq 2$ be an even integer and $n \geq 5$ such that $n - 1 = 2k$. Now let H be a k-regular graph of order n. Then $\overline{H}$ is also k-regular. Corollary 2.6 implies that $d_{kSS}(H) \leq 1$ and thus $d_{kSS}(H) = 1$. It follows that

$$d_{kSS}(H) + d_{kSS}(\overline{H}) = 2 = \frac{n-1}{k} \ .$$

Corollary 2.11. *Let G be a graph of order n with $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$. If $\delta(G) < 3k$ and $\delta(\overline{G}) < 3k$ or $n < 4k + 1$, then*

$$d_{kSS}(G) + d_{kSS}(\overline{G}) = 2 \ .$$

Proof. If $\delta(G) < 3k$ and $\delta(\overline{G}) < 3k$, then Corollary 2.8 implies the desired result immediately. If $n < 4k + 1$, then it follows from (2.3) that

$$d_{kSS}(G) + d_{kSS}(\overline{G}) \leq \frac{n-1}{k} < \frac{4k}{k} = 4 \ ,$$

and thus Theorem 2.7 leads to $d_{kSS}(G) + d_{kSS}(\overline{G}) = 2$. □

3. Signed star k-domatic number of regular graphs

In this section we determine values of the signed star k-domatic number for some classes of regular graphs.

Theorem 3.1. *Let G be an r-regular and 1-factorable graph and let $1 \leq k \leq r$ be an integer. Then*

$$d_{kSS}(G) = \begin{cases} \left\lfloor \frac{r}{k} \right\rfloor & \text{when } r \equiv k \pmod 2 \text{ and } \left\lfloor \frac{r}{k} \right\rfloor \text{ is odd}, \\ \left\lfloor \frac{r}{k} \right\rfloor - 1 & \text{when } r \equiv k \pmod 2 \text{ and } \left\lfloor \frac{r}{k} \right\rfloor \text{ is even}, \\ \left\lfloor \frac{r}{k+1} \right\rfloor & \text{when } r \equiv k+1 \pmod 2 \text{ and } \left\lfloor \frac{r}{k+1} \right\rfloor \text{ is odd}, \\ \left\lfloor \frac{r}{k+1} \right\rfloor - 1 & \text{when } r \equiv k+1 \pmod 2 \text{ and } \left\lfloor \frac{r}{k+1} \right\rfloor \text{ is even}. \end{cases}$$

Proof. By Observation 1.2 and Theorem 2.1 we may assume $k \leq r-2$. Let $\{M_0, M_1, \ldots, M_{r-1}\}$ be a 1-factorization of G. We distinguish two cases.

Case 1: Assume that $r \equiv k \pmod 2$. Suppose that $r = kq + t$, where q is a positive integer and $0 \leq t \leq k-1$. By Corollary 2.6 and Theorem 2.7, $d_{kSS}(G) \leq q$ if q is odd and $d_{kSS}(G) \leq q-1$ if q is even.

Subcase 1.1: Assume that q is odd. Then t is even. Define the functions $f_1, f_2, \ldots, f_q$ as follows.

$$f_1(e) = \begin{cases} 1 & \text{if } e \in M_i \text{ where } 0 \leq i \leq \frac{k(q-1)}{2} + k - 1, \\ -1 & \text{if } e \in M_i \text{ and } \frac{k(q-1)}{2} + k \leq i \leq kq - 1, \end{cases}$$

and for $2 \leq j \leq q$ and $0 \leq i \leq kq-1$,

$$f_j(M_i) = f_{j-1}(M_{i+2k}),$$

where the sum is taken modulo kq. In addition, if $t > 0$,

$$f_j(M_i) = (-1)^{i+j} \text{ for } 1 \leq j \leq q \text{ and } kq \leq i \leq r-1.$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \leq j \leq q$ and $\{f_1, f_2, \ldots, f_q\}$ is a signed star k-dominating family of G. Hence $d_{kSS}(G) \geq q$. Therefore $d_{kSS}(G) = q$, as desired.

Subcase 1.2: Assume that q is even. Then $t + k$ is even. Define the functions $f_1, f_2, \ldots, f_{q-1}$ as follows.

$$f_1(M_i) = \begin{cases} 1 & \text{if } 0 \leq i \leq \frac{k(q-2)}{2} + k - 1, \\ -1 & \text{if } \frac{k(q-2)}{2} + k \leq i \leq k(q-1) - 1, \end{cases}$$

and for $2 \leq j \leq q-1$ and $0 \leq i \leq k(q-1)-1$,

$$f_j(M_i) = f_{j-1}(M_{i+2k}),$$

where the sum is taken modulo $k(q-1)$. In addition,

$$f_j(M_i) = (-1)^{i+j} \text{ for } 1 \leq j \leq q \text{ and } k(q-1) \leq i \leq r-1.$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, f_2, \ldots, f_{q-1}\}$ is a signed star k-dominating family on G. Hence $d_{kSS}(G) \ge q-1$ and so $d_{kSS}(G) = q-1$.

