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SOME RIGID MOIETIES OF HOMOGENEOUS GRAPHS

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Abstract. Any countable K_n-free graph T embeds as a moiety into the universal homogeneous K_n-free graph $\mathbf{K_n}$ in such a way that every automorphism of T extends to a unique automorphism of $\mathbf{K_n}$. Furthermore, there are 2^ω such embeddings which are pairwise not conjugate under $Aut(\mathbf{K_n})$.

1. Introduction

There is an interesting class of structures for a given relational language, the so called Fraïssé limits. They are countable homogeneous objects, or *ultrahomogeneous* as Hodges calls them [3, Section 7.1]. Homogeneous in this sense means that any isomorphism between finitely generated substructures extends to an automorphism of the universal structure. Fraïssé limits are not restricted to relational languages but our focus here is on homogeneous graphs. By a *graph*, we mean a set of vertices V and a relation on $V \times V$ which is irreflexive and symmetric.

Given a countable set X, by a *moiety* we mean a subset $Y \subset X$ which is countable and co-countable. If X carries a structure in a relational language, a *rigid* moiety of X is a moiety $Y \subset X$ where each automorphism of Y (with the induced structure) extends uniquely to an automorphism of X. A question several people studied was whether rigid moieties exist for a given countable homogeneous structure. Henson showed in [2, Theorem 3.1] that the answer was positive for the random graph and, moreover, any countable graph can be embedded as a rigid moiety into the random graph. Macpherson and Woodrow gave another proof of this statement in [6, Lemma 2.1 and the following paragraph]. In the case of directed graphs, Jaligot showed in [4] that the answer was positive for the random tournament and, as in Henson's case, any countable tournament can be embedded as a rigid moiety into the random tournament.

Lachlan and Woodrow classified all homogeneous countable graphs in [5], and Cherlin has given another proof for the classification in [1]. The

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classification shows that all countably infinite homogeneous graphs, or their complements, have one of the following forms:

(1) Disjoint union (finite or countable) of copies of K_n, where K_n denotes the complete graph on n vertices (here n can be infinite),
(2) The universal K_n-free graph $\mathbf{K_n}$, for n finite and $n \geq 3$,
(3) The random graph, also called the Rado graph.

For K_n-free graphs, Henson also claimed in [2, Theorem 3.3] that any countably infinite K_n-free graph can be embedded as a rigid moiety in the corresponding universal K_n-free graph $\mathbf{K_n}$. But there was a small error in the proof, and actually he only proved it for graphs which satisfied an extra property. For graphs which did not satisfy that property, Henson extended them to supergraphs, which did satisfy the property, and proved the theorem for the supergraph. The problem was that even though the original graph was embedded rigidly into the supergraph, and the supergraph was embedded rigidly into the universal object $\mathbf{K_n}$, it does not follow that the original graph is embedded rigidly into $\mathbf{K_n}$. In this paper we give a nontrivial modification of Henson's proof to make it work for all countable K_n-free graphs, and we fully prove the following statement.

Theorem 1.1. *Let T be any countably infinite K_n-free graph. Then T embeds into the universal K_n-free graph $\mathbf{K_n}$ as a rigid moiety.*

2. Proofs

Throughout the paper, if T is a graph, and A is a subset of vertices of T, $T|A$ is going to denote the induced subgraph of T on A.

Lemma 2.1. *The class $\mathcal{K}_n$ of finite K_n-free graphs has a Fraïssé limit for each $n \geq 3$, denoted $\mathbf{K_n}$, and it is characterized by the property that for any finite disjoint subsets A and B such that A is K_{n-1}-free, there exists a vertex $v \in \mathbf{K_n}$ such that v is adjacent to every vertex in A and not adjacent to any vertex in B. See, e.g.,* [3, Exercise 7.4-7].

Lemma 2.2. *There is a countably infinite graph N, that has a trivial automorphism group and is K_n-free for all $n \geq 3$.*

There are many examples of graphs as in Lemma 2.2, and one is the graph consisting of the set N of natural numbers as vertices, and where edges are exactly of the form $(k, k+1)$ up to symmetry. The fact that it has a trivial automorphism group should be clear because 0 is the only vertex which is connected to only one vertex.

Here starts our proof of Theorem 1.1. Let T be a countably infinite K_n-free graph and let $T' = T \sqcup N$, where we add the specific graph N just described above as a new connected component to T. We are going to construct a supergraph of T'. Let $1 \leq n_1 < n_2 < \cdots$ be any strictly increasing sequence of positive integers. Construct an increasing chain of graphs $T' = T_0 \subset T_1 \subset T_2 \cdots$ as follows. For $k \geq 1$, obtain T_k by adding a new vertex $v(A, k)$ to T_{k-1} for each finite subset $A \subset T_{k-1}$ satisfying the following three conditions:

(1) $T_{k-1}|A$ is K_{n-1}-free,
(2) $A \cap T_0$ has exactly n_k elements,
(3) $A \cap T$ has at least one element.

