Contributions to Discrete Mathematics

Volume 5, Number 2, Pages 67–75
ISSN 1715-0868

HOMOTOPY TYPES OF INDEPENDENCE COMPLEXES OF FORESTS

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Abstract. Ehrenborg and Hetyei [3] proved that the independence complex of an arbitrary forest is either contractible or homotopy equivalent to a sphere. The present paper provides an inductive method of detecting the contractibility of the complex or the dimension of the sphere to which the complex is homotopy equivalent. Compared with a previous solution obtained by a general framework due to Marietti and Testa [7, 8], our approach is straightforward in compensation for restricting the consideration to independence complexes.

1. Introduction and Preliminaries

The present paper deals with the homotopy types of the independence complexes of forests. For a finite simple graph G, the *independence complex* $\mathrm{I}(G)$ is an abstract simplicial complex whose simplexes are the independent sets of vertices of G. Topology of independence complexes has been studied by several authors; see, for example, [3, 4, 5, 9]. For example, the independence complex of the path P_n with n edges is contractible if $n \equiv 0 \pmod 3$ and is homotopy equivalent to the sphere of dimension $\lfloor \frac{n}{3} \rfloor$ otherwise (see Example 3.4). Furthermore, Ehrenborg and Hetyei [3] proved that the independence complex of an arbitrary forest is either contractible or is homotopy equivalent to a sphere. They posed a question of detecting the contractibility or the dimension of the associated sphere. The present paper provides an answer to this question: for each tree T, we assign $n(T) \in \{0, 1, 2, \ldots, \infty\}$ which is determined by an induction on the number of edges of T such that $\mathrm{I}(T)$ is homotopy equivalent to the $n(T)$-dimensional sphere if $n(T) < \infty$ and is contractible if $n(T) = \infty$ (See Section 2.1 and Main Theorem). The general case for forests follows easily from the result (Observation 1.3).

After submitting the first manuscript of the present paper, the author got aware of the papers [7], in which the question of Ehrenborg-Hetyei above had been answered. Marietti and Testa introduced the notion of the core of a simplicial complex to study the homotopy types of dominance complexes and independence complexes of forests simultaneously. The answer to the above

Received by the editors March 22, 2009, and in revised form February 14, 2010.
2000 *Mathematics Subject Classification.* 05C10, 55P10, 05C05, 05C69, 05C99.
Key words and phrases. Independence complex, forest, tree, homotopy.

question is given as an application of their general result: the independence complex of a forest F is contractible or is homotopy equivalent to a sphere of dimension $(\gamma(F)-1)$, where $\gamma(F)$ denotes the domination number of the forest F. They proceeded further in [8] to provide a general framework for the study of various complexes associated with forests. In contrast with these general and strong results of [7] and [8], the feature of our approach is its simplicity. In compensation for confining ourselves to independence complexes of forests, our inductive method is rather straightforward and the proof is an easy application of Mayer-Vietoris sequences.

In the rest of this section, we fix notation and state some preliminary results. We follow [1] and [2] for terminologies on graphs. Throughout the paper, all graphs are finite, simple and undirected. A forest is a graph which contains no cycles. A tree is a connected forest. The complete bipartite graph with partite sets consisting of m and n elements is denoted by $K_{m,n}$. For a graph G with the vertex set $V(G)$, a subset A of $V(G)$ is said to be *independent* if no two vertices of A are adjacent. For a vertex v of G, let $N(v) = \{w \in V(G) \mid v \text{ and } w \text{ are adjacent}\}$, the set of neighbors of v and also let $\deg_G(v) := |N(v)|$, the degree of v of G. For a subset U of the vertex set $V(G)$ of G, $G - U$ denotes the subgraph of G induced by the vertex set $V(G) - U$ [2, p. 4]. For two graphs G and H, $G \cup H$ denotes the union of G and H in the sense of [2, p. 3].

An abstract simplicial complex K over the vertex set V is a collection of subsets of V such that

(1) $\{v\} \in K$ for each $v \in V$, and
(2) if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.

