



## ON A THEOREM OF G. D. CHAKERIAN

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ABSTRACT. Among all bodies of constant width in the Euclidean plane, the Reuleaux triangle of the same width has minimal area. But Reuleaux triangles are also minimal in another sense: if a convex body can be covered by a translate of a Reuleaux triangle, then it can be covered by a translate of any convex body of the same constant width. The first result is known as the Blaschke-Lebesgue theorem, and it was extended to an arbitrary normed plane by Ohmann and, independently, Chakerian. In the present paper we extend the second minimal property, known as Chakerian's theorem, to all normed planes. Some corollaries from this generalization are also given.

## 1. INTRODUCTION

Due to Blaschke [5] and Lebesgue [15] any plane convex body (i.e., a compact, convex set with nonempty interior) of constant width  $\lambda > 0$  has area not less than the area of a Reuleaux triangle of width  $\lambda$ ; see also [8, p. 128] and [13]. Note that a compact, convex set  $\mathcal{K}$  is a *set of constant width* if the distance between any pair of parallel supporting lines of  $\mathcal{K}$  is the same. The intersection of three congruent circular discs, such that the boundary of each of them passes through the centers of the other two, is called *Reuleaux triangle* and well known as a classical example of a non-circular body of constant width. But Reuleaux triangles in the Euclidean plane are also “minimal” in another sense. Namely, if  $\mathcal{RT}$  is Reuleaux triangle of width  $\lambda$  and any congruent copy  $\mathcal{P}'$  of a compact, convex set  $\mathcal{P}$  can be covered by a translate of  $\mathcal{RT}$ , then  $\mathcal{P}$  can also be covered by a translate of an arbitrary convex body of constant width  $\lambda$ . Here a congruent copy  $\mathcal{P}'$  of a set  $\mathcal{P}$  means that there exists a translation, or a rotation, or a product of translations and rotations mapping  $\mathcal{P}$  onto  $\mathcal{P}'$ . This property, known as Chakerian's theorem, was proved in [7]. Another proof was given later by Bezdek and Connelly; see [3]. The first result of Blaschke and Lebesgue was

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extended by Ohmann [21] and, independently, by Chakerian [6] to arbitrary normed planes; see also the survey [18, §2.8]. In this paper we present an extension of the second result to arbitrary normed planes, and we also give some corollaries of this generalization.

By a (normed or) *Minkowski plane*  $(\mathbb{X}, \|\cdot\|)$  we mean a two-dimensional real linear space  $\mathbb{X}$  with norm  $\|\cdot\|$ . As usual, the *unit disc*  $\mathfrak{D}$  and the *unit circle*  $\mathfrak{C}$  of  $(\mathbb{X}, \|\cdot\|)$  are defined by

$$\mathfrak{D} := \{x \in \mathbb{X} : \|x\| \leq 1\} \quad \text{and} \quad \mathfrak{C} := \{x \in \mathbb{X} : \|x\| = 1\},$$

where  $\mathfrak{D}$  is a two-dimensional convex body centered at the origin. If the unit circle is a strictly convex curve (i.e., it does not contain a non-degenerate line segment), the plane is called *strictly convex*, and if  $\mathfrak{C}$  is a smooth curve, the plane is said to be *smooth*. A homothetic copy  $p + \lambda \mathfrak{D}$ ,  $\lambda \in \mathbb{R}^+$ , of the unit disc is called the (*Minkowskian*) *disc* with *center*  $p$  and *radius*  $\lambda$  and denoted by  $\mathcal{D}(p, \lambda)$ . Analogously, the (*Minkowskian*) *circle*  $\mathcal{C}(p, \lambda)$  is defined by  $\mathcal{C}(p, \lambda) := p + \lambda \mathfrak{C}$ . If the points  $p_1, p_2$  lie on the circle  $\mathcal{C}(x, \lambda)$  and are not opposite in  $\mathcal{C}(x, \lambda)$ , then the *short arc* of  $\mathcal{C}(x, \lambda)$  with endpoints  $p_1$  and  $p_2$ , denoted by  $\text{arc}(p_1, p_2; x)$ , is that arc which belongs to the closed half-plane with bounding line through  $p_1, p_2$  which does not contain the center  $x$ . If  $p_1$  and  $p_2$  are opposite in  $\mathcal{C}(x, \lambda)$ , then by  $\text{arc}(p_1, p_2; x)$  we simply mean one of the two semi-circles of  $\mathcal{C}(x, \lambda)$  determined by  $p_1$  and  $p_2$ .

