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CONES OF PARTIAL METRICS

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Abstract. A *partial semimetric* on a set X is a function $(x,y) \mapsto p(x,y) \in \mathbb{R}_{\geq 0}$ satisfying $p(x,y) = p(y,x)$, $p(x,y) \geq p(x,x)$ and $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ for all $x,y,z \in X$. Using computations done for $n \leq 6$, we study the polyhedral convex cone PMET_n of all partial semimetrics on n points and its subcone l_1-PMET_n generated by all $\{0,1\}$-valued, i.e., contaning $\{0,1\}$-valued element, extreme rays. The elements of this subcone correspond to natural quasi-metric analogue of l_1-semimetrics.

We present data on those cones and their relatives: the number of facets, of extreme rays, of their orbits, incidences, characterize $\{0,1\}$-valued extreme rays and some classes of facets, including analoques of the *hypermetric* ones.

1. Convex cones under consideration

There are following two main motivations for this study. Partial semimetrics are generalization of semimetrics, having important applications in Computer Science (domain theory, analysis of data flow deadlock, complexity analysis of programs, etc.). This is the first polyhedral appoach to them. Also, we explore which part of rich theory of l_1-semimetrics versus cuts (see, for example, [6]) can be extended on quasi-metric analog of l_1-semimetrics versus oriented cuts. See [4] for generalizing of MET_n–CUT_n pair to other contexts.

A *convex cone in* $\mathbb{R}^m$ (see, for example, [16]) is defined either by *generators* $v_1, \ldots, v_N$, as $\{\sum \lambda_i v_i : \lambda_i \geq 0\}$, or by *linear inequalities* $f_1, \ldots, f_M$, as $\{x \in \mathbb{R}^m : f_i(x) \geq 0\}$. We consider only *polyhedral* convex cones, i.e., the number of generators and, alternatively, the number of defining inequalities is finite. If a convex cone has dimension m', then the ranks of the set of its generators and the set of defining inequalities are m'.

Let C be an m'-dimensional convex cone in $\mathbb{R}^m$. Given $v \in \mathbb{R}^m$, the inequality $\sum_{i=1}^m v_i x_i \geq 0$ is said to be *valid* for C if it holds for all $x \in C$. Then the set $\{x \in C : \sum_{i=1}^m v_i x_i = 0\}$ is called the *face* of C, *induced by the valid inequality* $\sum_{i=1}^m v_i x_i \geq 0$. A face of dimension $m'-1$, $m'-2$, 1 are

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called a *facet*, *ridge*, *extreme ray* of C, respectively (a *ray* is a set $\mathbb{R}_{\geq 0}x$ with $x \in C$).

The *incidence number* of a facet (or of an extreme ray) is the number of extreme rays lying on this facet (or, respectively, of facets containing this extreme ray).

Two extreme rays (or facets) of C are said to be *adjacent on* C if they span a 2-dimensional face (or, respectively, their intersection has dimension $m' - 2$). The *skeleton graph* of the convex cone C is the graph G_C, whose vertices are the extreme rays of C and with an edge between two vertices if they are adjacent on C. The *ridge graph* of C is the graph G_C^*, whose vertices are the facets of C and with an edge between two vertices if they are adjacent on C. So, the ridge graph of a convex cone is the skeleton of its dual cone.

Set $V_n = \{1, \ldots, n\}$ and consider a function $f = ((f_{ij})) : V_n^2 \longrightarrow \mathbb{R}_{\geq 0}$ such that

$$\mathrm{Tr}_{ij,k} : f_{ik} + f_{kj} - f_{ij} - f_{kk} \geq 0 \tag{1.1}$$

holds for all $i, j, k \in V_n$ (called *triangle inequality*). The function f is called *weak partial semimetric* if

$$f_{ij} = f_{ji} \text{ for all } i, j \in V_n \ (\textit{symmetry}). \tag{1.2}$$

A weak partial semimetric f is called *partial semimetric* if

$$N_{ij} : f_{ij} \geq f_{ii} \text{ for all } i, j \in V_n. \tag{1.3}$$

A partial semimetric f is called *semimetric* if

$$f_{ii} = 0 \text{ for all } i \in V_n. \tag{1.4}$$

The function f is called *quasi-semimetric* if (1.4) holds; so, it is a semimetric if, moreover, (1.2) holds. Clearly, for a quasi-semimetric f, the function $((f_{ij} + f_{ji}))$ is a semimetric; it called *symmetrization semimetric* of f.

A *weak partial metric*, *partial metric*, *quasi-metric*, or *metric* f is respectively weak partial, partial, quasi-, or simply semimetric, such that

$$f_{ii} = f_{ij} = f_{jj} \text{ implies } i = j \tag{1.5}$$

for all different $i, j \in V_n$ (*separation axiom*).

Let us denote the function f by p, q, or d if it is a weak partial semimetric, quasi-semimetric, or semimetric, respectively.

The quasi-metrics (or asymmetric, directed, oriented distances) appeared already in [11, pp. 145–146]. Examples of quasi-metrics on $\mathbb{R}$ are *Sorgenfrey quasi-metric* (equal to $y - x$ if $y \geq x$ and equal to 1, otherwise) and l_1 *quasi-semimetric* $\max\{y - x, 0\}$; see the next section. Real world examples: one-way streets mileage, travel time, transportation costs (up/downhill or up/downstream).

A quasi-semimetric q is *weightable* if there exists a (weight) function $w = (w_i) : V_n \longrightarrow \mathbb{R}_{\geq 0}$ such that

$$q_{ij} + w_i = q_{ji} + w_j \text{ for all different } i, j \in V_n, \tag{1.6}$$

i.e., $((2q_{ij}+w_i-w_j))$ is the symmetrization semimetric of q.

Partial metrics were introduced by Matthews in [14] for treatment of partially defined objects in computer science. He also remarked that a quasi-semimetric $q=((q_{ij}))$ is weightable if and only if the function $((q_{ij}+w_i))$ is a partial semimetric. (Moreover, $((q_{ij}+w_i))$ is a partial metric if q is an weightable *Albert quasi-metric*, i.e., $x=y$ whenever $q(x,y)=q(y,x)=0$.) Weak partial semimetrics were studied in [12]; an example: $p(x,y)=x+y$ for $x,y\in\mathbb{R}_{\geq 0}$. If $p(x,y)$ is a weak partial semimetric, then $p'(x,y)=\max\{p(x,y),p(x,x),p(y,y)\}$ is a partial semimetric. In fact, $p(x,y)$ is a weak partial semimetric if and only if $d(x,y)=2p(x,y)-p(x,x)-p(y,y)\}$ is a semimetric.

Scott's domain theory (see, for example, [8]) gives partial order and non-Hausdorff topology on partial objects in computation. In quantitative domain theory, a "distance" between programs (points of a semantic domain) is used to quantify speed (of processing or convergence) or complexity of programs and algorithms. For instance, $x\preceq y$ (program y contains all information from program x) is the *specialization preorder* ($x\preceq y$ if and only if $p(x,y)$=$p(x,x)$ for a partial metric p on X. In computation over a metric space of totally defined objects, partial metric models partially defined information: $p(x,x)>0$ or $p(x,x)=0$ mean that object x is partially or totally defined, respectively. For example, for *vague real numbers* x (i.e., non-empty segments of $\mathbb{R}$ as, say, decimals approximating π), the self-distance $p(x,x)$ can be the length of the segment measuring the extent of ambiguity at point x.

Any topology on a finite set X is defined by $cl\{x\}=\{y\in X: y\preceq x\}$ for $x\in X$, where $x\preceq y$ is the specialization preorder, meaning $p(x,y)$=$p(x,x)$), for some partial semimetric p on X ([10]). Not every finite topology is defined from a semimetric on X by this way.

Consider the following polyhedral convex cones in $\mathbb{R}^{n^2}$ with apex in (0).

(1) $\binom{n+1}{2}$-dimensional cone wPMET$_n$ of weak partial semimetrics p on V_n; its facets are n facets $M_{ii}: p_{ii}\geq 0$ with $i\in V_n$ and $3\binom{n}{3}$ facets Tr$_{ij,k}$ with with $k\in V_n$, $1\leq i<j\leq n$.
(2) $\binom{n+1}{2}$-dimensional cone PMET$_n$ of partial semimetrics p; its facets are n facets $M_{ii}: p_{ii}\geq 0$, $n(n-1)$ facets $N_{ij}: p_{ij}\geq p_{ii}$ with $i,j\in V_n$ and $3\binom{n}{3}$ facets Tr$_{ij,k}$ with $k\in V_n$, $1\leq i<j\leq n$ (the inequalities Tr$_{ii,k}$ are implied by $p_{ii}\leq p_{ik}=p_{ki}\geq p_{kk}$).
(3) $\binom{n}{2}$-dimensional cone MET$_n$ of semimetrics d; its facets are $3\binom{n}{3}$ facets Tr$_{ij,k}: d_{ik}+d_{kj}-d_{ij}\geq 0$ with $k\in V_n$, $1\leq i<j\leq n$ (the inequalities $d_{ij}\geq 0$ are implied by Tr$_{ij,k}$ and Tr$_{ik,j}$).
(4) $n(n-1)$-dimensional cone QMET$_n$ of quasi-semimetrics q; its facets are $n(n-1)$ facets $N_{ij}: q_{ij}\geq 0$ with different $i,j\in V_n$ and $6\binom{n}{3}$ facets Tr$_{ij,k}: q_{ik}+q_{kj}-q_{ij}\geq 0$ with with $k\in V_n$, $1\leq i\neq j\leq n$ (now the order of k and j matters). This cone was introduced and studied in [3], [5].

