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THE COPS AND ROBBER GAME ON GRAPHS WITH FORBIDDEN (INDUCED) SUBGRAPHS

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Abstract. The two-player, complete information game of Cops and Robber is played on undirected finite graphs. A number of cops and one robber are positioned on vertices and take turns in sliding along edges. The cops win if, after a move, a cop and the robber are on the same vertex. The minimum number of cops needed to catch the robber on a graph is called the cop number of that graph.

In this paper, we study the cop number in the classes of graphs defined by forbidding one or more graphs as either subgraphs or induced subgraphs. In the case of a single forbidden graph we completely characterize (for both relations) the graphs which force bounded cop number.

In closing, we bound the cop number in terms of the tree-width.

1. Introduction

Graphs studied in this paper are finite, undirected, without loops and multiple edges. We use standard notation and terminology; for what is not defined here, we refer the reader to Diestel [7].

The game of *Cops and Robber* is played on a connected graph by two players: the cops and the robber. The cop player has at her disposal k pieces (cops), for some integer $k \geq 1$, and the robber player has only one piece (the robber). The game begins with the cop player placing her k cops on (not necessarily distinct) vertices of the graph. Next, the robber player chooses a vertex for his piece. Now, starting with the cop player, the two players move their pieces alternately. In the cops' move, she decides for each of her cops whether it stands still or is moved to an adjacent vertex. In the robber's move, he can choose to move or not to move the piece. The game ends when a cop and the robber are on the same vertex (that is, the cops

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catch the robber); in this case the cop player wins. The robber wins if he can never be caught by the cops. Both players have complete information, that is, they know the graph and the positions of all the pieces.

The key problem in this game is to know how many cops are needed to catch a robber on a given graph. For a connected graph G, the smallest integer k such that with k cops, the cop player has a winning strategy is called the *cop number* of G and is denoted by $\mathrm{cop}(G)$. We follow Berarducci and Intrigila [4] in defining the cop number of a non-connected graph as the maximum cop number of its connected components. Nowakowski and Winkler [14] and Quilliot [15, 16] characterized the class of graphs with cop number 1. Finding a combinatorial characterization of graphs with cop number k (for $k \geq 2$) is a major open problem in the field. On the other hand, algorithmic characterizations of such graphs, which are polynomial in the size of the graph but not in k, do exist [4, 11, 13]. However, determining the cop number of a graph is a computationally hard problem [8]. For literature review we refer the reader to a recent survey on graph searching [9] (see also [12]).

In this paper we study the cop number for different types of graph classes. Our motivation is to learn what structural properties of graphs force the cop number to be bounded. (We say that the cop number is *bounded* for a class of graphs, if there exists a constant C such that the cop number of every graph from the class is at most C; otherwise the cop number is *unbounded* for this class.) We consider several containment relations and study the cop number for classes of graphs with a single forbidden graph with respect to these relations.

Families of graphs with unbounded cop number have been constructed [1]. For every fixed $d \geq 3$, there even exist families of d-regular graphs with unbounded cop number [2]. On the other hand, Aigner and Fromme [1] proved that the cop number of a planar graph is at most 3. This result has been generalized to the class of graphs with genus g; Schroeder [18] proved that the cop number of a graph is bounded by $\lfloor 3g/2 \rfloor + 3$ (improving an earlier bound of Quilliot [17]), and conjectured that this bound can be reduced to $g + 3$.

A graph is called *H-minor-free* (*H-topological minor-free*) if it does not contain H as a minor (as a topological minor). Andreae [3] studied classes of H-minor-free graphs and showed that the cop number of a K_5-minor-free graph (or $K_{3,3}$-minor-free graph) is at most 3. Since a planar graph does not have a K_5 or $K_{3,3}$ as a minor this result extends the result on planar graphs. However, for our purposes the most interesting result of Andreae [3] is that for any graph H the cop number is bounded in the class of H-minor-free graphs. In other words, forbidding a minor is enough to bound the cop number.

Andreae [3] also observed that excluding a topological minor does not necessarily bound the cop number. In fact, it is an easy corollary of his work that the class of H-topological minor-free graphs has bounded cop number if and only if the maximum degree of H is at most 3.

Inspired by these results we study other containment relations: subgraphs and induced subgraphs. A graph is called *H-subgraph-free* (*H-free*) if it does not contain H as a subgraph (as an induced subgraph). We give necessary and sufficient conditions for the class of H-subgraph-free graphs and H-free graphs to have bounded cop number. First we present our results for induced subgraphs.

Theorem 1.1. *The class of H-free graphs has bounded cop number if and only if every connected component of H is a path.*

Let us remark that a single vertex is considered to be a path. The graph consisting of a path on ℓ ($\ell \geq 1$) vertices is denoted by P_ℓ. The backward implication of Theorem 1.1 is a consequence of the following proposition.