The abbreviation  $\text{conv}\mathcal{K}$  is used for the *convex hull* of a set  $\mathcal{K}$ , and  $\text{bd } \mathcal{K}$  for the *boundary* of  $\mathcal{K}$ . We denote the *segment* between  $x, y \in \mathbb{X}$  by  $[x, y]$ , the *line* through  $x$  and  $y$  by  $\langle x, y \rangle$ , and the *ray* with origin  $x$  and passing through  $y$  by  $[x, y)$ . For the closed half-plane bounded by the line  $\langle p_1, p_2 \rangle$  and containing the point  $q \notin \langle p_1, p_2 \rangle$  we write  $\mathcal{HP}(p_1, p_2; q)$ .

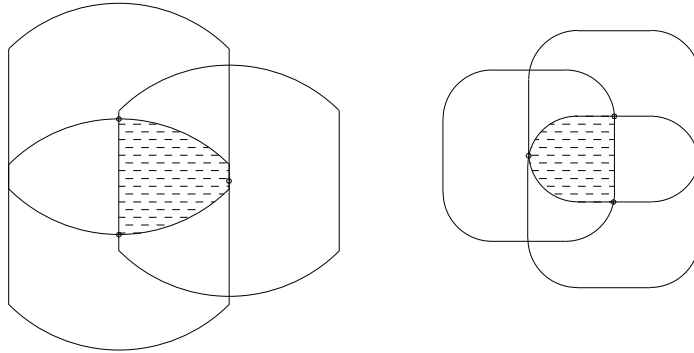


Figure 1: Reuleaux triangles in normed planes.

Let  $\mathcal{K} \subset \mathbb{X}$  be a compact, convex set and let the points  $p_1$  and  $p_2$  belong to the boundary of  $\mathcal{K}$ . The segment  $[p_1, p_2]$  is called an *affine diameter* of  $\mathcal{K}$  if there exist two different parallel supporting lines  $H_1$  and  $H_2$  of  $\mathcal{K}$  such that  $p_1 \in H_1$  and  $p_2 \in H_2$ . If all affine diameters of  $\mathcal{K}$  have the same Minkowskian length, then  $\mathcal{K}$  is said to be of *constant Minkowskian width*. A *Reuleaux triangle* in a normed plane is defined as the intersection of three discs  $\mathcal{D}(p_i, \lambda)$ ,  $i = 1, 2, 3$ , where  $p_i \in \mathcal{C}(p_j, \lambda) \cap \mathcal{C}(p_k, \lambda)$  for  $\{i, j, k\} = \{1, 2, 3\}$ ; see Figure 1. The so-defined Reuleaux triangle is a body of constant width  $\lambda$  (see [6], [16] and [21]), and we denote it by  $\mathcal{RT}\{p_1, p_2, p_3; \lambda\}$ . It should be noticed that there exist also other ways to define a Reuleaux triangle for normed planes; see, e.g., [18, §2.4].

The *Minkowskian diameter*  $\text{diam } \mathcal{K}$  of a set  $\mathcal{K}$  is defined by

$$\text{diam } \mathcal{K} := \sup \{\|x - y\| : x, y \in \mathcal{K}\}.$$

If  $\mathcal{K}$  is a compact, convex set and  $p_1, p_2 \in \text{bd } \mathcal{K}$  are such that  $\|p_1 - p_2\| = \text{diam } \mathcal{K}$ , then the segment  $[p_1 p_2]$  is called a *diametrical chord* of  $\mathcal{K}$ . Any diametrical chord of  $\mathcal{K}$  is also an affine diameter, but not vice versa; see [2, Theorem 2, IV].

## 2. PRELIMINARIES FROM MINKOWSKI GEOMETRY

In this section we give some results from the geometry of normed planes which are necessary for our considerations. The first statement is known as the *Monotonicity Lemma*; see, e.g., [10] and [19, Proposition 31].

**Proposition 2.1** (Monotonicity Lemma). *Let  $p, q, r$  be different points in a normed plane  $(\mathbb{X}, \|\cdot\|)$  belonging to a circle  $\mathcal{C}(x, \lambda)$  such that the center  $x$  does not belong to the open half-plane bounded by the line  $\langle p, q \rangle$  and containing  $r$ . Then*

$$\|p - q\| \geq \|p - r\|.$$

**Proposition 2.2** ([19, Lemma 13]). *In a normed plane  $(\mathbb{X}, \|\cdot\|)$ , any two circles  $\mathcal{C}(p_1, \lambda)$  and  $\mathcal{C}(p_2, \lambda)$  with  $\|p_1 - p_2\| \leq 2\lambda$  have non-empty intersection.*

*Remark 2.3:* It is easy to check that not all points of the intersection described by Proposition 2.2 lie in the same open half-plane bounded by the line  $\langle p_1, p_2 \rangle$ .