(5) $\binom{n+1}{2}$-dimensional (since $q_{ji} = q_{ij} + w_i - w_j$ by (6) above) cone WQMET_n of weightable quasi-semimetrics q; its facets are $n(n-1)$ facets $N_{ij} : q_{ij} \geq 0$ with different $i, j \in V_n$ and $3\binom{n}{3}$ facets $\mathrm{Tr}_{ij,k}$: $q_{ik} + q_{kj} - q_{ij} \geq 0$ with $k \in V_n$, $1 \leq i < j \leq n$ ($\mathrm{Tr}_{ij,k} = \mathrm{Tr}_{ji,k}$ for weightable quasi-semimetrics). Clearly,

$$\mathrm{MET}_n = \mathrm{PMET}_n \cap \mathrm{WQMET}_n \,.$$

Given an ordered partition $\{S_1, \ldots, S_t\}$, $2 \leq t \leq n$, of V_n into non-empty subsets, the *oriented multicut quasi-semimetric* (or *o-multicut*) $\delta^{'}(S_1, \ldots, S_t)$ on V_n is:

$$\delta^{'}_{ij}(S_1, \ldots, S_t) = \begin{cases} 1, & \text{if } i \in S_h, j \in S_m, m > h; \\ 0, & \text{otherwise.} \end{cases}$$

The *oriented anti-multicut quasi-semimetric* (or *o-anti-multicut*) $\alpha^{'}(S_1, \ldots, S_t)$ on V_n is $\alpha^{'}_{ij}(S_1, \ldots, S_t) = 1 - \delta^{'}(S_1, \ldots, S_t)$ if $1 \leq i \neq j \leq n$ and $= 0$ if $1 \leq i = j \leq n$.

The o-multicut $\delta^{'}(S_1, S_2) = \delta^{'}(S, \overline{S})$ with $t = 2$ and $S = S_1$ is called *o-cut* and denoted by $\delta^{'}(S)$; the o-anti-multicut $\alpha^{'}(S_1, S_2) = \alpha^{'}(S, \overline{S})$ is called *o-anti-cut* and denoted by $\alpha^{'}(S)$. Set $\delta'(\varnothing) = ((0))$; so, $\alpha'(\varnothing) = d(K_n)$, the path metric of the complete graph.

Given an ordered partition $\{S_1, \ldots, S_t\}$, $2 \leq t \leq n$, the *multicut semimetric* $\delta_{S_1,\ldots,S_t}$ is the symmetrization $\delta^{'}(S_1, \ldots, S_t) + \delta^{'}(S_t, \ldots, S_1)$ of the quasi-semimetric $\delta^{'}(S_1, \ldots, S_t)$. The *anti-multicut semimetric* $\alpha(S_1, \ldots, S_t)$ is the symmetrization $\alpha^{'}(S_1, \ldots, S_t) + \alpha^{'}(S_t, \ldots, S_1)$ of the quasi-semimetric $\alpha^{'}(S_1, \ldots, S_t)$; in fact, it is the path metric $d(K_{|S_1|,\ldots,|S_t|})$ of the complete multipartite graph. In the case $t = 2$, the multicut and anti-multicut semimetrics are called *cut* and *anti-cut* semimetrics and denoted by $\delta(S)$ and $\alpha(S)$, respectively. Set $\delta(\varnothing) = ((0))$ (it is *zero cut*) and $\alpha(\varnothing) = d(K_n)$.

It was shown in [5] that none of semimetrics but all non-zero o-multicuts represent extreme rays of QMET_n. For $n \geq 4$, this cone has other $\{0, 1\}$-valued extreme ray representatives, including (conjecture, checked for $n \leq 5$) all o-anti-multicuts, except those of Lemma 2(3) and (4) below.

Lemma 1.1. *o-multicuts and o-anti-multicuts are $\{0, 1\}$-valued quasi-semimetrics, which are weightable if and only if $t \leq 2$. The weight functions of o-cut $\delta^{'}(S)$ and o-anti-cut $\alpha^{'}(S)$ are $w_i = 1_{i \notin S}$ and $w_i = 1_{i \in S}$, respectively.*

Proof. In fact, let $i \in S_1$, $j \in S_2$, $k \in S_3$ in the quasi-semimetric $q{=}\delta^{'}(S_1, \ldots, S_q)$. If q is weightable, then $q_{ij} = (q_{ji} + w_j) - w_i = w_j - w_i$. Impossible, since also $q_{ik} = w_k - w_i = 1, q_{jk} = w_k - w_j = 1$. The proof for o-anti-multicuts is similar. □

The following equalities are easy to check.

Lemma 1.2.

(1) $\delta(S_1, \ldots, S_t) = \sum_{i=1}^{t} \delta^{'}(S_i) = \sum_{i=1}^{t} \delta^{'}(\overline{S_i}) = \frac{1}{2} \sum_{i=1}^{t} \delta(S_i)$.

(2) $\alpha(\varnothing) = d(K_n)$ *and* $\alpha(S_1, \ldots, S_t) = d(K_{|S_1|,\ldots,|S_t|})$.
(3) $\alpha^{'}(\{i\}) = \sum_{j\in\overline{\{i\}}} \delta^{'}(\{j\})$.
(4) *If* $t = n$, *i.e., all* $|S_i| = 1$, *then* $\alpha^{'}(S_1, \ldots, S_t) = \delta^{'}(S_t, \ldots, S_1)$ *(the reversal of the ordered partition).*

The $\binom{n}{2}$-dimensional cone generated by all non-zero cuts on V_n is denoted by CUT_n. It holds $\mathrm{CUT}_n \subset \mathrm{MET}_n$ with equality only for $n = 3, 4$. The $n(n-1)$-dimensional cone generated by all non-zero o-multicuts on V_n is denoted by OMCUT_n. It holds $\mathrm{OMCUT}_n \subset \mathrm{QMET}_n$ with equality only for $n = 3$. Denote by OCUT_n the $\binom{n+1}{2}$-dimensional subcone of WQMET_n generated by all non-zero o-cuts; this is different from OCUT_n as defined in [5]. It holds $\mathrm{OCUT}_n \subset \mathrm{WQMET}_n$ with equality only for $n = 3$. Denote by l_1-PMET_n the $\binom{n+1}{2}$-dimensional subcone of PMET_n generated by all its $\{0, 1\}$-valued extreme rays. Section 3.1 below imply that $p = ((p_{ij})) \in l_1$-PMET_n if and only if $((p_{ij} - p_{ii})) \in \mathrm{OCUT}_n$.

A mapping $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is called a *symmetry* of a cone C if it is an isometry, satisfying $f(C) = C$ (*an isometry* of $\mathbb{R}^m$ is a linear mapping preserving the Euclidean distance). Every permutation of V_n induce a symmetry of above cones MET_n, CUT_n, QMET_n and PMET_n; so, the group $\mathrm{Sym}(n)$ is a symmetry group of them. It is the full symmetry groups of MET_n and CUT_n for $n \geq 5$ (see [2]). In QMET_n, OMCUT_n appears also a *reversal* symmetry (see [5]), corresponding to transposition of matrix $((q_{ij})))$. We expect $Z_2 \times \mathrm{Sym}_{(n)}$ and $\mathrm{Sym}_{(n)}$ to be the full symmetry groups of WQMET_n, OCUT_n and PMET_n, l_1-PMET_n, respectively.

In Table 1 we summarize the most important numeric information on cones under consideration. The column 2 indicates the dimension of the cone, the columns 3 and 4 give the number of extreme rays and facets, respectively; in parenthesis are given the numbers of their orbits.