The geometric realization of the complex K is denoted by the same symbol K for simplicity. The join of (abstract) simplicial complexes K and L is denoted by $K * L$.

Throughout the present paper, S^n denotes the n-dimensional sphere and Δ^n denotes the n-dimensional simplex. The symbol $X \simeq Y$ means that two topological spaces X and Y are homotopy equivalent.

For a graph G with the vertex set $V(G)$, the *independence complex* $\mathrm{I}(G)$ of G is an abstract simplicial complex defined by

$$\mathrm{I}(G) = \{A \mid A \text{ is an independent subset of } V(G)\}.$$

The following theorem is fundamental of the present paper.

Theorem 1.1 ([3, Theorem 5.3 and Corollary 6.1])**.** *For each forest F, $\mathrm{I}(F)$ is either contractible or is homotopy equivalent to a sphere.*

For a topological space X, $\tilde{\mathrm{H}}_*(X)$ denotes the reduced singular homology of X with the integer coefficients. The above theorem leads to the following observation which reduces the detection of the homotopy type of $\mathrm{I}(F)$ to the computation of the homology $\tilde{\mathrm{H}}_*(\mathrm{I}(F))$.

Observation 1.2. *For a forest F, $\tilde{\mathrm{H}}_q(\mathrm{I}(F)) = 0$ for all but at most one q. If $\tilde{\mathrm{H}}_q(\mathrm{I}(F)) = 0$ for all q, then $\mathrm{I}(F)$ is contractible. Otherwise, $\mathrm{I}(T)$ is homotopy equivalent to the n-dimensional sphere, where n is the unique non-negative integer such that $\tilde{\mathrm{H}}_n(\mathrm{I}(F)) \neq 0$. Moreover, we have $\tilde{\mathrm{H}}_n(\mathrm{I}(F)) \cong \mathbb{Z}$.*

For a graph G with the components $G_1, \dots, G_m$, the independence complex $\mathrm{I}(G)$ is easily seen to be isomorphic to the m-fold join $\mathrm{I}(G_1) * \cdots * \mathrm{I}(G_m)$. Hence we have:

Observation 1.3. *The complex $\mathrm{I}(G)$ is contractible if $\mathrm{I}(G_i)$ is contractible for some i and is homotopy equivalent to the $(m-1+\sum_{i=1}^{m} n_i)$-dimensional sphere if $\mathrm{I}(G_i) \simeq S^{n_i}$ $(n_i \geq 0)$ for each $i = 1, \dots, m$.*

As was mentioned earlier, the above reduces the detection of homotopy type of $\mathrm{I}(F)$ for a forest F to the one of the independence complex of a tree.

2. Main theorem

For a tree T with the vertex set $V(T)$ and for two vertices u, v of T, the unique path joining u with v is denoted by $P_T(u, v)$. The length of the path $P_T(u, v)$, that is, the number of its edges, is denoted by $d_T(u, v)$. In what follows, we specify a vertex r of T, referred as a *root* of T, and $P_T(r, v)$ is simply denoted by $P_T(v)$. Let $l_T(v) = d_T(r, v)$ for $v \in V(T)$. This defines a function $l_T : V(G) \to [0, \infty)$, called the *level function*. Let $D_T = \max_{v \in V(G)} l_T(v)$. While the level function and D_T depend on the choice of r, we will fix the root r throughout and omit the reference to r.

When $D_T = 0$, then T consists of the single vertex r and hence $\mathrm{I}(T)$ is a singleton. When $D_T = 1$, then T is the complete bipartite graph $K_{1,d}$ where $d = \deg_T(r)$, and we see that $\mathrm{I}(T)$ is homeomorphic to the disjoint union $\{r\} \amalg \Delta^{d-1}$ which is homotopy equivalent to S^0.

2.1. **An auxiliary construction.** In the sequel, we assume $D_T \geq 2$. For a tree T with $D := D_T \geq 2$ and for a vertex v with $l_T(v) = D$, we construct four subtrees T_0, T_1, T_2 and T_3 of T as follows. Their homotopy types will be examined in the next section.