**Proposition 2.4** ([19, Lemma 5]). *Let  $p, q, r$  be three collinear points in a normed plane  $(\mathbb{X}, \|\cdot\|)$  such that  $r$  lies strictly between  $p$  and  $q$ . Then, for any point  $x \in \mathbb{X}$ ,*

$$\|x - r\| \leq \max\{\|x - p\|, \|x - q\|\}.$$

**Lemma 2.5.** *Let  $\mathcal{RT}(p_1, p_2, p_3; \lambda)$  be a Reuleaux triangle in a normed plane. If  $x \in \mathcal{RT}(p_1, p_2, p_3; \lambda)$ , then*

$$\mathcal{RT}(p_1, p_2, p_3; \lambda) \subset \mathcal{D}(x, \lambda).$$

*Proof.* Let  $y \in \mathcal{RT}(p_1, p_2, p_3; \lambda)$ . Since  $\text{diam } \mathcal{RT}(p_1, p_2, p_3; \lambda) = \lambda$ , we have

$$\|x - y\| \leq \lambda \iff y \in \mathcal{D}(x, \lambda).$$

□

**Lemma 2.6.** *In a normed plane  $(\mathbb{X}, \|\cdot\|)$ , let there be given three discs  $\mathcal{D}(x_i, \lambda)$ ,  $i = 1, 2, 3$ , such that  $x_i, x_j \in \mathcal{D}(x_k, \lambda)$  for  $\{i, j, k\} = \{1, 2, 3\}$ . Then  $\bigcap_{i=1}^3 \mathcal{D}(x_i, \lambda)$  contains a Reuleaux triangle of width  $\lambda$ .*

*Proof.* Let  $\|x_1 - x_2\| = \max\{\|x_1 - x_2\|, \|x_2 - x_3\|, \|x_3 - x_1\|\}$  and  $x'_2 \in [x_1, x_2]$  such that  $\|x_1 - x'_2\| = \lambda$ . The intersection of the circles  $\mathcal{C}(x_1, \lambda)$  and  $\mathcal{C}(x'_2, \lambda)$  is not empty, and not all points of this intersection lie in the same half-plane with respect to the line  $\langle x_1, x'_2 \rangle$ ; see Proposition 2.2 and Remark 2.3. Let

$$x'_3 \in \mathcal{C}(x_1, \lambda) \cap \mathcal{C}(x'_2, \lambda) \cap \mathcal{HP}(x_1, x'_2; x_3);$$

see Figure 2. If we prove that  $x_3 \in \mathcal{RT}(x_1, x'_2, x'_3; \lambda)$ , then the statement of the lemma follows from Lemma 2.5. If  $\|x_1 - x_2\| = \mu$ , then

$$x_3 \in \mathcal{D}(x_1, \mu) \cap \mathcal{D}(x_2, \mu) \cap \mathcal{HP}(x_1, x_2; x'_3).$$

Consider the point  $x''_3$  on  $[x_1, x'_3]$  such that  $\|x_1 - x''_3\| = \mu$ . Then  $x''_3 \in \mathcal{C}(x_1, \mu) \cap \mathcal{C}(x_2, \mu)$ , by Thales' Theorem; see again Figure 2. Therefore

$$x_3 \in \text{conv}\{x_1, x_2, x''_3\} \cup \text{conv arc}\{x_2, x''_3; x_1\} \cup \text{conv arc}\{x''_3, x_1; x_2\}.$$

If  $x_3 \in \text{conv}\{x_1, x_2, x''_3\}$ , then  $x_3 \in \mathcal{RT}(x_1, x'_2, x'_3; \lambda)$ . Let now

$$x_3 \in \text{conv arc}\{x_2, x''_3; x_1\} \cup \text{conv arc}\{x''_3, x_1; x_2\}.$$

We will prove that

$$(2.1) \quad \text{conv arc}\{x_2, x''_3; x_1\} \cup \text{conv arc}\{x''_3, x_1; x_2\} \subset \mathcal{D}(x_1, \lambda) \cap \mathcal{D}(x'_2, \lambda).$$

We omit the case  $\lambda = \mu$ , which is obvious. Consider the homothety  $\varphi$  mapping the circle  $\mathcal{C}(x_1, \mu)$  onto the circle  $\mathcal{C}(x_1, \lambda)$ . Clearly, if  $x$  is an arbitrary point of  $\text{arc}\{x_2, x''_3; x_1\}$ , then  $x' = \varphi(x)$  is a point belonging to  $\text{arc}\{x'_2, x'_3; x_1\}$ . Moreover,  $x$  is strictly between  $x_1$  and  $x'$ , i.e.,  $x \in \mathcal{D}(x_1, \lambda)$ . On the other hand, Propositions 2.1 and 2.4 imply

$$\|x'_2 - x\| \leq \max\{\|x'_2 - x_1\|, \|x'_2 - x'\|\} = \lambda,$$

which means that  $x \in \mathcal{D}(x'_2, \lambda)$ . Thus we have proved that

$$\text{conv arc}\{x_2, x''_3; x_1\} \subset \mathcal{D}(x_1, \lambda) \cap \mathcal{D}(x'_2, \lambda).$$