2. Weightable, l_1- and digraphic quasi-semimetrics

We introduce the following short notation for the cyclic sum

$$\sum_{1\leq i\leq k-1} q(x_i, x_{i+1}) + q(x_k, x_1) = x_1 x_2 \cdots x_k x_1.$$

A quasi-semimetric q on X has *relaxed symmetry* if for different $x, y, z \in X$ it holds $xyzx = xzyx$, i.e.,

$$q(x, y) + q(y, z) + q(z, x) = q(x, z) + q(z, y) + q(y, x)$$

implying

$$\mathrm{Tr}_{xz,y} = q(x, y) + q(y, z) - q(x, z) = q(z, y) + q(y, x) - q(z, x) = \mathrm{Tr}_{zx,y}\,.$$

Lemma 2.1 ([21])**.** *A quasi-semimetric q on X has relaxed symmetry if and only if it is weightable.*

TABLE 1. Some parameters of cones for $n \leq 6$

Cone	Dim.	Nr. of extreme rays (orbits)	Nr. of facets (orbits)
CUT_3=MET_3	3	3 (1)	3 (1)
CUT_4=MET_4	6	7 (2)	12 (1)
CUT_5	10	15 (2)	40 (2)
MET_5	10	25 (3)	30 (1)
CUT_6	15	31 (3)	210 (4)
MET_6	15	296 (7)	60 (1)
$OMCUT_3$=$QMET_3$	6	12 (2)	12 (2)
$OMCUT_4$	12	74 (5)	72 (4)
$QMET_4$	12	164 (10)	36 (2)
$OMCUT_5$	20	540 (9)	35320 (194)
$QMET_5$	20	43590 (229)	80 (2)
$OMCUT_6$	30	4682 (19)	$\geq$ 217847040 ($\geq$ 163822)
$QMET_6$	30	$\geq$ 492157440 ($\geq$ 343577)	150 (2)
l_1-$PMET_3$=$PMET_3$	6	13 (5)	12 (3)
l_1-$PMET_4$	10	44 (9)	46 (5)
$PMET_4$	10	62 (11)	28 (3)
l_1-$PMET_5$	15	166 (14)	585 (15)
$PMET_5$	15	1696 (44)	55 (3)
l_1-$PMET_6$	21	705 (23)	
$PMET_6$	21	337092 (734)	96 (3)

Proof. Relaxed symmetry means

$$q(x, y) - q(y, x) = (q(z, y) - q(y, z)) - (q(z, x) - q(x, z)).$$

Equivalently, q is weightable: fix point $z_0 \in X$ and define $w(x) = q(z_0, x) - q(x, z_0) + \max_z(q(z, z_0) - q(z_0, z)) \geq 0$ for all $x \in X$. On the other hand, it is easy to see that the above equality holds if q is weightable. □

Given $k \geq 3$, a quasi-semimetric q is called *k-cyclically symmetric* if it holds

$$x_1x_2x_3 \cdots x_kx_1 = x_1x_kx_{k-1} \cdots x_2x_1$$

for any different $x_1x_2 \cdots x_k \in X$. So, a quasi-semimetric has relaxed symmetry if and only if it 3-cyclically symmetric, respectively.

Lemma 2.2. *A quasi-semimetric q on X has relaxed symmetry if and only if it is k-cyclically symmetric for any $k \geq 3$.*

Proof. In fact, it holds

$$\begin{aligned}(x_1x_2x_3x_1 - x_1x_3x_2x_1) + (x_1x_3x_4x_1 - x_1x_4x_3x_1) + \cdots \\ + (x_1x_{k-1}x_kx_1 - x_1x_kx_{k-1}x_1) = (x_1x_2\cdots x_kx_1 - x_1x_k\cdots x_2x_1)\end{aligned}$$

for any $k \geq 4$. For example, for $k = 4$ it holds:

$$(x_1x_2x_3x_1 - x_1x_3x_2x_1) + (x_1x_3x_4x_1 - x_1x_4x_3x_1) = x_1x_2x_3x_4x_1 - x_1x_4x_3x_2x_1.$$

In the other direction, we have:

$$\begin{aligned}(k-2)\cdot(x_1x_2\cdots x_{k-1}x_1 - x_1x_{k-1}\cdots x_2x_1) \\ = (x_1x_2\cdots x_{k-1}x_kx_1 - x_1x_k\cdots x_2x_1) \\ + (x_1x_2\cdots x_kx_{k-1}x_1 - x_1x_{k-1}x_k\cdots x_2x_1) \\ + \cdots (x_1x_kx_2\cdots x_{k-1}x_1 - x_1x_{k-1}\cdots x_2x_kx_1).\end{aligned}$$

For example,

$$\begin{aligned}2(x_1x_2x_3x_1 - x_1x_3x_2x_1) = (x_1x_2x_3x_4x_1 - x_1x_4x_3x_2x_1) \\ + (x_1x_2x_4x_3x_1 - x_1x_3x_4x_2x_1) + (x_1x_4x_2x_3x_1 - x_1x_3x_2x_4x_1).\end{aligned}$$

□

Clearly, any finite quasi-semimetric q can be realized as the (shortest directed) path quasi-semimetric of a $\mathbb{R}_{\geq 0}$-weighted digraph: take the complete digraph and put on each arc (ij) the weight q_{ij}. The earliest known to us occurrence of the notion, but not the term, of relaxed symmetry was in [15].

Theorem 1 ([15, Theorem 5])**.** *A finite quasi-metric q can be realized as the path quasi-metric of a $\mathbb{R}_{\geq 0}$-weighted* bidirectional tree *(a tree with all edges replaced by 2 oppositely directed arcs) if and only if q has relaxed symmetry and its symmetrization $((q_{ij} + q_{ji}))$ can be realized as the path metric of a $\mathbb{R}_{\geq 0}$-weighted tree.*

Example. Consider random walks on a connected graph $G = (V, E)$, where at each step walk moves with uniform probability from current vertex to a neighboring one. The *hitting time quasi-metric* $H(u, v)$ on V is the expected number of steps (edges) for a random walk on beginning at vertex u to reach v for the first time; put $H(u, u) = 0$. The *cyclic tour* property of reversible Markov chains implies that $H(u, v)$ is weightable. The *commuting time metric* $C(u, v) = H(u, v) + H(v, u)$ is ([20]) $2|E|R(u, v)$, where $R(u, v)$ is the *effective resistance metric*, i.e., 0 if $u = v$ and, otherwise, $\frac{1}{R(u,v)}$ is the current flowing into v, when grounding v and applying 1 volt potential to u (each edge is seen as a resistor of 1 ohm). It holds

$$R(u, v) = \sup_{f: V \to \mathbb{R},\, D(f) > 0} \frac{(f(u) - f(v))^2}{D(f)},$$

where $D(f)=\sum_{st\in E}(f(s)-f(t))^2$.

Define for $p\geq 1$ *oriented l_p-norm* on $\mathbb{R}^m$ as

$$||x||_{p,\,or}=\Big(\sum_{k=1}^{m}(\max(x_k,0))^p\Big)^{\frac{1}{p}}$$

and *oriented l_∞-norm* as $||x||_{\infty,\,or}=\max_{k=1}^{m}\max(x_k,0)$, where x_k is the k-th coordinate of x. Then, l_p *quasi-semimetric* $l_{p,or}$ is defined as $||x-y||_{p,\,or}$. The quasi-semimetric space $(\mathbb{R}^m,l_{p,\,or})$ is abbreviated as $l^m_{p,\,or}$.

Theorem 2 ([5, Theorem 1])**.**

(i) *Any quasi-semimetric q on V_n is embeddable in $l^m_{1,\,or}$ for some m if and only if $q\in \mathrm{OCUT}_n$.*
(ii) *Any quasi-semimetric q on V_n is embeddable in $l^n_{\infty,\,or}$.*

In [1], *oriented l_p semimetric* on $\mathbb{R}^m$ was given as

$$||x-y||^{CMM}_{p,\,or}=\Big(\sum_{i=1}^{m}|x_i-y_i|^p\Big)^{\frac{1}{p}}+\Big(\sum_{i=1}^{m}|y_i|^p\Big)^{\frac{1}{p}}-\Big(\sum_{i=1}^{m}|x_i|^p\Big)^{\frac{1}{p}}.$$

Those two definitions are very similar on $\mathbb{R}^m_{\geq 0}$ for $p=1$:

$$||x-y||^{CMM}_{1,\,or}=2||y-x||_{1,\,or}.$$

Given a *measure space* $(\Omega,\mathcal{A},\mu)$, then the *measure semimetric* on the set $\mathcal{A}_\mu=\{A\in\mathcal{A}:\mu(A)<\infty\}$ is $\mu(A\triangle B)$, where $A\triangle B=(A\cup B)\backslash(A\cap B)=(A\backslash B)\cup(B\backslash A)$ is the symmetric difference of sets A and B. If $\mu(A)=|A|$, then $\mu(A\triangle B)=|A\triangle B|$ a metric. A *measure quasi-semimetric* on the set $\mathcal{A}_\mu$ is $q(A,B)=\mu(A\triangle B)+\mu(B)-\mu(A)=\mu(B\backslash A)$. In fact (as well as for semimetrics), the measure quasi-semimetrics are exactly l_1-quasi-semimetrics ([3, p. 780]).

Example. Let q be a quasi-metric on V_3 with $q_{21}{=}q_{23}{=}2$ and $q_{ij}{=}1$ for other $i\neq j$. In fact, $q=\delta'(\{1\})+2\delta'(\{2\})+\delta'(\{3\})\in \mathrm{OCUT}_3=\mathrm{WQMET}_3$. So, q is weightable ($w_i=1,0,1$ for $i=1,2,3$) and $P(q)=P(\delta'(\{1\})+\delta'(\{2\}))+P(\delta'(\{2\})+\delta'(\{3\}))$ ($P(q)$ is defined in Section 3 below). The quasi-metric q is a l_1-quasi-semimetric with $q_{ij}=\sum_{s=1}^{5}\max(0,x_s^{(i)}-x_s^{(j)})$ $(1\leq i,j\leq 3)$ for $x^{(1)}=(1,1,0,0,0)$, $x^{(2)}=(1,0,0,1,1)$, $x^{(3)}=(1,0,1,0,0)\in\mathbb{R}^5$. In other words, q is a measure quasi-semimetric with $q_{ij}=\mu(A^{(j)}\backslash A^{(i)})$ (for the counting measure $\mu(A)=|A|$) on the following subsets of V_5: $A^{(1)}=\{1,2\}$, $A^{(2)}=\{1,4,5\}$, $A^{(3)}=\{1,3\}$.