For a vertex $u \in V(T)$, let $V(T)_{\geq u} = \{w \in V(T) \mid u \in P_T(w)\}$ and let $V(T)_{>u} = V(T)_{\geq u} \setminus \{u\}$. The subtree of T induced by $V(T)_{\geq u}$ is denoted by $T_{\geq u}$. Observe that

$$V(T)_{\geq u} \subset l_T^{-1}([l_T(u), D]) \quad \text{and} \quad V(T)_{>u} \subset l_T^{-1}([l_T(u)+1, D]).$$

Also let

$$N_T^{+1}(u) = \{w \in N(u) \mid l_T(w) = l_T(u) + 1\} \subset N_T(u).$$

Fix a vertex v with $l_T(v) = D$ and let

$$P_T(v) : \; r = v_0 \; v_1 \; \cdots \; v_{D-2} \; v_{D-1} \; v_D = v$$

be the unique path joining r with v.

Step 1. Consider the set $N_T^{+1}(v_{D-2})$ and let

$$N_T^{+1*}(v_{D-2}) = \{u \in N_T^{+1}(v_{D-2}) \mid N_T^{+1}(u) \neq \varnothing\},$$

the set of vertices which "cover v_{D-2}" and "are covered by vertices of level D." Observe that $v_{D-1} \in N^{+1*}(v_{D-2}) \subset N_T^{+1}(v_{D-2}) \subset l_T^{-1}(D-1)$. Also let

$$V^{+1*}_{>v_{D-2}} = N_T^{+1*}(v_{D-2}) \cup \bigcup_{u \in N^{+1*}(v_{D-2})} N_T^{+1}(u),$$

and notice $\{v_{D-1}, v\} \subset V^{+1*}_{>v_{D-2}} \subset l_T^{-1}([D-1, D])$. For each $u \in N_T^{+1*}(v_{D-2})$, fix a vertex $a_u \in N_T^{+1}(u) \subset l_T^{-1}(D)$. We make a convention that $a_{v_{D-1}} = v$. Now let T_0 be the tree defined as

$$T_0 = (T - V^{+1*}_{>v_{D-2}}) \cup \bigcup_{u \in N_T^{+1*}(v_{D-2})} v_{D-2}\ u\ a_u.$$

This is a subtree of T obtained by "replacing each tree $T_{\geq u}, u \in N_T^{+1*}(v_{D-2})$, with a single edge $u\ a_u$." Notice that $P(v) \subset T_0$ (Figure 1).

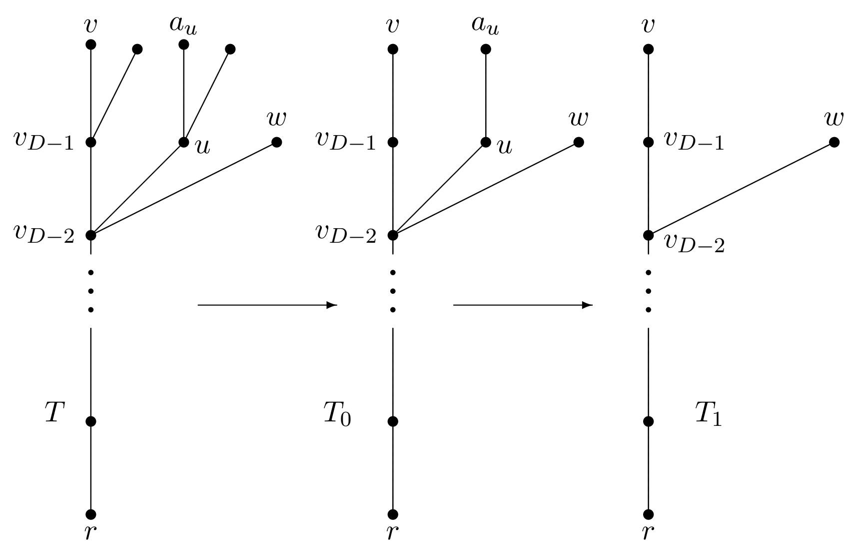

FIGURE 1. T, T_0 ($\nu(v) = 1$) and T_1

Step 2. Let T_1 be the tree defined as

$$T_1 = \left(T_0 - \bigcup_{u \in N^{+1*}(v_{D-2})} v_{D-2}\ u\ a_u \right) \cup v_{D-2}\ v_{D-1}\ v.$$