Case 2: Assume that $r \equiv k+1 \pmod 2$. Suppose that $r = (k+1)q + t$, where q is a positive integer and $0 \le t \le k$. By Corollary 2.6 and Theorem 2.7, $d_{kSS}(G) \le q$ if q is odd and $d_{kSS}(G) \le q-1$ if q is even.

Subcase 2.1: Assume that q is odd. Then t is even. Define the functions $f_1, f_2, \ldots, f_q$ as follows.

$$f_1(M_i) = \begin{cases} 1 & \text{if } 0 \le i \le \dfrac{(k+1)(q-1)}{2} + k \ , \\ -1 & \text{if } \dfrac{(k+1)(q-1)}{2} + k + 1 \le i \le (k+1)q - 1 \ , \end{cases}$$

and for $2 \le j \le q$ and $0 \le i \le (k+1)q - 1$,

$$f_j(M_i) = f_{j-1}(M_{i+2(k+1)}) \ ,$$

where the sum is taken modulo $(k+1)q$. In addition, if $t > 0$,

$$f_j(M_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)q \le i \le r-1 \ .$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q$ and $\{f_1, f_2, \ldots, f_q\}$ is a signed star k-dominating family of G. Hence $d_{kSS}(G) \ge q$. Therefore $d_{kSS}(G) = q$, as desired.

Subcase 2.2: Assume that q is even. Then $t + k + 1$ is even. Define the functions $f_1, f_2, \ldots, f_{q-1}$ as follows.

$$f_1(M_i) = \begin{cases} 1 & \text{if } 0 \le i \le \dfrac{(k+1)(q-2)}{2} + k \ , \\ -1 & \text{if } \dfrac{(k+1)(q-2)}{2} + k + 1 \le i \le (k+1)(q-1) - 1 \ , \end{cases}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+1)(q-1) - 1$,

$$f_j(M_i) = f_{j-1}(M_{i+2(k+1)}) \ ,$$

where the sum is taken modulo $(k+1)(q-1)$. In addition,

$$f_j(M_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)(q-1) \le i \le n-1 \ .$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, f_2, \ldots, f_{q-1}\}$ is a signed star k-dominating family of G. Hence $d_{kSS}(G) \ge q-1$ and so $d_{kSS}(G) = q-1$, as desired. □

Applying Theorem 3.1 and the well-known classical Theorem of König [4] that a k-regular bipartite graph is 1-factorable, we obtain the next result.

Corollary 3.2. *If G is an r-regular bipartite graph and $1 \le k \le r$ is an integer, then*

$$d_{kSS}(G) = \begin{cases} \left\lfloor \frac{r}{k} \right\rfloor & \text{when } r \equiv k \pmod 2 \text{ and } \left\lfloor \frac{r}{k} \right\rfloor \text{ is odd ,} \\ \left\lfloor \frac{r}{k} \right\rfloor - 1 & \text{when } r \equiv k \pmod 2 \text{ and } \left\lfloor \frac{r}{k} \right\rfloor \text{ is even ,} \\ \left\lfloor \frac{r}{k+1} \right\rfloor & \text{when } r \equiv k+1 \pmod 2 \text{ and } \left\lfloor \frac{r}{k+1} \right\rfloor \text{ is odd ,} \\ \left\lfloor \frac{r}{k+1} \right\rfloor - 1 & \text{when } r \equiv k+1 \pmod 2 \text{ and } \left\lfloor \frac{r}{k+1} \right\rfloor \text{ is even .} \end{cases}$$

Theorem 3.3. *Let G be a graph of order n and factorable into r Hamiltonian cycles and let $1 \le k \le 2r$ be an integer. Then*

$$d_{kSS}(G) = \begin{cases} \left\lfloor \frac{2r}{k} \right\rfloor & \text{when } k \text{ is even and } \left\lfloor \frac{2r}{k} \right\rfloor \text{ is odd ,} \\ \left\lfloor \frac{2r}{k} \right\rfloor - 1 & \text{when } k \text{ and } \left\lfloor \frac{2r}{k} \right\rfloor \text{ are even ,} \\ \left\lfloor \frac{2r}{k+1} \right\rfloor & \text{when } k \text{ and } \left\lfloor \frac{2r}{k+1} \right\rfloor \text{ are odd ,} \\ \left\lfloor \frac{2r}{k+1} \right\rfloor - 1 & \text{when } k \text{ is odd and } \left\lfloor \frac{2r}{k+1} \right\rfloor \text{ is even .} \end{cases}$$

Proof. Let G be a Hamiltonian factorable graph, and let $\{C_0, C_1, \ldots, C_{r-1}\}$ be a Hamiltonian factorization of G. We distinguish two cases.

Case 1: Assume that k is even. Suppose that $2r = kq + t$, where q is a positive integer and $0 \le t \le k-1$. By Corollary 2.6 and Theorem 2.7, $d_{kSS}(G) \le q$ if q is odd and $d_{kSS}(G) \le q-1$ if q is even.