3. The cone PMET_n

Call a partial semimetric $p=((p_{ij}))\in \mathrm{PMET}_n$ *reducible* if $\min_{1\leq i\leq n}p_{ii}=0$. For any quasi-semimetric $q=((q_{ij}))\neq((0))$ in the cone $WQMET_n$, there exist a (weight) function $w=(w_i):V_n\longrightarrow\mathbb{R}_{\geq 0}$ such that $((q_{ij}+w_i))$ is a partial semimetric. Clearly, $((q_{ij}+w'_i))$ is also a partial semimetric for any (weight) function $w'=(w'_i)$ with $w'_i=w_i-\min_{1\leq j\leq n}w_j+\lambda$, $\lambda\geq 0$. In other words, there is a bijection P between all weightable non-zero quasi-semimetrics $q \in \mathrm{WQMET}_n$ and all reducible partial semimetrics $p = P(q) \in \mathrm{PMET}_n$. For $q = ((0))$, set $P(q) = J$, where $J = ((J_{ij}))$ with $J_{ij} = 1$. So, P is a bijection between WQMET_n and the set of rays $\{P(q) + \lambda J\}$, where $\lambda \geq 0$. Here, for $q = ((0))$, $P(q) = J$ and, otherwise, it is a reducible partial semimetric. For an element c of a convex cone C denote by $rank(c)$ the rank of the set of defining inequalities of the cone, which become equalities for c, i.e., the set of facets to which c belongs.

Lemma 3.1. *If $q = ((q_{ij})) \in \mathrm{WQMET}_n$ with a weight function $w = (w_i)$ and $p = ((q_{ij} + w_i))$, then $rank(p) = rank(q) + |\{1 \leq i \leq n : w_i = 0\}|$.*

Proof. Clearly, p lies on the facet $N_{ij} : p_{ij} - p_{ii} \geq 0$ with $i \neq j$ of PMET_n if and only if q lies on the facet $N_{ij} : q_{ij} \geq 0$ of WQMET_n. Also, p lies on the facet $\mathrm{Tr}_{ik,j} : p_{ij} + p_{jk} - p_{ik} - p_{jj} \geq 0$ of PMET_n if and only if q lies on the facets $\mathrm{Tr}_{ik,j} : q_{ij} + q_{jk} - q_{ik} \geq 0$ and $\mathrm{Tr}_{ki,j} = q_{kj} + q_{ji} - q_{ki} \geq 0$ of WQMET_n. The equality $\mathrm{Tr}_{ik,j} = \mathrm{Tr}_{ki,j}$ is exactly relaxed symmetry characterizing weightable quasi-semimetrics. So, all triangle equalities for q stay for p, while equalities $q_{ij} = 0$ for $1 \leq i \neq j \leq n$ transform into $p_{ij} - p_{ii} = 0$. Only the equalities $p_{ii} = 0$, corresponding to the equalities $w_i = 0$ became new and they increase rank. □

The previous lemma implies that if $q \in \mathrm{WQMET}_n$ represents an extreme ray, then $P(q) \in \mathrm{PMET}_n$ also represents an extreme ray. In fact, the cones have the same dimension. But more extreme rays appear in PMET_n. For $q^{(1)}, q^{(2)} \in \mathrm{WQMET}_n$, $P(q^{(1)}) + P(q^{(2)})$ belongs to the ray $\{P(q^{(1)} + q^{(2)}) + \lambda J\}$, i.e., $P(q^{(1)} + q^{(2)}) = P(q^{(1)}) + P(q^{(2)}) - \lambda J$ with $\lambda \geq 0$, which is weaker than linearity, corresponding to $\lambda = 0$.

Recall that $\mathrm{MET}_n = \mathrm{PMET}_n \cap \mathrm{WQMET}_n$. The previous lemma also implies that a semimetric represents an extreme ray in $\binom{n+1}{2}$-dimensional cone PMET_n if and only if it represents an extreme ray in $\binom{n}{2}$-dimensional cone MET_n. In fact, exactly n new valid equalities $p_{ii} = 0$ appear for it in PMET_n. See the number of such extreme rays for $n \leq 6$ in Table 1. For $n \leq 6$, the orbit-representing semimetrics, besides cuts $\delta(S)$ with $1 \leq |S| \leq \lfloor \frac{n}{2} \rfloor$, are $d(K_{\{1,2\},\{3,4,5\}})$ and $d(K_{\{1=6,2\},\{3,4,5\}})$, $d(K_{\{1,2\},\{3,4,5=6\}})$, $d(K_{\{1,2\},\{3,4,5,6\}})$, $d(K_{\{1,2,3\},\{4,5,6\}})$, $d(K_{\{1,2,3\},\{4,5,6\}} - e_{14})$. Here notation $-e_{ij}$ means that the edge ij is deleted.

3.1. $\{0,1\}$**-valued elements of** PMET_n**.** For any subset $S_0 \subset V_n$, denote by $J(S_0)$ the function $a : V_n^2 \longrightarrow \{0,1\}$ with $a_{ij} = 1$ exactly when $i, j \in S_0$; so, $J(V_n) = J$, where, as above, J is the partial semimetric with all values 1.

Given integer $i \geq 0$, let $Q(i)$ denote i-th *partition number*, i.e., the number of ways to write i as a sum of positive integers; $Q(i)$ form the sequence A000041 in [19]: 1, 1, 2, 3, 5, 7, 11, 15, ... Given integer $i \geq 0$, let $B(i)$ denote i-th *Bell number*, i.e., the number of partitions of $V_i = \{1, \dots, i\}$; $B(i)$ form the sequence A000110 in [19]: 1, 1, 2, 5, 15, 52, 203, 877, ... So, $B(n)$ is the number of all multicuts on V_n, while the number of all cuts is

2^{n-1}. Note that the number of all o-cuts on V_n is 2^n, while the number of all o-multicuts is n-th *ordered Bell number* $Bo(n)$, i.e., the number of ordered partitions of V_n; $Bo(n)$ form the sequence A000670 in [19]: 1, 1, 3, 13, 75, 541, 4683, 47293, ...

Theorem 3.

(i) *All* $\{0,1\}$*-valued elements of* PMET_n *are* $\sum_{0\le i\le n}\binom{n}{i}B(n-i)$ *(organized into* $\sum_{0\le i\le n}Q(i)$ *orbits under* $\mathrm{Sym}(n)$*) elements of the form* $J(S_0)+\delta(S_0,S_1,\dots,S_t)=P(\sum_{1\le i\le t}\delta'(S_i))$*, where* S_0 *is any subset of* V_n *and* $S_1,\dots,S_t$ *is any partition of* $\overline{S_0}$*.*

(ii) *The incidence number (defined in Section 1) of* $\{0,1\}$*-valued element* $p=J(S_0)+\delta(S_0,S_1,\dots,S_t)$*, is:* $n-|S_0|$ *(to facets* $M_{ii}: p_{ii}\ge 0$*) plus* $\sum_{1\le k\le t}|S_k|(|S_k|-1)+(|S_0|(|S_0|-1)+|S_0|(n-|S_0|)$*) (to facets* $N_{ij}: p_{ij}-p_{ii}\ge 0,\ i\ne j,$ *with* $0-0=0$ *and* $1-1=0$*, respectively) plus* $3\sum_{1\le k\le t}\binom{|S_k|}{3}+\sum_{1\le k\le t}|S_k|(|S_k|-1)(n-|S_k|)+|S_0|\sum_{1\le k\le k'\le t}|S_k||S_{k'}|$ *(to facets* $\mathrm{Tr}_{ij,k}: p_{ik}+p_{kj}-p_{ij}-p_{kk}\ge 0$ *with* $0+0-0-0=0$, $1+0-1-0=0$ *and* $1+1-1-1=0$*).*

(iii) *All* $\{0,1\}$*-valued representatives of extreme rays of* PMET_n *are* $2^{n-1}+\sum_{1\le i\le n-1}\binom{n}{i}B(n-i)$ *(organised into* $1+\lfloor\frac{n}{2}\rfloor+\sum_{1\le i\le n-1}Q(i)$ *orbits under* $\mathrm{Sym}(n)$*) elements of the form* $J(S_0)+\delta(S_0,S_1,\dots,S_t)$*, where* $t=2$ *if* $S_0=\varnothing$ *(w.l.o.g. suppose* $S_i\ne\varnothing$ *for* $1\le i\le t$*).*

Proof.