It is a subtree of T_0 obtained by replacing the subtree

$$\bigcup_{u \in N_T^{+1*}(v_{D-2})} v_{D-2}\ u\ a_u$$

of T_0 with a single path $v_{D-2}\, v_{D-1}\, v$ (Figure 1). Again observe that $P(v) \subset T_1$. For later use, it is convenient to introduce the following notation. We define

$$\nu(v) = |N_{T_0}^{+1*}(v_{D-2})| - 1 \ (= |N_T^{+1*}(v_{D-2})| - 1).$$

For example, $\nu(v) = 1$ for the tree T_0 (and T) in Figure 1.

Step 3. Let $T_2 = T_1 - v$ and $T_3 = T_2 - v_{D-1}$. Then we have a decreasing sequence of trees with root r:

$$T \supset T_0 \supset T_1 \supset T_2 \supset T_3 \supset (P(v) - \{v_{D-1}, v\})$$

such that $D_{T_3} \leq D$.

We assign $n(T), n(T_i) \in \{0, 1, \dots, \infty\}$ for these trees T and T_i $(i = 0, \dots, 3)$ satisfying the following conditions:

(1) If $n(T_3) < \infty$, then $n(T_2) \in \{n(T_3), \infty\}$. With such $n(T_2)$ and $n(T_3)$ being given, $n(T_1)$, $n(T_0)$ and $n(T)$ are determined as follows.

(2) We take
   (a) $n(T_1) = n(T_2)$ if $n(T_3) = \infty$, and
   (b) if $n(T_3) < \infty$, then

$$n(T_1) = \begin{cases} n(T_3) + 1 & \text{if } n(T_2) = \infty, \\ \infty & \text{if } n(T_2) = n(T_3). \end{cases}$$

(3) $n(T) = n(T_0) = n(T_1) + \nu(v)$. Here we make a convention that $n + \infty = \infty$ for each non-negative integer n.

Also we make the following convention.

(4) For a subtree U of T with root r such that $D_U \leq 1$, $n(U)$ is defined by:

$$n(U) = \begin{cases} \infty & \text{if } D_U = 0, \text{ i.e. } U = r, \\ 0 & \text{if } D_U = 1. \end{cases}$$

2.2. **Main theorem.** For a tree T with root r such that $D_T \geq 2$, we pick up a vertex v with $l_T(v) = D$. We carry out the construction in Section 2.1 to obtain a decreasing sequence of trees with root r:

$$T \supset T_0 \supset T_1 \supset T_2 \supset T_3$$

such that $D_{T_3} \leq D_T$. Next we take a vertex $v' \in V(T_3)$ with $l_{T_3}(v') = D_{T_3}$ and repeat the above construction. In this way, we obtain a decreasing sequence of trees

$$T \supset T_0 \supset T_1 \supset T_2 \supset T_3 \supset T_4 \supset \cdots \supset T_{N-1} \supset T_N = r$$

such that $D_{T_{N-1}} = 1$. The conditions (1)–(4) in Section 2.1 determine $n(T)$.

Main Theorem. *Under the above notation,* $\mathrm{I}(T)$ *is contractible if* $n(T) = \infty$ *and is homotopy equivalent to* $S^{n(T)}$ *if* $n(T) < \infty$.

3. Proof of Main Theorem

We will compare the homotopy types of $\mathrm{I}(T)$, $\mathrm{I}(T_0), \ldots, \mathrm{I}(T_3)$ and show that the associated $n(T), n(T_i)$ $(i = 0, \ldots, 3)$ satisfy the conditions (1)–(3) in Section 2.1. Since we already know at the beginning of Section 2 that (4) in Section 2.1 gives the correct answer for every tree U with $D_U \leq 1$, the main theorem follows immediately.