In order to prove that  $\text{arc}\{x''_3, x_1; x_2\}$  also belongs to  $\mathcal{D}(x_1, \lambda) \cap \mathcal{D}(x'_2, \lambda)$ , we consider the homothety  $\psi$  mapping  $\mathcal{C}(x_2, \mu)$  onto  $\mathcal{C}(x'_2, \lambda)$ . It is easy to check that the center of  $\psi$  is the point

$$(2.2) \quad s = \frac{\lambda}{\lambda - \mu} x_2 - \frac{\mu}{\lambda - \mu} x'_2.$$

Since  $\frac{\lambda}{\lambda - \mu} > 1$ , the point  $s$  lies on the opposite ray of  $[x_2, x'_2]$ . By (2.2) we get

$$\|s - x_2\| = \frac{\mu}{\lambda - \mu} \|x_2 - x'_2\| = \mu,$$

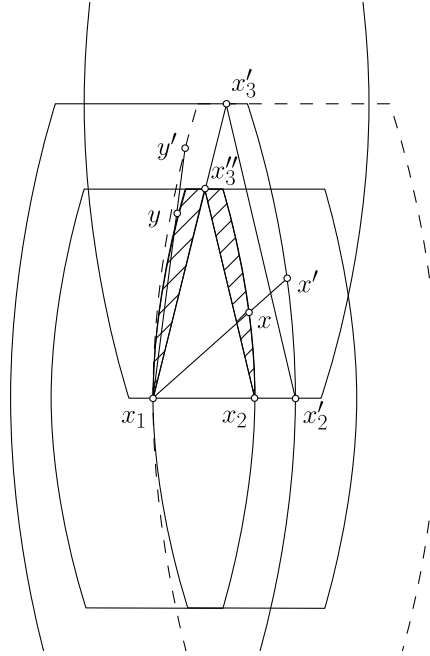


Figure 2: Proof of Lemma 2.6.

i.e.,  $s \equiv x_1$ . If  $y \in \text{arc}\{x_3'', x_1; x_2\}$  and  $\psi(y) = y'$ , then  $y' \in \mathcal{C}(x_2', \lambda)$  and  $y$  lies strictly between  $x_1$  and  $y'$ , yielding  $y \in \mathcal{D}(x_1, \lambda)$ . Besides this, Proposition 2.4 implies

$$\|x_2' - y\| \leq \max\{\|x_2' - x_1\|, \|x_2' - y'\|\} = \lambda.$$

Thus the inclusion (2.1) is proved and  $x_3 \in \mathcal{D}(x_1, \lambda) \cap \mathcal{D}(x_2', \lambda)$ . So it remains to show that if

$$x_3 \in \text{conv arc}\{x_2, x_3''; x_1\} \cup \text{conv arc}\{x_3'', x_1; x_2\},$$

then  $x_3 \in \mathcal{D}(x_3', \lambda)$ . If  $x_3 \in \text{conv arc}\{x_2, x_3''; x_1\}$  and  $[x_1, x_3] \cap C(x_1, \lambda) = \{x_3^*\}$ , then  $x_3$  is strictly between  $x_1$  and  $x_3^*$ . Thus, by Proposition 2.4 and the Monotonicity Lemma we have

$$\|x_3' - x_3\| \leq \max\{\|x_3' - x_1\|, \|x_3' - x_3^*\|\} = \lambda.$$

In the same way, we can prove that

$$\|x_3' - x_3\| \leq \lambda,$$

in the case  $x_3 \in \text{conv arc}\{x_3'', x_1; x_2\}$ . □

The next statement, due to Kelly [11] and Eggleston [9], gives a characterization of bodies of constant Minkowskian width.

**Theorem 2.7.** *Let  $\mathcal{K}$  be a convex body in a Minkowski plane. Then  $\mathcal{K}$  is of constant Minkowskian width  $\lambda$  if and only if  $\mathcal{K}$  coincides with the intersection of all Minkowskian discs of radius  $\lambda$ , whose centers are in  $\mathcal{K}$ .*

*Remark 2.8:* The property that a convex body  $\mathcal{K}$  coincides with the intersection of all discs (or balls in a space of dimension  $\geq 3$ ) of radius  $\text{diam } \mathcal{K}$  and centered at  $\mathcal{K}$  is known as the *circular (spherical) intersection property*. In the Euclidean space  $\mathbb{E}^n$ , the property of constant width and the spherical intersection property are equivalent; see [9]. But in an  $n$ -dimensional normed space they are only equivalent in the case  $n = 2$ , i.e., for  $n \geq 3$ , Theorem 2.7 is no longer true; see [18, §2.6]. For every  $n$ -dimensional Minkowski space we have the spherical intersection property of a convex body  $\mathcal{K}$  is equivalent to the fact that  $\mathcal{K}$  is complete (i.e., it does not have a proper subset of the same diameter); see again [9] and [18, Theorem 6].