(i) Given an $\{0,1\}$-valued element p of PMET_n, let us construct partition $S_0,S_1,\dots,S_t$ such that $p=J(S_0)+\delta(S_0,S_1,\dots,S_t)$.

See, for example, below the partial semimetric

$$p=((p_{ij}))=J(\{67\})+\delta(\{67\},\{1\},\{23\},\{45\})=P(q)$$

($\{0,1\}$-valued extreme ray of PMET_7) and corresponding weightable quasi-semimetric

$$q=((q_{ij}=p_{ij}-p_{ii}))$$

($\{0,1\}$-valued non-extreme ray of WQMET_7).

$$\begin{array}{ccccccc}
\mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1\\
1 & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1\\
1 & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1\\
1 & 1 & 1 & \mathbf{0} & \mathbf{0} & 1 & 1\\
1 & 1 & 1 & \mathbf{0} & \mathbf{0} & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\qquad
\begin{array}{ccccccc}
\mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1\\
1 & \mathbf{0} & 0 & 1 & 1 & 1 & 1\\
1 & 0 & \mathbf{0} & 1 & 1 & 1 & 1\\
1 & 1 & 1 & \mathbf{0} & 0 & 1 & 1\\
1 & 1 & 1 & 0 & \mathbf{0} & 1 & 1\\
0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}
\end{array}$$

Set $S_0=\{1\le k\le n: p_{kk}=1\}$; then $p_{kk'}=p_{k'k}=1$ for any $k\in S_0$ and $1\le k'\le n$ by definition of the facets $N_{kk'},N_{k'k}$. Let S_1 be a maximal subset of $\overline{S_0}$ such that $p_{kk'}=0$ for $k,k'\in S_1$, then S_2 be a maximal subset of $\overline{S_0\cup S_1}$ such that $p_{kk'}=0$ for $k,k'\in S_2$

and so on. It remains to show that $p_{kk'} = 1$ if $k \in S_i$, $k' \in S_{i'}$ with different $1 \leq i, i' \leq t$. W.l.o.g. suppose $|S_i| \geq 2$ for some $1 \leq i \leq n$, since, otherwise, (i) holds by construction of sets S_i. The inequalities $\mathrm{Tr}_{kj,k'} \geq 0$ and $\mathrm{Tr}_{k'j,k} \geq 0$, where $k, k' \in S_i$ and $j \in S_{i'}$, imply $p_{kj} = p_{k'j}$, since $p_{kk'} = p_{kk} = p_{k'k'} = 0$. Now, $p_{kj} = p_{k'j} = 0$ is impossible by construction of sets S_i; so, $p_{kj} = p_{k'j} = 1$.

(ii) It can be checked by direct computation.

(iii) If $S_0 = \varnothing$, then p is a multicut; so, by first equality in Lemma 2, it represents an extreme ray if and only if it is a non-zero cut. Let $S_0 \neq \varnothing$; let order it as $S_0 = \{z_1, \dots, z_s\}$, where $s = |S_0|$. The following list of $\binom{n+1}{2} - 1$ linearly independent (as vectors) facets among those, to which p is incident, show (iii):
- $N_{ij} : p_{ij} - p_{ii} = 0 - 0 = 0$ with $i, j \in S_k$, $1 \leq k \leq t$;
- $\mathrm{Tr}_{ij,z_1} : p_{iz_1} + p_{jz_1} - p_{ij} - p_{z_1 z_1} = 1 + 1 - 1 - 1 = 0$ with $i \in S_k$, $j \in S_{k'}$ and different $1 \leq k, k' \leq t$;
- $n - 1 - s + k$ facets $N_{iz_k} : p_{z_k i} - p_{z_k z_k} = 1 - 1 = 0$ with $i \in \overline{S_0} \cup \{z_1, \dots, z_{k-1}\}$ for each $1 \leq k \leq s$;
- $s - 1$ facets $N_{z_k z_1} : p_{z_k z_1} - p_{z_1 z_1} = 1 - 1 = 0$ with $2 \leq k \leq s$;
- $n - s$ facets $M_{ii} : p_{ii} = 0$ with $i \in \overline{S_0}$.

So, $\{0,1\}$-valued partial semimetric

$$p = J(S_0) + \delta(S_0, S_1, \dots, S_t) = P(\textstyle\sum_{1 \leq i \leq t} \delta'(S_i))$$

consists of all ones if $S_0 = V_n$; it is a semimetric (moreover, the multicut $\delta(S_0, S_1, \dots, S_t)$) if $S_0 = \varnothing$.

For $S_0 = V_n, \varnothing$, exactly 2^{n-1} partial semimetrics p represent an extreme ray: $p = J(V_n) = J = P(\delta(\varnothing))$ (one orbit) and $2^{n-1} - 1$ non-zero cuts $\delta(S)$ ($\lfloor \frac{n}{2} \rfloor$ orbits).

The incidence number of the extreme ray represented by $P(\delta(S)) = \delta(S)$ (cut semimetric) is

$$3\binom{n}{3} - \frac{n(n - |S|)(|S| - 2)}{2} + |S|^2$$
$$= \left(3\binom{n}{3} + n^2\right) - \frac{n|S|(n - |S|)}{2} - |S|(n - |S|).$$

The incidence number of the extreme ray represented by $P(\delta'(S)) = J(\overline{S}) + \delta(S)$ is

$$(3\binom{n}{3} + n^2) - \frac{n|S|(n - |S|)}{2} - (n - |S|).$$

The case $S = \varnothing$ corresponds to the extreme ray $J = J(V_n) + \delta(\varnothing))$ of all-ones. The orbit size of $P(\delta(S))$ and $P(\delta'(S))$ is $\binom{n}{|S|}$, except the case $|S| = \frac{n}{2}$, when it is $\frac{1}{2}\binom{n}{|S|}$.

The incidence number of the extreme ray represented by $P(\sum_{1\le i\le t}\delta'(\{i\}))$ is

$$t+t(n-t)+(n-t)\binom{n-1}{2}.$$

The size of its orbit is $\binom{n}{t}$. For $t=n-1$, it is a simplicial extreme ray represented by $P(\alpha'\{n\})$. □

3.2. **Some other extreme rays of** PMET_n**.** The partial semimetric $((p_{ij}))$ $=P(\alpha'(S))$ have $p_{ij}=2$ if $|\{i,j\}\cap S|=2$, $p_{ii}=0$ if $i\notin S$ and $p_{ij}=1$, otherwise. So, it is the matrix J of all-ones if $S=\varnothing$. If $|S|\ge 2$, then the incidence number of $P(\alpha'(S))$ is $\frac{n|S|(n-|S|)}{2}+(n-|S|)$. It is the sum of:

- $(n-|S|)$ (to facets M_{ii}, $i\notin S$),
- $|S|(n-|S|)$ (to facets N_{ij}, $i\in S, j\notin S$), and
- $(|S|\binom{n-|S|}{2}+(n-|S|)\binom{|S|}{2})$ (to facets $T_{ij,k}=1+1-2-0$ or $=1+1-1-1$ for $i,j\in S, k\notin S$ or $i,j\notin S, k\in S$, respectively).

It can be shown, similarly to the proof of Theorem 3(iii), that $P(\alpha'(S))$ represents an extreme ray of PMET_n and this ray is simplicial if and only if $|S|=1$.

The partial semimetric $((p_{ij}))=P(\alpha(S))$ is the semimetric $\alpha(S)=d(K_{S,\overline{S}})$, which represent also an extreme ray in MET_n if $2\le|S|\le n-2$. The incidence number of it, as an extreme ray of PMET_n, is

$$n+0+\left(|S|\binom{n-|S|}{2}+(n-|S|)\binom{|S|}{2}\right)=\frac{n|S|(n-|S|)}{2}+n-|S|(n-|S|).$$

We conjecture that o-cuts $\delta^{'}(S)$ with $1\le|S|\le n-1$ and o-anticuts $\alpha^{'}(S)$ with $2\le|S|\le n-2$ are only representatives q of extreme ray in QMET_n such that $P(q)$ represent an extreme ray in PMET_n.

Above formulae for incidence numbers imply that, for any n, the partial metrics $P(\alpha'(\{i\}))=\sum_{j\in\overline{\{i\}}}\delta^{'}(\{j\})$ form unique orbit of $\{0,1\}$-valued representatives of a *simplicial* (i.e., with incidence number $\binom{n+1}{2}-1$) extreme ray in PMET_n. Besides, PMET_n with $n=4,5,6$ have, respectively, $0,1,16$ such orbits of size $n!$ and $1,3,8$ such orbits of size $\frac{n!}{2}$ and $0,1,1$ such orbits of size $\frac{n!}{3!}$. Also, PMET_4 has one such orbit of size 10. Hence, altogether PMET_n with $n=3,4,5,6$ have $3,16,340,14526$ simplicial extreme rays, organized in $1,2,7,26$ orbits, respectively.