Subcase 1.1: Assume that q is odd. Then t is even and $r = (k/2)q + (t/2)$. Define the functions $f_1, f_2, \ldots, f_q$ as follows.

$$f_1(C_i) = \begin{cases} 1 & \text{if } 0 \le i \le \frac{k(q-1)}{4} + \frac{k}{2} - 2 \,, \\ -1 & \text{if } \frac{k(q-1)}{4} + \frac{k}{2} - 1 \le i \le \frac{k}{2}q - 1 \,, \end{cases}$$

and for $2 \le j \le q$ and $0 \le i \le \frac{k}{2}q - 1$,

$$f_j(C_i) = f_{j-1}(C_{i+k}) \,,$$

where the sum is taken modulo $(k/2)q$. In addition, if $t > 0$,

$$f_j(C_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } \frac{k}{2}q \le i \le r-1 \,.$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q$ and $\{f_1, f_2, \ldots, f_q\}$ is a signed star k-dominating family of G. Hence $d_{kSS}(G) \ge q$. Therefore $d_{kSS}(G) = q$, as desired.

Subcase 1.2: Assume that q is even. Then $(k/2)+(t/2)$ is even. Define the functions $f_1, f_2, \ldots, f_{q-1}$ as follows.

$$f_1(C_i) = \begin{cases} 1 & \text{if } 0 \le i \le \dfrac{k(q-2)}{4} + \dfrac{k}{2} - 2 \ , \\ -1 & \text{if } \dfrac{k(q-2)}{4} + \dfrac{k}{2} - 1 \le i \le \dfrac{k}{2}(q-1) - 1 \ , \end{cases}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k/2)(q-1)-1$,

$$f_j(M_i) = f_{j-1}(M_{i+k}) \ ,$$

where the sum is taken modulo $(k/2)(q-1)$. In addition,

$$f_j(C_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } \frac{k}{2}(q-1) \le i \le r-1 \ .$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, f_2, \ldots, f_{q-1}\}$ is a signed star k-dominating family on G. Hence $d_{kSS}(G) \ge q-1$ and so $d_{kSS}(G) = q-1$.

Case 2: Assume that k is odd. Suppose that $2r = (k+1)q + t$, where q is a positive integer and $0 \le t \le k$. By Corollary 2.6 and Theorem 2.7, $d_{kSS}(G) \le q$ if q is odd and $d_{kSS}(G) \le q-1$ if q is even.

Subcase 2.1: Assume that q is odd. Then t is even. Define the functions $f_1, f_2, \ldots, f_q$ as follows.

$$f_1(C_i) = \begin{cases} 1 & \text{if } 0 \le i \le \dfrac{(k+1)(q-1)}{4} + \dfrac{k+1}{2} - 2 \ , \\ -1 & \text{if } \dfrac{(k+1)(q-1)}{4} + \dfrac{k+1}{2} - 1 \le i \le \dfrac{(k+1)}{2}q - 1 \ , \end{cases}$$

and for $2 \le j \le q$ and $0 \le i \le (k+1)q/2 - 1$,

$$f_j(C_i) = f_{j-1}(C_{i+(k+1)}) \ ,$$

where the sum is taken modulo $(k+1)q/2$. In addition, if $t > 0$,

$$f_j(C_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } \frac{(k+1)}{2}q \le i \le r-1 \ .$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q$ and $\{f_1, f_2, \ldots, f_q\}$ is a signed star k-dominating family of G. Hence $d_{kSS}(G) \ge q$. Therefore $d_{kSS}(G) = q$, as desired.

Subcase 2.2: Assume that q is even. Then $t/2 + (k+1)/2$ is even. Define the functions $f_1, f_2, \ldots, f_{q-1}$ as follows.

$$f_1(C_i) = \begin{cases} 1 & \text{if } 0 \le i \le \dfrac{(k+1)(q-2)}{4} + \dfrac{k+1}{2} - 2 \ , \\ -1 & \text{if } \dfrac{(k+1)(q-2)}{4} + \dfrac{k+1}{2} - 1 \le i \le \dfrac{(k+1)}{2}(q-1) - 1 \ , \end{cases}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+1)(q-1)/2-1$,

$$f_j(C_i) = f_{j-1}(C_{i+(k+1)}) ,$$

where the sum is taken modulo $(k+1)(q-1)/2$. In addition,

$$f_j(C_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } \frac{(k+1)}{2}(q-1) \le i \le r-1 .$$

It is easy to see that f_j is a signed star k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, f_2, \ldots, f_{q-1}\}$ is a signed star k-dominating family of G. Hence $d_{kSS}(G) \ge q-1$ and so $d_{kSS}(G) = q-1$, as desired. □

According to Theorems 3.1, 3.3 and the following two well-known results, we can determine the signed star k-domatic number of complete graphs.

Theorem. *The complete graph K_{2r} is 1-factorable.*

Theorem. *For every positive integer r, the graph K_{2r+1} is Hamiltonian factorable.*

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