**Proposition 2.9** ([19, Proposition 21]). *In a normed plane, let there be given a circle  $\mathcal{C}(x, \lambda)$  and  $\mathcal{C}(x', \lambda')$  be its homothetic copy with respect to a positive homothety  $\varphi$ . Then  $\mathcal{C}(x, \lambda) \cap \mathcal{C}(x', \lambda')$  is a union of two segments  $[p_1, p'_1]$  and  $[p_2, p'_2]$ , each of which may be a point or empty. Let both these segments be non-empty,  $q_i \in [p_i, p'_i]$  and  $r_i = \varphi^{-1}(q_i)$ ,  $r'_i = \varphi(q_i)$ , for  $i = 1, 2$ . If  $\gamma_1$  ( $\gamma_2$ ) is that arc of  $\mathcal{C}(x, \lambda)$  with endpoints  $q_1$  and  $q_2$  which lies on the same side (opposite side) of  $\langle q_1, q_2 \rangle$  as  $r_1$  and  $\gamma'_1$  ( $\gamma'_2$ ) is determined in the same way for  $\mathcal{C}(x', \lambda')$ , then*

$$\gamma_2 \subseteq \text{conv } \gamma'_1 \quad \text{and} \quad \gamma'_2 \subseteq \text{conv } \gamma_1.$$

*Remark 2.10:* Proposition 2.9 holds also if  $\mathcal{C}(x', \lambda')$  is a translate of  $\mathcal{C}(x, \lambda)$ ; see [19, Proposition 22].

**Lemma 2.11.** *In a normed plane  $(\mathbb{X}, \|\cdot\|)$ , let there be given a disc  $\mathcal{D}(x, \lambda_0)$  and two points  $p, q$  belonging to  $\mathcal{D}(x, \lambda_0)$ . Then every short arc of a circle with radius  $\lambda > \lambda_0$  and endpoints  $p$  and  $q$  also belongs to  $\mathcal{D}(x, \lambda_0)$ .*

*Proof.* By Proposition 2.1 we have that

$$\|p - q\| \leq 2\lambda_0.$$

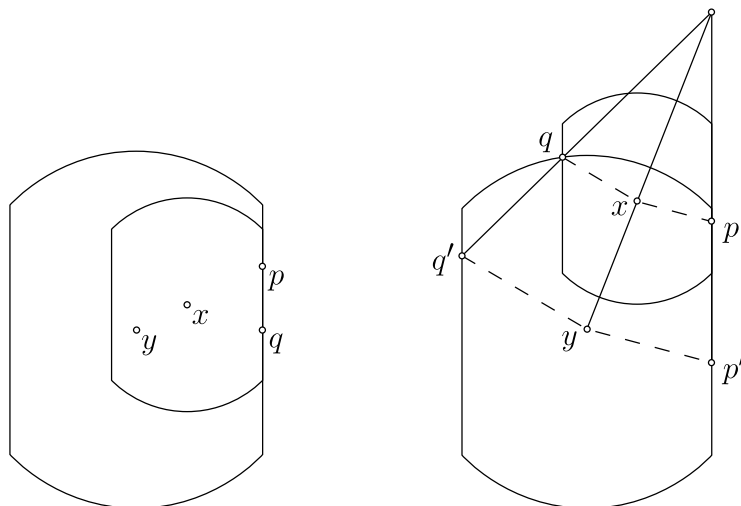
Thus Proposition 2.2 implies that for any  $\lambda > \lambda_0$  there exists a circle  $\mathcal{C}(y, \lambda)$  with  $y \neq x$  passing through  $p$  and  $q$ . Moreover, not all points of  $\mathcal{C}(y, \lambda)$  belong to  $\mathcal{D}(x, \lambda_0)$ . Indeed, if the opposite ray of  $[y, x)$  intersects  $\mathcal{C}(y, \lambda)$  in  $y_0$ , then

$$\|x - y_0\| = \|x - y\| + \lambda > \lambda_0.$$

Thus, by Jordan's curve theorem we have that

$$\mathcal{C}(x, \lambda_0) \cap \mathcal{C}(y, \lambda) \neq \emptyset.$$

According to Proposition 2.9 this intersection consists of two segments, each of them possibly degenerate or empty. If  $\mathcal{C}(x, \lambda_0) \cap \mathcal{C}(y, \lambda)$  consists of only one segment  $\mathcal{S}$  (see Figure 3(a)), then  $p$  and  $q$  cannot be interior with respect to  $\mathcal{C}(x, \lambda_0)$  and they have to belong to  $\mathcal{S}$ . Therefore the arc of  $\mathcal{C}(y, \lambda)$  with endpoints  $p$  and  $q$  belongs to  $\mathcal{D}(x, \lambda_0)$ . Let now  $\mathcal{C}(x, \lambda_0) \cap \mathcal{C}(y, \lambda)$  consist of two non-empty segments  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , possibly degenerate. If  $p, q \in \mathcal{S}_1$  or  $p, q \in \mathcal{S}_2$ , then the proof is done. Consider the case  $p \in \mathcal{S}_1$  and  $q \in \mathcal{S}_2$  (see Figure 3(b)). Let  $\varphi$  be the homothety mapping  $\mathcal{C}(x, \lambda_0)$  onto  $\mathcal{C}(y, \lambda)$



(a) The intersection consists of only one segment.