The diameter of the skeleton of PMET_n is, perhaps, 2, because the extreme ray J of all ones is incident to all facets incident to any extreme ray $\{p+\lambda J\}$, except M_{ii}, whenever $p_{ii}>0$. If p is any other $\{0,1\}$-valued partial semimetric, i.e., $p=P(\sum_{1\le k\le t}\delta'(S_k))$, then $n-|S_0|$ such facets are excluded. In particular, for simplicial extreme ray represented by $p=P(\alpha'(\{i\})=\sum_{j\in\overline{\{i\}}}\delta^{'}(\{j\})$, the common facets are $\binom{n+1}{2}-1$ facets of p, except $n-1$ facets $p_{jj}=0$ with $1\le j\le n, j\ne i$; so, they are not adjacent.

3.3. **Vertex-splitting.** The *vertex-splitting* of a function $f = ((f_{ij}))$ on V_n^2 is a function $f^{vs} = ((f_{ij}^{vs}))$ on V_{n+1}^2, defined, for $1 \le i, j \le n+1$, by

$$\begin{aligned} f^{vs}_{n+1\,n+1} &= f^{vs}_{n\,n+1} = f^{vs}_{n+1\,n} = 0, \\ f^{vs}_{i\,n+1} &= f_{i\,n}, \\ f^{vs}_{n+1\,i} &= f_{n\,i}, \text{ and} \\ f^{vs}_{i\,j} &= f_{i\,j}\ . \end{aligned}$$

The vertex-splitting of an o-multicut $\delta'(S_1, \dots, S_q)$ is the o-multicut $\delta'(S_1, \dots, S_l \cup \{n+1\}, \dots, S_q)$ if $n \in S_l$.

The vertex-splitting of a generic $\{0,1\}$-valued element $J(S_0) + \delta(S_0, S_1, \dots, S_t)$ of PMET_n is $J(S_0) + \delta(S_0, S_1, \dots, S_l \cup \{n+1\}, \dots, S_t) \in \mathrm{PMET}_{n+1}$ if $n \in S_l$ with $l \neq 0$, and it is not a partial metric, otherwise. So, the only $\{0,1\}$-valued elements, which are not vertex-splittings, are those with $|S_i| = 1$ for all $1 \le i \le t$.

Finally, the vertex-splitting of a ($\{0,1,2\}$-valued) extreme ray representative $P(\alpha'(S))$ is a $P(\alpha'(S) + e_{n\,n+1}) \in \mathrm{PMET}_{n+1}$ if $n \notin S$ and it is not a partial metric, otherwise. The orbit O_{18} of extreme ray representatives in PMET_5 consists of vertex-splittings of ones of the orbit O_{10} of $P(\alpha'(\{14\})$ in PMET_4. The orbits O_{28} and O_{29} of ($\{0,1,2,3\}$-valued) extreme ray representatives in PMET_5 consist of vertex-splittings (two ways) of ones of the orbit O_{11} in PMET_4.

Theorem 4. *If a partial semimetric p represents an extreme ray of* PMET_n *and has $p_{nn} = 0$, then its vertex-splitting p^{vs} represents an extreme ray of* PMET_{n+1}.

Proof. The condition $p_{nn} = 0$ is needed since, othervise, p^{vs} violate the inequality $f_{n+1\,n+1} - f_{nn} \ge 0$, which is valid in PMET_{n+1}. It suffice to present $n+1$ facets which, together with $\binom{n+1}{2} - 1$ linearly independent facets (seen as vectors) containing p, will form $\binom{n+2}{2} - 1$ linearly independent facets containing p^{vs}. Such facets are two of type N_{ij} ($p_{n+1\,n+1} \ge 0$ and $p_{n\,n+1} - p_{n+1\,n+1} \ge 0$) and $n-1$ of type $\mathrm{Tr}_{in,n+1} : p_{i\,n+1} + p_{n\,n+1} - p_{in} - p_{n+1\,n+1} \ge 0$. □

Above theorem gives another proof for the completeness of the list of $\{0,1\}$-valued extreme rays of PMET_n.

4. The cone l_1-PMET_n

The subcone l_1-PMET_n of PMET_n, generated by all its $\{0,1\}$-valued extreme rays consists of all partial semimetrics $p = ((p_{ij}))$ such that $q = ((p_{ij} - p_{ii})) \in \mathrm{OCUT}_n$, i.e., the quasi-semimetric q is l_1-embeddable. l_1-PMET_n coincides with PMET_n only for $n = 3$.

A *zero-extension of an inequality* $\sum_{1 \le i \neq j \le n-1} f_{ij} d_{ij} \ge 0$, is an inequality

$$\sum_{1 \le i \neq j \le n} f'_{ij} d_{ij} \ge 0 \text{ with } f'_{ni} = f'_{in} = 0 \text{ and } f'_{ij} = f_{ij}, \text{ otherwise.}$$

O_i	Representative $P(q)$	11	21	22	31	32	33	Inc.	$\|O_i\|$
O_1	$P(\delta'(\varnothing))$	1	1	1	1	1	1	9	1
O_2	$P(\delta'(\overline{\{1\}}))$	1	1	0	1	0	0	8	3
O_3	$P(\delta'(\{1\}))$	0	1	1	1	1	1	7	3
O_4	$P(\delta(\{1\}) = \delta'(\{1\}) + \delta'(\overline{\{1\}})$	0	1	0	1	0	0	7	3
O_5	$P(\alpha'(\{3\}) = \delta'(\{1\}) + \delta'(\{2\})$	0	1	0	1	1	1	5	3

TABLE 2. The representatives of orbits of extreme rays in PMET_3

O_i	Representative	11	21	22	31	32	33	Inc.	$\|O_i\|$
O_1	$\mathrm{NN}_{11} : p_{11} \geq 0$	1	0	0	0	0	0	8	3
O_2	$\mathrm{Tr}_{23,1} : p_{21} + p_{31} \geq p_{32} + p_{11}$	-1	1	0	1	-1	0	8	3
O_3	$\mathrm{NN}_{21} : p_{21} \geq p_{11}$	-1	1	0	0	0	0	7	6

TABLE 3. The representatives of orbits of facets in $\mathrm{PMET}_3 = l_1 - \mathrm{PMET}_3$

Easy to see that zero-extension of any facet-defining inequality of l_1-PMET_{n-1} is a valid inequality of l_1-PMET_n. We conjecture that, moreover, it is a facet-defining inequality of l_1-PMET_n.

Given a sequence of n integers $b = (b_1, \dots, b_n)$, let $\sum(b)$ denote $\sum_{i=1}^n b_i$ and, for any $p = ((p_{ij})) \in l_1$-PMET_n, denote

$$H_p(b) = -\sum_{1 \leq i < j \leq n} b_i b_j p_{ij},$$

$$\mathrm{Hyp}_p(b) = H_p(b) - \frac{1}{2}\sum_{i=1}^{n} b_i(b_i - 1)p_{ii}, \text{ and}$$

$$A_p(b) = H_p(b) - \frac{1}{2}\sum_{i=1}^{n} \max\{0, |b_i|(|b_i| + 1) - 2\}p_{ii}.$$

For $\sum(b) \in \{0, 1\}$, call $\mathrm{Hyp}_p(b) \geq 0$ and $A_p(b) \geq 0$ *hypermetric inequality* and *modular inequality*, respectively.

Lemma 4.1.

(i) *Any hypermetric inequality* $\mathrm{Hyp}_p(b) \geq 0$ *is valid on* l_1-PMET_n.
(ii) *Any modular inequality* $A_p(b) \geq 0$ *with* $\max_{1\leq i\leq n} |b_i| \leq 2$ *is valid on* l_1-PMET_n.

Proof. In fact, it suffices to check its validity for a typical extreme ray of l_1-PMET_n represented by $p = J(S_0) + \delta(S_0, S_1, \dots, S_t) = P(\sum_{1\leq i\leq t} \delta'(S_i))$.

O_i	Representative $P(q)$	11	21	22	31	32	33	41	42	43	44	Inc.	$\lvert O_i\rvert$
O_1	$P(\delta'(\varnothing))$	1	1	1	1	1	1	1	1	1	1	24	1
O_2	$P(\delta'(\overline{\{2\}})$	0	1	1	0	1	0	0	1	0	0	21	4
O_3	$P(\delta'(\{3\})$	1	1	1	1	1	0	1	1	1	1	19	4
O_4	$P(\delta(\{3\})$	0	0	0	1	1	0	0	0	1	0	19	4
O_5	$P(\delta'(\{3,4\})$	1	1	1	1	1	0	1	1	0	0	18	6
O_6	$P(\delta(\{3,4\})$	0	0	0	1	1	0	1	1	0	0	16	3
O_7	$P(\delta'(\{2\})+\delta'(\{3\})$	1	1	0	1	1	0	1	1	1	1	14	6
O_8	$P(\delta'(\{2\})+\delta'(\{3,4\})$	1	1	0	1	1	0	1	1	0	0	14	12
O_9	$P(\alpha'(\{4\})$	0	1	0	1	1	0	1	1	1	1	9	4
O_{10}	$P(\alpha'(\{1,4\})$	1	1	0	1	1	0	2	1	1	1	10	6
O_{11}	$P(\delta(\{3\})+2\delta'(\overline{\{4\}})+2d(K_{\{1,2\}})$	0	2	0	1	1	0	2	2	3	2	9	12