The argument is divided into several steps. Let us first recall the set up.

Set up. T is a tree with root r such that $D_T \geq 2$ and v is a vertex of T with $l_T(v) = D_T$.

Step 1. Repeated applications of [3, Proposition 6.2 (i)] imply that $\mathrm{I}(T_0)$ is homotopy equivalent to $\mathrm{I}(T)$. Also we may apply [3, Proposition 6.2 (iii)] to conclude that $\mathrm{I}(T_0)$ is homotopy equivalent to the $\nu(v)$-fold suspension of $\mathrm{I}(T_1)$. These two imply that the condition (3) of Section 2.1 gives the correct results for T, T_0 and T_1.

Step 2. We examine the homotopy types of $\mathrm{I}(T_1)$, $\mathrm{I}(T_2)$ and $\mathrm{I}(T_3)$ and the result is stated as follows. This corresponds to (1), (2.a) and (2.b) in Section 2.1.

Proposition 3.1.

(1) *If* $\mathrm{I}(T_3)$ *is contractible, then* $\mathrm{I}(T_2)$ *is homotopy equivalent to* $\mathrm{I}(T_1)$.

(2) *If* $\mathrm{I}(T_3)$ *is homotopy equivalent to the* n*-sphere* S^n*, then* $\mathrm{I}(T_2)$ *is contractible or is homotopy equivalent to* S^n*. Moreover*
   - (a) $\mathrm{I}(T_2)$ *is contractible if and only if* $\mathrm{I}(T_1) \simeq S^{n+1}$.
   - (b) $\mathrm{I}(T_2) \simeq S^n$ *if and only if* $\mathrm{I}(T_1)$ *is contractible.*

As is mentioned in Section 1, the proof of the above lemma is based on the computation of homology. For a graph G, a vertex u of G and a vertex v not in G, $G \cup uv$ denotes the graph obtained from G by attaching the edge uv to u.

Lemma 3.2. *Let G be a graph and u be a vertex of G. Take distinct vertices v, w, not in G and let $H = G \cup uv$ and $K = H \cup vw$. Then there exists an exact sequence*

$$\cdots \longrightarrow \tilde{\mathrm{H}}_q(\mathrm{I}(G)) \longrightarrow \tilde{\mathrm{H}}_q(\mathrm{I}(H)) \longrightarrow \tilde{\mathrm{H}}_q(\mathrm{I}(K)) \longrightarrow \tilde{\mathrm{H}}_{q-1}(\mathrm{I}(G)) \longrightarrow \cdots$$

Proof. Let $\mathrm{I}_w(K)$ be the subcomplex of $\mathrm{I}(K)$ generated by the simplices $\{A \in \mathrm{I}(K) \mid w \in A\}$. Then we can see easily that $\mathrm{I}_w(K)$ is isomorphic to the cone over $\mathrm{I}(G)$ with apex w. In particular, $\mathrm{I}_w(K)$ is contractible. Also we have

$$\mathrm{I}(K) = \mathrm{I}_w(K) \cup \mathrm{I}(H), \qquad \mathrm{I}_w(K) \cap \mathrm{I}(H) = \mathrm{I}(G).$$

Thus the Mayer-Vietoris sequence of the pair $(\mathrm{I}_w(K), \mathrm{I}(H))$ (see [10]) yields the desired sequence. □

Proof of Proposition 3.1. Applying Lemma 3.2 to $G = T_3$, $H = T_2 = T_3 \cup v_{D-2}v_{D-1}$ and $K = T_1 = T_2 \cup v_{D-1}v$, we obtain an exact sequence:

$$\cdots \longrightarrow \tilde{\mathrm{H}}_q(\mathrm{I}(T_3)) \longrightarrow \tilde{\mathrm{H}}_q(\mathrm{I}(T_2)) \longrightarrow \tilde{\mathrm{H}}_q(\mathrm{I}(T_1)) \longrightarrow \tilde{\mathrm{H}}_{q-1}(\mathrm{I}(T_3)) \longrightarrow \cdots$$