Then each new vertex $v(A, k)$ in T_k will be adjacent to the vertices in A and to no others.

Let $\mathcal{T}$ be the union of the chain T_k. Notice that $T_k - T_{k-1}$ is infinite for each $k \geq 1$.

Lemma 2.3. *$\mathcal{T}$ is countable and K_n-free.*

Proof. Since at each step we are adding countably many new vertices, the union $\mathcal{T}$ is countable. Notice that $T_0 = T \sqcup N$ is K_n-free. We now show that T_k is K_n-free for each k. By induction on k, assume that T_{k-1} is K_n-free. Each new vertex $v(A, k)$ we add is adjacent only to A inside T_{k-1}, where $T_{k-1}|A$ is K_{n-1}-free, and adjoining a new vertex to all vertices of a K_{n-1}-free graph cannot create a complete graph on n vertices. Since there are no edges in $T_k - T_{k-1}$, we get that T_k is K_n-free. □

Definition 2.4. *Let G be a graph. A subset I of vertices of G is said to be an* independent *set if the induced subgraph on I does not have any edge relations. G itself is called* independent *if it does not have any edge relations.*

Lemma 2.5. *$T' = T \sqcup N$ satisfies the following condition: if F is any finite subset of vertices of T', then there exists an infinite independent set of vertices $A \subset N - F$ such that no vertex in F is adjacent in T' to any vertex in A.*

Proof. Recall that N has the isomorphism type of the graph described after Lemma 2.2, and is a connected component of T'. Since F is finite, there are infinitely many vertices in $N - F$. Removing the finitely many vertices of $N - F$ connected to some element in F, we get an infinite set of elements of $N - F$ not adjacent to any vertex in F. Now choose A to be an independent set among these vertices, for example all odd numbered or all even numbered remaining vertices. □

Lemma 2.6. *Let A, B be two finite disjoint subsets of vertices of $\mathcal{T}$ such that $\mathcal{T}|A$ is K_{n-1}-free. Then there exists a vertex $v \in \mathcal{T}$ such that v is adjacent to every vertex in A and to none of the vertices in B.*

Proof. Choose k large enough so that $A \sqcup B \subset T_{k-1}$ and $A \cap T_0$ has at most $n_k - 1$ elements. Let $C \subset T_0$ consist of $A \cap T_0$ together with every vertex in T_0 which is connected to some member of $A - T_0$. Since A is finite, and each vertex in $\mathcal{T} - T_0$ is connected to only finitely many members of T_0, it follows that C is finite. Then by Lemma 2.5 there exists an infinite independent set A' in the subgraph N such that $A' \cap C = \emptyset$ and no vertex in C is connected in T_0 to any vertex in A'. Now, $\mathcal{T}|(A \cup A')$ is K_{n-1}-free. Since A' is infinite, we can choose a set $D \subset (A \cup A') \cap T_0$ such that $D \cap B = \emptyset$, $A \cap T_0 \subseteq D$ and D has exactly $n_k - 1$ elements. Now there are two cases:

Case 1: $A \cap T \neq \emptyset$.

In this case D has at least one element in T. Also add one more vertex to D to make its size exactly n_k (one may choose a new element in the set A' as above). Letting $E = A \cup D$, it follows that $\mathcal{T}|E$ is K_{n-1}-free and $E \cap T_0 = D$ has n_k elements with at least one element in T. Thus there exists $v(E, k)$ in T_k which is adjacent to every member of E, and in particular to every member of A, and to no member of B.

Case 2: $A \cap T = \emptyset$.

Since T is infinite under our standing assumptions for Theorem 1.1, $T - (B \cup C)$ is nonempty. Let x be any vertex in $T - (B \cup C)$. Look at the set $D' = D \sqcup \{x\}$, which is a disjoint union because $D \subset (A \cup A') \subseteq N$. Now D' has exactly n_k vertices with one vertex in T, $D' \cap B = \emptyset$ and $A \cap T_0 \subseteq D \subset D'$. Let $E = A \cup D'$. We need to check that E is K_{n-1}-free. But as in Case 1 we see that $E - \{x\} = A \cup D$ is K_{n-1}-free, and x is chosen in such a way that it is not adjacent to any vertex in $E - \{x\}$ because T and N are two distinct connected components of T_0, so E is K_{n-1}-free. Since $E \cap T_0 = D'$ has exactly n_k elements, with the element x in T, the rest of the proof follows exactly as in Case 1 by considering the vertex $v(E, k)$ of T_k.

□

So, $\mathcal{T}$ is a countable K_n-free graph such that for any finite disjoint subsets $A, B \subset \mathcal{T}$ such that A is K_{n-1}-free, there exists a vertex $v \in \mathcal{T}$ such that v is adjacent to each vertex in A and to none of the vertices in B. Hence $\mathcal{T}$ is isomorphic to $\mathbf{K_n}$.