TABLE 4. The representatives of orbits of extreme rays in PMET_4

O_i	Representative	11	21	22	31	32	33	41	42	43	44	Inc.	$\lvert O_i\rvert$
O_1	$\text{NN}_{11}: p_{11} \geq 0$	1	0	0	0	0	0	0	0	0	0	29	4
O_2	$\text{Hyp}_p(-1,1,1,0) = p_{21}+p_{31}-p_{32}-p_{11} \geq 0$	-1	1	0	1	-1	0	0	0	0	0	26	12
O_3	$\text{NN}_{21} = \text{Hyp}_p(1,-1,0,0) = p_{21}-p_{11} \geq 0$	-1	1	0	0	0	0	0	0	0	0	23	12
O_4	$\text{Hyp}_p(1,1,-1,-1) \geq 0$	0	-1	0	1	1	-1	1	1	-1	-1	16	6
O_5	$A_p(2,1,-1,-1) = H_p(2,1,-1,-1) - 2p_{11} \geq 0$	-2	-2	0	2	1	0	2	1	-1	0	9	12

TABLE 5. The representatives of orbits of facets in l_1-PMET_4

O_i	Representative $P(q)$	11	21	22	31	32	33	41	42	43	44	51	52	53	54	55	Inc.	$\lvert O_i\rvert$
O_1	$P(\delta'(\varnothing))$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	90	1
O_2	$P(\delta'(\overline{\{5\}})$	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	44	5
O_3	$P(\delta'(\{5\})$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	41	5
O_4	$P(\delta(\{1\})$	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0	41	5
O_5	$P(\delta'(\overline{\{1,5\}})$	1	1	0	1	0	0	1	0	0	0	1	1	1	1	1	38	10
O_6	$P(\delta'(\{1,5\})$	0	1	1	1	1	1	1	1	1	1	0	1	1	1	0	37	10
O_7	$P(\delta(\{1,5\})$	0	1	0	1	0	0	1	0	0	0	0	1	1	1	0	34	10
O_8	$P(\delta'(\{5\}) + \delta'(\{4\}))$	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	32	10
O_9	$P(\delta'(\{1\}) + \delta'(\overline{\{1,2\}}))$	0	1	1	1	1	0	1	1	0	0	1	1	0	0	0	32	20
O_{10}	$P(\delta'(\{1\} + \delta'(\{1,5\}))$	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0	29	30
O_{11}	$P(\delta'(\{1,5\} + \delta'(\{3,4\}))$	0	1	1	1	1	0	1	1	0	0	0	1	1	1	0	28	15
O_{12}	$P(\delta'(\{5\}+\delta'(\{4\})+\delta'(\{1\}))$	0	1	1	1	1	1	1	1	1	0	1	1	1	1	0	23	10
O_{13}	$P(\delta'(\{5\}+\delta'(\{4\})+\delta'(\{1,3\}))$	0	1	1	0	1	0	1	1	1	0	1	1	1	1	0	21	30
O_{14}	$P(\alpha'(\{1\})$	1	1	0	1	1	0	1	1	1	0	1	1	1	1	0	14	5
O_{15}	$P(\alpha'(\{2,3\})$	0	1	1	1	**2**	1	1	1	1	0	1	1	1	1	0	18	10
O_{16}	$P(\alpha'(\overline{\{4,5\}})$	1	2	1	2	2	1	1	1	1	0	1	1	1	1	0	17	10
O_{17}	$P(\alpha(\{4,5\}) = d(K_{2,3})$	0	2	0	2	2	0	1	1	1	0	1	1	1	2	0	14	10
O_{18}	$P(\alpha'(\{23\} + e_{14})$	0	1	1	1	2	1	1	1	1	0	0	1	1	1	0	24	30
O_{19}		1	1	1	2	2	1	1	1	1	0	1	1	1	1	0	23	30
O_{20}		2	2	1	2	2	1	2	1	1	0	2	1	1	1	0	19	30
O_{21}		1	2	2	1	2	0	2	2	1	1	2	2	1	2	1	17	20
O_{22}		0	1	1	2	2	1	1	1	1	0	1	1	1	1	0	17	60

Table 6. The representatives of orbits of extreme rays in PMET_5.

O_i	Representative $P(q)$	11	21	22	31	32	33	41	42	43	44	51	52	53	54	55	Inc.	$\|O_i\|$
O_{23}		1	2	1	1	2	0	2	1	1	0	1	1	1	1	0	16	60
O_{24}		1	2	2	1	2	0	2	2	1	1	2	2	1	2	0	16	60
O_{25}		1	2	2	1	2	0	2	2	1	0	2	2	1	2	0	15	60
O_{26}		0	2	2	1	2	0	2	2	1	0	2	2	1	2	0	14	20
O_{27}		0	2	2	1	**3**	0	2	2	1	0	2	2	**3**	2	2	22	30
O_{28}		0	2	2	1	3	0	2	2	1	0	0	2	1	2	0	22	60
O_{29}		0	3	2	0	3	0	1	2	1	0	1	2	1	2	0	21	30
O_{30}		0	2		2	3	3	2	1	2	2	0	1	2	2	0	18	30
O_{31}		0	2	2	1	3	0	2	2	1	0	2	2	1	2	0	16	20
O_{32}		0	2	2	3	3	0	2	2	1	0	2	2	1	2	0	16	60
O_{33}		3	3	2	3	3	0	3	2	1	0	3	2	1	2	0	16	60
O_{34}		1	3	2	1	3	0	2	2	1	0	1	2	1	2	0	16	120
O_{35}		0	3	2	2	3	0	1	2	1	0	1	2	1	2	0	15	30
O_{36}		2	3	2	2	3	0	3	2	1	0	3	2	1	2	0	15	60
O_{37}		0	2	2	2	3	1	1	2	1	0	1	2	2	2	0	15	120
O_{38}		0	2	2	2	3	0	1	2	1	0	1	2	1	2	0	14	60
O_{39}		0	2	2	3	3	2	1	2	2	0	1	2	2	2	0	14	60
O_{40}		2	3	2	2	3	0	3	2	1	0	1	1	1	1	0	14	120
O_{41}		2	**4**	2	3	3	0	2	2	1	0	2		1	2	0	16	30
O_{42}		0	2	2	3	4	2	1	2	2	0	1	2	2	2	0	15	60
O_{43}		2	3	2	2	3	0	3	2	1	0	4	3	2	3	2	14	60
O_{44}		0	4	4	3	**5**	0	2	4	1	0	2	4	3	4	2	15	120

TABLE 6. The representatives of orbits of extreme rays in PMET_5. (Continued.)

O_i	Representative	11	21	22	31	32	33	41	42	43	44	51	52	53	54	55	Inc.	$\|O_i\|$
O_1	$\mathrm{NN}_{11} : p_{11} \geq 0$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	114	5
O_2	$\mathrm{Hyp}_p(1, 1, -1, 0, 0) \geq 0$	-1	1	0	1	-1	0	0	0	0	0	0	0	0	0	0	92	30
O_3	$\mathrm{Hyp}_p(1, -1, 0, 0, 0) \geq 0$	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	81	20
O_4	$\mathrm{Hyp}_p(1, 1, 1, -1, -1) \geq 0$	0	-1	0	-1	-1	0	1	1	1	-1	1	1	1	-1	-1	62	10
O_5	$\mathrm{Hyp}_p(1, 1, -1, -1, 0) \geq 0$	0	-1	0	1	1	-1	1	1	-1	-1	0	0	0	0	0	54	30
O_6	$\mathrm{Hyp}_p(1, 1, 1, -1, -2) \geq 0$	0	-1	0	-1	-1	0	1	1	1	-1	2	2	2	-2	-3	36	20
O_7	$A_p(2, 1, -1, -1, 0) \geq 0$	-2	-2	0	2	1	0	2	1	-1	0	0	0	0	0	0	31	60
O_8	$\mathrm{Hyp}_p(2, 1, -1, -1, -1) \geq 0$	-1	-2	0	2	1	-1	2	1	-1	-1	2	1	-1	-1	-1	29	20
O_9	$A_p(3, 1, -1, -1, -1) \geq 0$	-5	-3	0	3	1	0	3	1	-1	0	3	1	-1	-1	0	23	20
O_{10}	$A_p(2, 2, -1, -1, -1) \geq 0$	-2	-4	-2	2	2	0	2	2	-1	0	2	2	-1	-1	0	20	10
O_{11}	$A_p(2, 1, 1, -1, -2) \geq 0$	-2	0	-2	-1	0	2	1	1	0	4	2	2	-2	-2	2	20	60
O_{12}		-5	-5	-2	3	2	0	3	2	-1	0	5	3	-2	-2	0	19	60
O_{13}		0	2	-2	-1	2	0	-1	0	1	0	2	-2	0	2	-2	18	60
O_{14}		-2	-3	0	4	2	-2	3	1	-2	0	4	2	-2	-2	-2	18	60
O_{15}		-2	-2	2	1	0	2	2	-1	-2	2	1	0	-2	0	2	17	120

