Contributions to Discrete Mathematics

Volume 4, Number 2, Pages 89–93
ISSN 1715-0868

ANOTHER SHORT PROOF OF THE JONI-ROTA-GODSIL INTEGRAL FORMULA FOR COUNTING BIPARTITE MATCHINGS

ERIN E. EMERSON AND P. MARK KAYLL

ABSTRACT. How many perfect matchings are contained in a given bipartite graph? An exercise in Godsil's 1993 *Algebraic Combinatorics* solicits proof that this question's answer is an integral involving a certain rook polynomial. Though not widely known, this result appears implicitly in Riordan's 1958 *An Introduction to Combinatorial Analysis*. It was stated more explicitly and proved independently by S. A. Joni and G.-C. Rota [*JCTA* **29** (1980), 59–73] and C. D. Godsil [*Combinatorica* **1** (1981), 257–262]. Another generation later, perhaps it's time both to revisit the theorem and to broaden the formula's reach.

INTRODUCTION

This note considers the relation between the number of perfect matchings of a bipartite graph G and the number of matchings of various sizes in its 'bipartite complement' $\widetilde{G}$. These numbers are related by a surprising integral formula involving the rook polynomial of $\widetilde{G}$. Though not widely known, this result appears implicitly in Riordan's book [10]. It was first stated more explicitly, using an integral, by Joni and Rota [8], although it was Godsil [5] who cast it in the form treated here. See also [4], which predates the later results in addressing the special case when G is a disjoint union of complete bipartite graphs. Our purpose is twofold: to present a simple, stand-alone proof and to broaden the formula's reach. Our proof, using inclusion-exclusion, is at once more direct than Godsil's and more transparent than the others'; the remarks following the statement of Theorem 2 elaborate. Readers might appreciate how this proof ties together the sign alternation in the rook polynomial's definition with that in the inclusion-exclusion formula.

Received by the editors February 10, 2009, and in revised form October 3, 2009.

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05C70, 33B15.

Key words and phrases. perfect matching, rook polynomial, bipartite complement, inclusion-exclusion.

Direct correspondence to P.M.K.; based on work done while E.E.E. attended University of Montana.

Notation and terminology

Given a graph G and an integer k, we denote by $\mu_G(k)$ the number of matchings in G containing exactly k edges; naturally, $\mu_G(0) = 1$. If k is half the number of vertices, i.e. if $\mu_G(k)$ counts perfect matchings, then we write $\Xi(G)$ for $\mu_G(k)$.[1] If G is a spanning subgraph of $K_{n,n}$, then the *rook polynomial* of G is defined by $\rho_G(t) := \sum_{k=0}^{n}(-1)^k\mu_G(k)t^{n-k}$ (see [6] or [10] for etymology), and the *bipartite complement* $\widetilde{G}$ shares its vertex set with G and has for edges all the edges of $K_{n,n}$ that are not in G. Most standard graph theory texts should furnish any omitted definitions; we generally follow [2].

Results

The formula under consideration is the conclusion of the first result.

Theorem 1 ([5, 8])**.** *If G is a spanning subgraph of $K_{n,n}$, then*

$$\Xi(G) = \int_0^\infty \rho_{\widetilde{G}}(t)e^{-t}\,dt. \tag{1}$$

In our statement of the Principle of Inclusion-Exclusion (PIE), we remind the reader of the shorthand $[m]$ for $\{1, 2, \ldots, m\}$ when m is a nonnegative integer.

PIE. *If $\{A_i\}_{i=1}^m$ is a family of subsets of a finite set $\mathcal{X}$, then*

$$\Big|\,\mathcal{X} \setminus \bigcup_{i=1}^{m} A_i\,\Big| \;=\; \sum_{I\subseteq[m]} (-1)^{|I|}\,\Big|\bigcap_{i\in I} A_i\Big|. \tag{2}$$

Any elementary combinatorics text, such as [3] (from which we borrowed the catchy abbreviation), is likely to present a proof of PIE.