Lemma 2.7. *Every automorphism of T extends uniquely to an automorphism of $\mathcal{T}$.*

Proof. First, to prove the existence of extensions, given an automorphism ϕ of T, extend ϕ to an automorphism of T_0 by setting $\phi(v) = v$ for $v \in N$. Then proceed by induction on k as follows: given that ϕ is extended to an automorphism of T_{k-1}, extend it further to an automorphism of T_k by setting $\phi(v(A, k)) = v(\phi(A), k)$. This takes care of the existence of the extension.

Now, for the uniqueness, let $\phi \in Aut(\mathcal{T})$ such that $\phi(T) = T$. Then it follows that $\phi(N) = N$ since the only vertices in $\mathcal{T}$ which are not adjacent to any vertex in T are the vertices of N. So we have $\phi(T') = \phi(T \sqcup N) = \phi(T) \sqcup \phi(N) = T \sqcup N = T'$. And also notice that ϕ is the identity on N since N only has the trivial automorphism. So ϕ is uniquely determined for vertices in T_0.

Notice that any $\phi \in Aut(\mathcal{T})$ which setwise stabilizes $T_0 = T'$ has to setwise stabilize each $T_k - T_{k-1}$ for $k \geq 1$. This is true because vertices in

$T_k - T_{k-1}$ are adjacent to exactly n_k many vertices in T_0 and $n_k \neq n_{k'}$ for $k \neq k'$. Thus, each T_k is setwise stabilized by ϕ as well.

We will proceed by induction. Let $v \in \mathcal{T} - T_0$. Then $v \in T_k - T_{k-1}$ for some k. By induction assume that ϕ is uniquely determined for vertices in T_{k-1}. But $v = v(A, k)$ for some $A \subset T_{k-1}$ such that

(1) $T_{k-1}|A$ is K_{n-1}-free,
(2) $A \cap T_0$ has exactly n_k elements,
(3) $A \cap T$ has at least one element,

and v is adjacent to all vertices of A and to no other vertex in T_{k-1}.

By induction $\phi(A)$ is uniquely determined, and because ϕ is an automorphism which setwise stabilizes both T and T_0, we have

(1) $T_{k-1}|\phi(A)$ is K_{n-1}-free,
(2) $\phi(A) \cap T_0$ has exactly n_k elements,
(3) $\phi(A) \cap T$ has at least one element.

Hence there is a unique vertex $v(\phi(A), k) \in T_k$ which is adjacent to $\phi(A)$ and to no other vertices in T_{k-1}. So $\phi(v) = v(\phi(A), k)$. □

So this concludes the proof of Theorem 1.1. But notice that we actually proved a little more. There are exactly 2^ω such rigid embeddings of T which are not conjugate in $Aut(\mathbf{K_n})$.

Theorem 2.8. *Given a countably infinite K_n-free graph T, there are 2^ω many rigid embeddings of T into $\mathbf{K_n}$ which are pairwise non-conjugate under $Aut(\mathbf{K_n})$.*

Proof. Notice from the above proof that for each strictly increasing sequence $1 \leq n_1 < n_2 < \cdots$ of positive integers we get a rigid embedding of the form

$$X \sqcup X_0 \sqcup X_{n_1} \sqcup \cdots \sqcup X_{n_k} \sqcup \cdots$$

where $X = T, X_0 = T_0 - T, \ldots, X_{n_k} = T_k - T_{k-1}$. Furthermore, X_0 is the set of elements connected to no vertices in T, and X_{n_k} is the set of elements connected to exactly n_k many vertices in $X \sqcup X_0$.

Now for two distinct strictly increasing sequences of natural numbers, we get two rigid embeddings of T into $\mathbf{K_n}$ which are not conjugate by an automorphism of $\mathbf{K_n}$. Since there are 2^ω many such sequences, we conclude that there are 2^ω non-conjugate rigid embeddings of T into $\mathbf{K_n}$. □

Remark 2.9. In our proof of Theorem 2.8, we varied the sequence of integers $1 \leq n_1 < n_2 < \cdots$, but we could also have varied the isomorphism type of the graph X_0, there isomorphic to the specific graph N as described after Lemma 2.2, as long as it is a countably infinite K_n-free graph with a trivial automorphism group and satisfying the statement of Lemma 2.5.

Of course, Theorems 1.1 and 2.8 remain valid if one works in the class of graphs whose complements are K_n-free. So they cover all cases of homogeneous graphs listed in item 2 of the classification given in the introduction. We also remark that the existence of 2^ω pairwise non-conjugate embeddings as a rigid moiety, as in Theorem 2.8, holds similarly in the case of the random graph: the proof given in [2, Theorem 3.1] directly yields the same remark as in the proof of Theorem 2.8.

To conclude, we also remark that the maximal number of non-conjugate embeddings as a rigid moiety can also be obtained in the case of the random tournament. Actually, a slight modification of the argument in [4], along the lines of [2, Theorem 3.1], yields the same result as in Theorem 2.8 in the case of tournaments: for any countably infinite tournament T, there are 2^ω embeddings of T as a rigid moiety of the random tournament which are pairwise non-conjugate in the automorphism group of the random tournament.

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