If $\mathrm{I}(T_3)$ is contractible, then the above implies that $\tilde{\mathrm{H}}_q(\mathrm{I}(T_2)) \cong \tilde{\mathrm{H}}_q(\mathrm{I}(T_1))$ for each q and hence $\mathrm{I}(T_2) \simeq \mathrm{I}(T_1)$ by Observation 1.2. Assume that $\mathrm{I}(T_3)$ is homotopy equivalent to S^n. By the above exact sequence, we obtain

$$\tilde{\mathrm{H}}_q(\mathrm{I}(T_2)) \cong \tilde{\mathrm{H}}_q(\mathrm{I}(T_1)) \text{ for each } q \neq n, n+1. \tag{3.1}$$

Also the above sequence yields

$$\begin{array}{ccccc} 0 \longrightarrow \tilde{\mathrm{H}}_{n+1}(\mathrm{I}(T_2)) & \longrightarrow & \tilde{\mathrm{H}}_{n+1}(\mathrm{I}(T_1)) & & \\ & & \downarrow & & \\ & & \tilde{\mathrm{H}}_n(\mathrm{I}(T_3)) \cong \mathbb{Z} & & \\ & & \downarrow & & \\ & & \tilde{\mathrm{H}}_n(\mathrm{I}(T_2)) & \longrightarrow & \tilde{\mathrm{H}}_n(\mathrm{I}(T_1)) \longrightarrow 0 \end{array} \tag{3.2}$$

We conclude this proof by way of a series of three claims.

Claim. $\mathrm{I}(T_2)$ *is contractible or is homotopy equivalent to* S^n.

Proof. The complex $\mathrm{I}(T_2)$ cannot be homotopy equivalent to S^{n+1}. For, if it could be the case, then the sequence (3.2) would reduce to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\mathrm{H}}_{n+1}(\mathrm{I}(T_1)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

which is impossible as rank $\tilde{\mathrm{H}}_{n+1}(\mathrm{I}(T_1)) \leq 1$. If $\mathrm{I}(T_2) \simeq S^p$ for some $p \neq n, n+1$, then the sequence (3.2) reduces to

$$0 \longrightarrow \tilde{\mathrm{H}}_{n+1}(\mathrm{I}(T_1)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

Hence, by Observation 1.2, we have $\mathrm{I}(T_1) \simeq S^{n+1}$ and (3.1) implies that $\tilde{\mathrm{H}}_q(\mathrm{I}(T_2)) = 0$ for each $q \neq n, n+1$. This contradicts our assumption $\mathrm{I}(T_2) \simeq S^p, p \neq n, n+1$. This proves the claim. □

Next we show:

Claim. *If* $\mathrm{I}(T_2)$ *is contractible, then* $\mathrm{I}(T_1) \simeq S^{n+1}$.

Proof. This follows easily from (3.2) and Observation 1.2. □

Finally, we verify:

Claim. *If* $\mathrm{I}(T_2) \simeq S^n$*, then* $\mathrm{I}(T_1)$ *is contractible.*

Proof. By (3.1), we see that $\tilde{\mathrm{H}}_q(\mathrm{I}(T_1)) = 0$ for each $q \neq n, n+1$. Hence $\mathrm{I}(T_1)$ is either contractible or is homotopy equivalent to S^n or S^{n+1}. Suppose that $\mathrm{I}(T_1) \simeq S^n$. Then the sequence (3.2) reduces to a sequence

$$0 \longrightarrow \tilde{\mathrm{H}}_n(\mathrm{I}(T_3)) \cong \mathbb{Z} \longrightarrow \tilde{\mathrm{H}}_n(\mathrm{I}(T_2)) \cong \mathbb{Z} \longrightarrow \tilde{\mathrm{H}}_n(\mathrm{I}(T_1)) \cong \mathbb{Z} \longrightarrow 0$$