(b) The intersection consists of two segments  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , possibly degenerate, and  $p \in \mathcal{S}_1$  and  $q \in \mathcal{S}_2$ .

Figure 3

and  $\varphi(\{p, q\}) = \{p', q'\}$ . Assume that  $y$  lies in the half-plane with respect to  $\langle p, q \rangle$  which does not contain  $p'$  and  $q'$ . Then the Monotonicity Lemma implies

$$\|p - q\| \geq \|p - q'\| \geq \|p' - q'\|,$$

contradicting the fact that  $\lambda_0 < \lambda$ . Therefore the short arc of  $\mathcal{C}(y, \lambda)$  with endpoints  $p$  and  $q$  is that which does not contain  $p'$  and  $q'$ , and according to Proposition 2.9 it belongs to  $\mathcal{D}(x, \lambda)$ .

In order to complete the proof, it remains to consider the case that at least one of  $p$  and  $q$ , say  $p$ , is interior with respect to  $\mathcal{C}(x, \lambda_0)$ ; see Figure 4. Let  $\mathcal{S}_1 = [p_1, p_2]$  and  $\mathcal{S}_2 = [q_1, q_2]$ , such that  $p_2, p_1, p, q, q_1, q_2$  are successive on  $\mathcal{C}(y, \lambda)$ . Note that it is possible that  $p_2 \equiv p_1$ , or  $q \equiv q_1$ , or  $q_1 \equiv q_2$ , or  $q \equiv q_1 \equiv q_2$ . The inclusion  $\text{arc}(p, q; y) \subset \mathcal{D}(x, \lambda_0)$  can be proved as in the above case.  $\square$

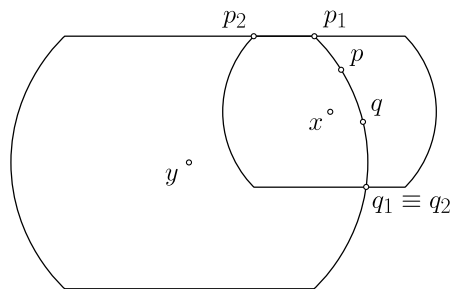


Figure 4: The point  $p$  is interior with respect to  $\mathcal{C}(x, \lambda_0)$ .

**Lemma 2.12.** *In a strictly convex normed plane  $(\mathbb{X}, \|\cdot\|)$ , let there be given a disc  $\mathcal{D}(x, \lambda_0)$  and two points  $p, q$  belonging to  $\mathcal{D}(x, \lambda_0)$ . Then every short arc of a circle with radius  $\lambda \geq \lambda_0$  and endpoints  $p$  and  $q$  also belongs to  $\mathcal{D}(x, \lambda_0)$ .*

*Proof.* In view of Lemma 2.11 we need to prove the statement only for  $\lambda = \lambda_0$ . We omit the trivial case that  $p$  and  $q$  are opposite in  $\mathcal{C}(x, \lambda_0)$ . Let  $\mathcal{C}(y, \lambda_0)$  be a circle through  $p$  and  $q$ . Note that, except for  $\mathcal{C}(x, \lambda_0)$ , there exists exactly one circle of radius  $\lambda_0$  passing through  $p$  and  $q$ ; see [23, p. 104]. Since in a strictly convex normed plane two circles intersect in at most two points ([19, Proposition 14]), we have that  $\mathcal{C}(x, \lambda_0)$  and  $\mathcal{C}(y, \lambda_0)$  intersect in exactly two points  $p_1$  and  $q_1$ . Consider the points

$$p_2 = p_1 + (y - x) \quad \text{and} \quad q_2 = q_1 + (y - x)$$

on  $\mathcal{C}(y, \lambda_0)$ ; see Figure 5. Since  $y$  is the intersection point of the diagonals

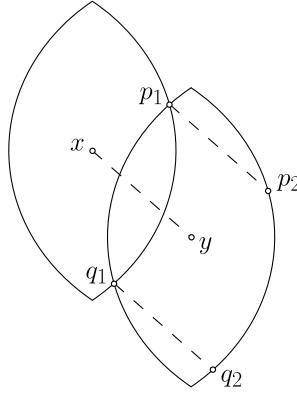


Figure 5: The intersection of two circles in a strictly convex normed plane.

of the parallelogram with vertices  $p_1, p_2, q_2, q_1$ , the short arc of  $\mathcal{C}(y, \lambda_0)$  with endpoints  $p_1$  and  $q_1$  does not contain the points  $p_2$  and  $q_2$ . Thus, by Remark 2.10 we get that this short arc belongs to  $\mathcal{D}(x, \lambda_0)$ .  $\square$

*Remark 2.13:* Figure 6 shows that Lemma 2.12 is not true in a normed plane which is not strictly convex.