Proof of Theorem 1. To determine $\Xi(G)$, let $\mathcal{X}$ denote the set of perfect matchings of $K_{n,n}$, and suppose that $\widetilde{G}$ has $m \geq 0$ edges; say $E(\widetilde{G}) = [m]$. For $i \in E(\widetilde{G})$, let $A_i = \{M \in \mathcal{X} : i \in M\}$. The elements of $\mathcal{X} \setminus \bigcup_{i=1}^m A_i$ are precisely the perfect matchings of G; whence $\Xi(G)$ is given by the right side of (2), which we proceed to simplify.

First note that when $I \subseteq E(\widetilde{G}) = [m]$ is not a matching in $\widetilde{G}$, we have $\bigcap_{i\in I} A_i = \varnothing$, so the only sets $I \subseteq [m]$ contributing nonzero terms to the sum in (2) are matchings in $\widetilde{G}$. For a fixed such I, we have $\big|\bigcap_{i\in I} A_i\big| = (n - |I|)!$ because the left side counts those $M \in \mathcal{X}$ containing each $i \in I$ and so effectively counts the perfect matchings of $K_{n-|I|,n-|I|}$. Now, given an integer k, with $0 \leq k \leq m$, there are $\mu_{\widetilde{G}}(k)$ matchings in $\widetilde{G}$ of size k;

[1] We chose this notation because the Greek letter Xi (Ξ) resembles a perfect matching in a graph of order six, and, conveniently enough, six is a perfect number.

this is the number of nonzero terms in (2) when $|I| = k$. Thus, if we sum instead over the possible sizes k of I, we obtain

$$\Xi(G) = \sum_{k=0}^{m} (-1)^k \mu_{\widetilde{G}}(k)(n-k)!.$$

Since $\widetilde{G}$ spans $K_{n,n}$, each $\mu_{\widetilde{G}}(k)$ with $k > n$ is zero, and since $|E(\widetilde{G})| = m$, each $\mu_{\widetilde{G}}(k)$ with $k > m$ is zero. This implies that the "m" in the preceding identity may be replaced by "n". On introducing Euler's gamma function (see, e.g., [1]) to rewrite the factorials, we finally obtain

$$\Xi(G) = \sum_{k=0}^{n} (-1)^k \mu_{\widetilde{G}}(k) \int_0^\infty t^{n-k} e^{-t}\, dt = \int_0^\infty \Big(\sum_{k=0}^{n} (-1)^k \mu_{\widetilde{G}}(k) t^{n-k} \Big) e^{-t}\, dt,$$

which is (1). $\square$

For general (simple but not necessarily bipartite) graphs G (with n vertices), Theorem 1 has an analogue in which the rook polynomial is replaced by the *matchings polynomial* $\alpha_G(t) := \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \mu_G(k) t^{n-2k}$, the bipartite complement is replaced by the ordinary complement $\overline{G}$, and the integration is with respect to a different measure.

Theorem 2 ([5]). *Each graph G satisfies* $\Xi(G) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \alpha_{\overline{G}}(t) e^{-t^2/2}\, dt.$

We mention Theorem 2 because it admits a proof closely paralleling our proof of Theorem 1. See also [9, Exercise 5.18(a)] which takes the same approach to a related result.

Remarks

As noted above, Riordan's book includes Theorem 1 implicitly. The result is a consequence of a generating-function identity, also derived using inclusion-exclusion (see [10, Theorem 2, p. 180]). Godsil's proofs of Theorems 1 and 2 (see [5, 6]) use induction leaning on the basic properties of $\rho_G(t)$ and $\alpha_G(t)$. As suggested above, Joni and Rota [8] actually proved a generalization of Theorem 1; they applied Möbius inversion to a related simplicial complex.

Theorems 1 and 2 have many applications, both in combinatorics and in the theory of orthogonal polynomials. For example, Theorem 1 "is perhaps the fundamental tool in" [7] (the quotation being from *op. cit.*). We present one combinatorial application below and cite [6] for further discussion and references.