which fails to be exact. If $\mathrm{I}(T_1) \simeq S^{n+1}$, then (3.2) yields a sequence

$$0 \longrightarrow \tilde{\mathrm{H}}_{n+1}(\mathrm{I}(T_1)) \cong \mathbb{Z} \longrightarrow \tilde{\mathrm{H}}_n(\mathrm{I}(T_3)) \cong \mathbb{Z} \longrightarrow \tilde{\mathrm{H}}_n(\mathrm{I}(T_2)) \cong \mathbb{Z} \longrightarrow 0$$

which is not exact either. Thus we conclude that $\mathrm{I}(T_1)$ is contractible. □

This completes the proof of Proposition 3.1 and therefore completes the proof of Main Theorem. □

Example 3.3. *Here we determine the homotopy type of* $\mathrm{I}(T)$ *with* $D_T = 2$.

Let T be a tree with root r such that $D_T = 2$ and take a vertex v with $l_T(v) = 2$ (see Figure 2). Then $v_{D-2} = v_0 = r$ and the neighbors $N_T(r)$ coincides with $N_T^{+1}(r)$. Let $d_1 = |N_T^{+1*}(r)|$ and $d_2 = \deg_T(r) - d_1 = |N_T(r) \setminus N_T^{+1*}(r)|$. Notice that $\nu(v) = d_1 - 1$.

Case 1: $d_2 > 0$:

We carry out the construction in Section 2.1 and see that T_2 and T_3 are complete bipartite subgraphs: $T_2 = K_{1,d_2+1}, T_3 = K_{1,d_2}$. As we see at the beginning of Section 2.1, $\mathrm{I}(T_2)$ and $\mathrm{I}(T_3)$ are both homotopy equivalent to S^0 and hence $n(T_2) = n(T_3) = 0$. By condition (2.b) of Section 2.1, we see $n(T_1) = \infty$. Thus by condition (3), $n(T) = \infty$ and $\mathrm{I}(T)$ is contractible.

Case 2: $d_2 = 0$:

We see that T_1 is the path of length 2 and hence $\mathrm{I}(T_1) \simeq S^0$. Thus by condition (3), we obtain $n(T) = n(T_0) = n(T_1) + \nu(v) = d_1 - 1$ and hence $I(T) \simeq S^{d_1-1}$.

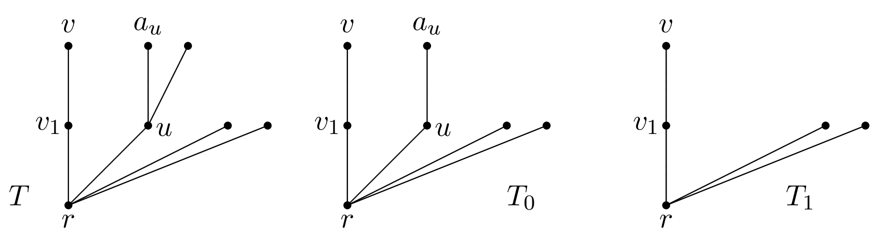

FIGURE 2. T, T_0 and T_1

Example 3.4. *Independence complexes of paths.*

Let P_n be the path of length n. We may apply our main theorem with a simple induction to prove the following:

$$\mathrm{I}(P_n) \simeq \begin{cases} \text{a point (i.e. contractible)} & \text{if } n = 3k, \\ S^k & \text{if } n = 3k+1 \text{ or } n = 3k+2. \end{cases}$$

This has already been proved in [5]. See also [6, Proposition 11.6, p. 193] in which P_n is denoted by L_{n+1}.

Remark: We may choose the root r of a tree T so that D_T with respect to r is as small as possible. Such a vertex is called a center of T. It is known that every tree T has either exactly one center, or exactly two, adjacent, centers [1, Exercise 4.1.8].

Acknowledgments

The author expresses his sincere appreciation to Professor M. Hachimori for many helpful suggestions, especially for pointing out an error of the first manuscript. Also he also expresses his deep gratitude to the referee for helpful suggestions which improved the exposition of the paper.

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