*Remark 2.14:* In  $n$ -dimensional Euclidean space  $\mathbb{E}^n$ , Lemma 2.12 also holds for  $\lambda = \lambda_0$ . Moreover, if  $\mathcal{K}$  is a complete body of diameter  $\lambda_0$  in  $\mathbb{E}^n$ , then it contains every short circular arc of radius  $\lambda_0$  joining two of its points ([24, p. 373, Theorem 7.6.4]). By our Remark 2.8 and Lemmas 2.11 and 2.12, we obtain that any complete body of diameter  $\lambda_0$  in a normed plane contains every short circular arc of radius  $\lambda > \lambda_0$ . If, in addition, the plane is strictly convex then  $\mathcal{K}$  contains every short circular arc of radius  $\lambda \geq \lambda_0$ .



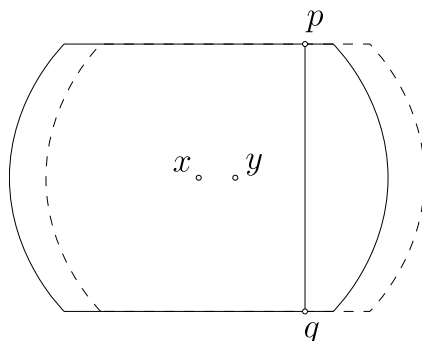


Figure 6: A counterexample of Lemma 2.12 in a non-strictly convex normed plane.

### 3. THE MAIN RESULT AND SOME COROLLARIES

In order to prove the main result, we also need the following generalization of Helly’s theorem; see, e.g., [12] and [7, Theorem 1].

**Lemma 3.1** (A generalization of Helly’s theorem). *Let  $\mathcal{P}$  be a fixed compact, convex set in the plane and  $\mathfrak{F}$  be a family of compact, convex sets having the property that each three or less members of  $\mathfrak{F}$  have a translate of  $\mathcal{P}$  in common. Then all the members of  $\mathfrak{F}$  have a translate of  $\mathcal{P}$  in common.*

If  $\psi$  is an isometry in a normed plane  $(\mathbb{X}, \|\cdot\|)$  preserving the orientation of  $\mathbb{X}$  and  $\mathcal{K}$  is a point set in  $\mathbb{X}$ , then  $\psi(\mathcal{K})$  is called a *congruent copy* of  $\mathcal{K}$ . It should be noticed that the only maps of  $\mathbb{X}$  that are isometries with respect to all norms are translations, reflections with respect to a point, and the identity map; see [1] and [17]. But there exist normed planes (e.g., the Euclidean plane), where the group of isometries is richer.

**Theorem 3.2.** *In a Minkowski plane, let there be given a compact, convex set  $\mathcal{P}$  such that every congruent copy of  $\mathcal{P}$  can be covered by a translate of any Reuleaux triangle of Minkowskian width  $\lambda$ . Then each congruent copy of  $\mathcal{P}$  can be covered by a translate of any convex body of constant Minkowskian width  $\lambda$ .*

*Proof.* Let  $\mathcal{K}$  be an arbitrary convex body of constant Minkowskian width  $\lambda$  and  $x_1, x_2, x_3$  be three arbitrary points of  $\mathcal{K}$ . Then  $\bigcap_{i=1}^3 \mathcal{D}(x_i, \lambda)$  contains a Reuleaux triangle  $\mathcal{RT}$  of width  $\lambda$ , see Lemma 2.6. By the assumption of the theorem there exists a translate  $\mathcal{P}'$  of any congruent copy of  $\mathcal{P}$  such that  $\mathcal{P}' \subseteq \mathcal{RT}$ . Therefore, by Lemma 3.1, all discs of radius  $\lambda$  centered at  $\mathcal{K}$  have a translate of  $\mathcal{P}$  in common. Since  $\mathcal{K}$  is of constant Minkowskian width, i.e.,  $\mathcal{K} = \bigcap_{x \in \mathcal{K}} \mathcal{D}(x, \lambda)$  by Theorem 2.7, we conclude that  $\mathcal{K}$  contains a translate of any congruent copy of  $\mathcal{P}$ , and the proof is done.  $\square$

The next statement can be obtained as an elementary corollary of Theorem 3.2.