An application to derangements

Recall that a *derangement* of a set S is a permutation of S admitting no fixed points. If $|S| = n \geq 1$, then the number d_n of derangements of S can be written as $d_n = n! \sum_{k=0}^{n} (-1)^k/k!$ or described as the integer closest to $n!/e$. Typical derivations of these facts apply either inclusion-exclusion or generating functions (see, e.g., [3, 11]) but Godsil [6] took the following novel approach using Theorem 1.

Fix an integer $n \geq 1$ and consider the bipartite graph G obtained from $K_{n,n}$ by removing a perfect matching M from $K_{n,n}$. Notice that the perfect matchings of G correspond bijectively to the derangements of an n-set; thus, $d_n = \Xi(G)$. The bipartite complement $\widetilde{G}$, being induced by M, satisfies $\mu_{\widetilde{G}}(k) = \binom{n}{k}$, for $0 \leq k \leq n$, which implies that $\rho_{\widetilde{G}}(t) = (t-1)^n$. Now Theorem 1 shows that $d_n = \int_0^\infty (t-1)^n e^{-t}\,dt$. If we separate the integral and change variables on the first subinterval, another evaluation of the gamma function Γ presents itself:

$$\begin{aligned} d_n &= \int_1^\infty (t-1)^n e^{-t}\,dt + \int_0^1 (t-1)^n e^{-t}\,dt \\ &= \int_0^\infty x^n e^{-(x+1)}\,dx + \int_0^1 (t-1)^n e^{-t}\,dt \\ &= e^{-1}\Gamma(n+1) + E_n, \end{aligned} \tag{3}$$

where we now view the second integral as an error term E_n. It turns out that E_n doesn't contribute much to d_n; since $e^{-t} < 1$ on the interval $(0,1)$, we obtain

$$|E_n| \leq \int_0^1 \left|(t-1)^n e^{-t}\right| dt < \int_0^1 (1-t)^n\,dt = \frac{1}{n+1}.$$

This shows that for each $n \geq 1$, the error $|E_n| < 1/2$, and it follows from (3) that d_n is the integer closest to $e^{-1}\Gamma(n+1)$, i.e., to $n!/e$.

Acknowledgement

This note was written while the second author was on sabbatical at Université de Montréal. Many thanks to Geňa Hahn for being such a generous host.

References

1. E. Artin, *The gamma function*, Holt, Rinehart and Winston, New York, 1964, MR0165148 (29 #2437).
2. J.A. Bondy and U.S.R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008, MR2368647 (2009c:05001).
3. P.J. Cameron, *Combinatorics: topics, techniques, algorithms*, Cambridge University Press, Cambridge, 1994, MR1311922 (95j:05002).
4. S. Even and J. Gillis, *Derangements and Laguerre polynomials*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 135–143, MR0392590 (52 #13407).
5. C.D. Godsil, *Hermite polynomials and a duality relation for matching polynomials*, Combinatorica **1** (1981), 257–262, MR0637830 (83e:05015).
6. ———, *Algebraic combinatorics*, Chapman & Hall, New York, 1993, MR1220704 (94e:05002).
7. C.D. Godsil and B.D. McKay, *Asymptotic enumeration of Latin rectangles*, J. Combin. Theory Ser. B **48** (1990), 19–44, MR1047551 (91j:05009).

8. S.A. Joni and G.-C. Rota, *A vector space analog of permutations with restricted position*, J. Combin. Theory Ser. A **29** (1980), 59–73, MR0577543 (83j:05005a).
9. L. Lovász, *Combinatorial problems and exercises*, second ed., North-Holland, Amsterdam, 1993, MR1265492 (94m:05001).
10. J. Riordan, *An introduction to combinatorial analysis*, John Wiley & Sons, Inc., New York, 1958, MR0096594 (20 #3077).
11. J.H. van Lint and R.M. Wilson, *A course in combinatorics*, Cambridge University Press, Cambridge, 1992, MR1207813 (94g:05003).

Ann Arbor, MI 48104, USA
E-mail address: `erinbeth@umich.edu`

Department of Mathematical Sciences, University of Montana,
Missoula, MT 59812-0864, USA
E-mail address: `mark.kayll@umontana.edu`