TABLE 7. The representatives of orbits of facets in l_1-PMET$_5$

For any $0 \le k \le t$, let $\alpha_k = \sum_{i \in S_k} b_i$; so, $\sum(b) = \sum_{k=0}^{t} \alpha_k$. It holds that

$$\begin{aligned}
2H_p(b) &= \sum_{i=1}^{n} b_i^2 p_{ii} - \sum_{1 \le i,j \le n} b_i b_j p_{ij} \\
&= \sum_{i \in S_0} b_i^2 - \sum_{i \in S_0} b_i \sum(b) - \sum_{k=1}^{t} \Big(\sum_{i \in S_k} b_i\Big)\Big(\sum_{i \notin S_k} b_i\Big) \\
&= \sum_{i \in S_0} b_i^2 - \alpha_0 \sum(b) - \sum_{k=1}^{t} \alpha_k \Big(\sum(b) - \alpha_k\Big) \\
&= \sum_{i \in S_0} b_i^2 + \sum_{k=1}^{t} (\alpha_k)^2 - \Big(\sum(b)\Big)^2 \\
&= \sum_{i \in S_0} b_i(b_i - 1) + \sum_{k=1}^{t} \alpha_k(\alpha_k - 1) - \sum(b)\Big(\sum(b) - 1\Big).
\end{aligned}$$

So, $2\,\mathrm{Hyp}_p(b) = \sum_{k=1}^{t} \alpha_k(\alpha_k - 1) \ge 0$, i.e., (i) holds.

Now,

$$\begin{aligned}
2A_p(b) &= \Big(\sum_{i \in S_0} b_i^2 + \sum_{k=1}^{t} (\alpha_k)^2 - \big(\sum(b)\big)^2\Big) - \Big(\sum_{i \in S_0} |b_i|^2 - \sum_{i \in S_0} |b_i| + 2|S_0^{'}|\Big) \\
&= \sum_{k=1}^{t} (\alpha_k)^2 - \Big(\sum(b)\Big)^2 + \sum_{i \in S_0} |b_i| - 2|S_0^{'}|,
\end{aligned}$$

where $S_0^{'} = \{i \in S_0 : b_i \neq 0\}$. If $\sum(b) = 0$, then $2A_p(b) \ge 0$. If $\sum(b) = 1$, then either $\sum_{k=1}^{t} (\alpha_k)^2$, or $2|S_0^{'}| - \sum_{i \in S_0} |b_i|$ is at least 1. So, (ii) holds. $\square$

In fact, the typical facet-defining inequalities $N_{12} : p_{12} - p_{22} \ge 0$ and $\mathrm{Tr}_{12,3} : p_{13} + p_{23} - p_{12} - p_{33} \ge 0$ of PMET_n are instances of $\mathrm{Hyp}_p(b) \ge 0$ for $b = (1, -1, 0, \dots, 0)$ and $b = (1, 1, -1, 0, \dots, 0)$), respectively.

The cone l_1-PMET_4 (besides orbits O_1, O_2, O_3 of sizes 4, 12, 12 of facets of PMET_4) has orbits O_4, O_5 (of sizes 6, 12) of facets, represented by:

$$\mathrm{Hyp}_p((1, 1, -1, -1) = (p_{13} + p_{23} + p_{14} + p_{24}) - (p_{12} + p_{34}) - (p_{33} + p_{44}) \ge 0$$

and

$$A_p(2, 1, -1, -1) = (2p_{13} + p_{23} + 2p_{14} + p_{24}) - (2p_{12} + p_{34}) - 2p_{11} \ge 0.$$

Note that the orbits O_{10} and O_{11} of extreme rays in PMET_4 excluded in l_1-PMET_4 by orbits O_4 and O_5, respectively. In fact, $P(\alpha'\{1, 2\})$ violates $\mathrm{Hyp}_p(1, 1, -1, -1) \ge 0$, while $P(\delta(\{1\}) + 2\delta'(\overline{\{2\}}) + 2d(K_{\{3,4\}})$ violates $A_p(2, 1, -1, -1) \ge 0$.

The cone l_1-PMET_5 has 585 facets in 15 orbits, represented in Table 8 up to a permutation (orbits O_1, O_2, O_3, O_5, O_7 consist of 0-extensions of facets of l_1-PMET_4).

O_i	Size	Representative
O_1	5	$N_{11} : p_{11} \geq 0$
O_2	30	$\mathrm{Tr}_{12,3} = \mathrm{Hyp}_p(1,1,-1,0,0) \geq 0.$
O_3	20	$N_{12} : \mathrm{Hyp}_p(1,-1,0,0,0) \geq 0.$
O_4	10	$\mathrm{Hyp}_p(1,1,1,-1,-1) \geq 0.$
O_5	30	$\mathrm{Hyp}_p(1,1,-1,-1,0) \geq 0.$
O_6	20	$\mathrm{Hyp}_p(1,1,1,-1,-2) \geq 0.$
O_7	20	$A_p(2,1,-1,-1,0) = H_p(2,1,-1,-1,0) - 2p_{11} \geq 0.$
O_8	20	$\mathrm{Hyp}_p(2,1,-1,-1,-1) \geq 0.$
O_9	20	$A_p(3,1,-1,-1,-1) = H_p(3,1,-1,-1,-1) - 5p_{11} \geq 0.$
O_{10}	10	$A_p(2,2,-1,-1,-1) = H_p(2,2,-1,-1,-1) - 2(p_{11} + p_{22}) \geq 0.$
O_{11}	60	$A_p(2,1,1,-1,-2) = H_p(2,1,1,-1,-2) - 2(p_{11} + p_{55}) \geq 0.$
O_{12}	60	$H_p(3,2,-1,-1,-2) \geq -p_{12} + 5p_{11} + 2p_{22} + p_{15} + p_{25}.$
O_{13}	60	$2p_{12} + 2p_{23} + p_{34} + 2p_{45} + 2p_{51} \geq 2p_{22} + p_{13} + 2p_{25} + p_{41} + 2p_{55}.$
O_{14}	60	$(4p_{13} + 3p_{14} + 4p_{15}) + (2p_{23} + p_{24} + 2p_{25}) \geq 2(p_{34} + p_{35} + p_{45}) + 3p_{12} + 2(p_{11} + p_{33} + p_{55}).$
O_{15}	120	$H_p(-2,-1,1,2,1) \geq 2(p_{14} + p_{34} + p_{11} + p_{44}),$ i.e., $2(p_{13} + p_{14} + p_{15}) + (p_{23} + 2p_{24} + p_{25}) \geq (p_{34} + 2p_{45}) + 2p_{12} + 2(p_{11} + p_{44}).$

TABLE 8. The cone l_1-PMET$_5$.

Denote by INHYP_n the *inhomogeneous hypermetric cone* of all symmetric $n \times n$ matrices $((a_{ij}))$ with $a_{ij} \geq 0$ defined by the inequalities $\mathrm{Hyp}_a(b) \geq 0$ for all sequence of n integers $b = (b_1, \dots, b_n)$, with $\sum_{i=1}^n b_i \in \{0, 1\}$. Clearly,

$$l_1 - \mathrm{PMET}_n \subset \mathrm{INHYP}_n \subset \mathrm{PMET}_n,$$

generalizing

$$\mathrm{CUT}_n \subset \mathrm{HYP}_n \subset \mathrm{MET}_n,$$

for the restrictions of cones on the semi-metrics.

Remind Theorem 2(i) that the cone OCUT_n of all quasi-semimetrics on V_n embeddable into $l^m_{1,or}$ for some m consists of all $n \times n$ matrices $((q_{ij} = p_{ij} - p_{ii}))$, where $((p_{ij})) \in l_1 - \mathrm{PMET}_n$; so, any $((q_{ij}))$ is a weightable quasi-semimetric with weights $w_i = p_{ii}$, $1 \leq i \leq n$. Using

$$\begin{aligned} 2H_p(b) - \sum_{i=1}^n b_i^2 p_{ii} &= - \sum_{1 \leq i,j \leq n} b_i b_j p_{ij} \\ &= - \sum_{1 \leq i,j \leq n} b_i b_j q_{ij} - \sum_{i=1}^n b_i p_{ii} \sum_{j=1}^n b_j \\ &= - \sum_{1 \leq i,j \leq n} b_i b_j q_{ij} - \sum(b) \sum_{i=1}^n b_i w_i, \end{aligned}$$

we can reformulate above Lemma as follows.

Corollary 1. *Given a sequence of n integers $b = (b_1, \dots, b_n)$, with $\sum(b) = \sum_{i=1}^n b_i \in \{0, 1\}$, the following two inequalities are valid on* OCUT_n*:*

$$- \sum_{1 \leq i,j \leq n} b_i b_j q_{ij} + (1 - \sum(b)) \sum_{i=1}^n b_i w_i \geq 0, \tag{4.1}$$

$$- \sum_{1 \leq i,j \leq n} b_i b_j q_{ij} + \sum_{i=1: b_i \neq 0}^n (2 - |b_i| - b_i \sum(b)) w_i \geq 0 \tag{4.2}$$

if $\max_{1 \leq i \leq n} |b_i| \leq 2$.

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