**Corollary 3.3.** *In a Minkowski plane, let there be given a finite point set  $\mathcal{P}$  such that every congruent copy of  $\mathcal{P}$  can be covered by a translate of any Reuleaux triangle of Minkowskian width  $\lambda$ . Then any congruent copy of the convex hull of  $\mathcal{P}$  can be covered by a translate of any convex body of constant Minkowskian width  $\lambda$ .*

Under the same assumptions as in Corollary 3.3 we are even able to state that something more than  $\text{conv } \mathcal{P}$  can be covered by a translate of  $\mathcal{K}$ . In order to determine this “something more,” we need some preliminaries.

In a normed plane  $(\mathbb{X}, \|\cdot\|)$ , let there be given two different points  $p$  and  $q$ . For each  $\lambda \geq \frac{\|p-q\|}{2}$  there exist at least two circles of radius  $\lambda$  passing through  $p$  and  $q$ ; see Proposition 2.2 and Remark 2.3. The union of all short arcs of the circles through  $p$  and  $q$  with radii at least  $\lambda$  is called the  $\lambda$ -spindle of  $p$  and  $q$ , and it is denoted by  $\text{sp}_\lambda[p, q]$ . The union of all short arcs of the circles through  $p$  and  $q$  with radii strictly larger than  $\lambda$  is called the *open  $\lambda$ -spindle* of  $p$  and  $q$ . We will use the notation  $\text{sp}_\lambda(p, q)$ . A set  $\mathcal{P} \subset \mathbb{X}$  is  $\lambda$ -spindle convex if for any  $p, q \in \mathcal{P}$  we have  $\text{sp}_\lambda[p, q] \subseteq \mathcal{P}$ . We define the  $\lambda$ -spindle convex hull of  $\mathcal{P}$ , denoted by  $\text{sp conv}_\lambda[\mathcal{P}]$ , as the intersection of all  $\lambda$ -spindle convex sets which contain  $\mathcal{P}$ . If for any  $p, q \in \mathcal{P}$  the set  $\mathcal{P}$  also contains the open  $\lambda$ -spindle of  $p$  and  $q$ , we say that  $\mathcal{P}$  is *open  $\lambda$ -spindle convex*. The *open  $\lambda$ -spindle convex hull*  $\text{sp conv}_\lambda(\mathcal{P})$  of a set  $\mathcal{P}$  is defined as the intersection of all open  $\lambda$ -spindle convex sets containing  $\mathcal{P}$ . The notion of spindle convexity in a normed plane, being in addition strictly convex and smooth, appeared in [20], called there “*Überkonvexität*.” Our definition refers to all normed planes, but it is not our aim here to study spindle convexity in an arbitrary normed plane. We give only one property (see Proposition 3.4) related to this notion in order to prove that under the assumptions of Corollary 3.3 not only the convex hull of  $\mathcal{P}$  can be covered by a translate of  $\mathcal{K}$ , but also its  $\lambda$ -spindle convex hull. Note also that the convex hull of any set of diameter  $2\lambda$  is contained in the  $\lambda$ -spindle convex hull of this set. The notion of spindle convexity in Euclidean space is also investigated in [4], [14], and [22].

**Proposition 3.4.** *Every disc of radius  $\lambda$  in a normed plane is open  $\lambda$ -spindle convex. If the plane is strictly convex, then it is  $\lambda$ -spindle convex.*

*Proof.* This proposition follows immediately from Lemmas 2.11 and 2.12.  $\square$

The conditions in the next corollary are the same as in Corollary 3.3, but the conclusion is stronger.

**Corollary 3.5.** *In a Minkowski plane, let there be given a finite point set  $\mathcal{P}$  such that every congruent copy of  $\mathcal{P}$  can be covered by a translate of any Reuleaux triangle of Minkowskian width  $\lambda$ . Then any congruent copy of the open  $\lambda$ -spindle convex hull of  $\mathcal{P}$  can be covered by a translate of any convex body  $\mathcal{K}$  of constant Minkowskian width  $\lambda$ . If the plane is strictly convex, then any congruent copy of the  $\lambda$ -spindle convex hull of  $\mathcal{P}$  can be covered by a translate of  $\mathcal{K}$ .*

*Proof.* We begin by assuming that  $\mathcal{P}$  can be covered by the Reuleaux triangle  $\mathcal{RT}(p_1, p_2, p_3; \lambda)$ , i.e.,

$$\mathcal{P} \subseteq \bigcap_{i=1}^3 \mathcal{D}(p_i, \lambda).$$

Since every disc of radius  $\lambda$  is open  $\lambda$ -spindle convex (and  $\lambda$ -spindle convex if the plane is strictly convex), we have

$$\text{sp conv}_\lambda(\mathcal{P}) \subseteq \mathcal{D}(p_i, \lambda) \quad \left( \text{sp conv}_\lambda[\mathcal{P}] \subseteq \mathcal{D}(p_i, \lambda) \right)$$

for each  $i = 1, 2, 3$ . Thus the desired statement follows from Theorem 3.2.  $\square$

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