Proposition 1.2. *For every $\ell \geq 3$, every P_ℓ-free graph has cop number at most $\ell - 2$.*

Using the same technique, it is in fact possible to show the following stronger result.

Proposition 1.3. *For every $\ell \geq 3$, every graph with no induced cycle of length at least ℓ has cop number at most $\ell - 2$.*

Notice that it is possible to rephrase the condition of Theorem 1.1 and say that every connected component of H is a tree with at most two leaves. Here is our result for H-subgraph-free graphs.

Theorem 1.4. *The class of H-subgraph-free graphs has bounded cop number if and only if every connected component of H is a tree with at most three leaves.*

It is easy to see that the cop number of a tree is 1. As an intermediate step towards Theorem 1.4, we study how the cop number of a graph G is related to its tree-width, which is denoted by $\mathrm{tw}(G)$.

Proposition 1.5. *The cop number of a graph G is at most* $\mathrm{tw}(G)/2 + 1$.

This bound is sharp for tree-width up to 5. (This is easy to prove for $\mathrm{tw}(G) \leq 3$; for $\mathrm{tw}(G) = 4$ and 5, the Petersen graph and the disjoint union of the Petersen graph and K_6 plus an edge linking them are tight examples, respectively.)

2. Forbidding induced subgraphs

Our goal in this section is to prove Theorem 1.1. Notice that a graph whose every connected component is a path, is an induced subgraph of some sufficiently long path. Hence, the following proposition proves the backward implication of Theorem 1.1.

Proposition 1.2. *For every $\ell \geq 3$, every P_ℓ-free graph has cop number at most $\ell - 2$.*

Let us remark that, for $\ell = 1, 2$, the cop number of a P_ℓ-free graph is trivially 1.

Proof of Proposition 1.2. Let G be a P_ℓ-free graph and let us also assume, without loss of generality, that G is connected. We will give a winning strategy for $\ell - 2$ cops. Initially all $\ell - 2$ cops are on the same arbitrary vertex. The strategy is divided into stages. The distance between the cops and the robber is the minimum distance from the robber to a cop. The goal of each stage is to decrease the distance between the cops and the robber. Once the distance is decreased we begin the next stage. We will show that a stage lasts a finite number of rounds.

At the beginning of each stage we choose a *lead cop* (for this stage) among the pieces which are at the minimum distance from the robber. All distances in this proof are measured after the robber's and before the cops' move. We route the lead cop and instruct the other pieces to follow the lead cop in single file; the cops should form a path of length $\ell - 2$.

If the distance between the cops and the robber is at most one, then the cops clearly win. Suppose that the distance between the lead cop on vertex x and the robber on vertex y is $d \geq 2$. We order the lead cop to travel along the shortest path from x to y and then follow the exact route the robber took from vertex y. Notice that since the graph is P_ℓ-free the distance between the cops and the robber will decrease after at most $\ell - d - 1$ moves. Once the distance decreased, we move to the next stage. □

We mention the following result which can be derived using almost the same strategy as in Proposition 1.2.

Proposition 1.3. *For every $\ell \geq 3$, every graph with no induced cycle of length at least ℓ has cop number at most $\ell - 2$.*

Before completing the proof of Theorem 1.1, we look at bipartite graphs with no long induced paths. A simple modification of the proof of Proposition 1.2 yields a better bound for the bipartite case. Here is how the cops' strategy needs to be modified: the cops follow the lead cop in such a way that the distance between any two consecutive cops is 2. We leave the details of this proof to the reader.

Proposition 2.1. *For every $\ell \geq 1$, every $P_{2\ell}$-free bipartite graph has cop number at most ℓ.*

To prove the forward implication of Theorem 1.1, we need to introduce two graph operations which do not decrease the cop number: clique substitution and edge subdivision. Let $N(v)$ be the the set of neighbors of a vertex v. A *clique substitution* at a vertex v consists in replacing v with a clique of size $|N(v)|$ and creating a matching between vertices of the clique and the vertices of $N(v)$. The graph obtained from a graph G by substituting a clique at each vertex of G will be denoted by G^+. More formally, the vertex set of G^+ is $\bigcup_v(\{v\} \times N(v))$ and two vertices (v_1, u_1) and (v_2, u_2) are adjacent if and only if $v_1 = v_2$, or $v_1 = u_2$ and $u_1 = v_2$.

Lemma 2.2. *Clique substitution does not decrease the cop number.*

Proof. Let G be a graph. To each vertex $v \in V(G)$ there corresponds a clique in G^+, which we denote by $\phi(v)$. We simultaneously play two games: one on G and another on G^+. We assume that we have a winning strategy for the cop player on G^+. We use the same number of cops in G as in G^+.

At the beginning, the cops are placed on G^+ according to the strategy, and the corresponding cops in G are placed in the obvious way: If a cop in G^+ is on a vertex of the clique $\phi(v)$ for some vertex $v \in V(G)$, then the corresponding cop in G is put on vertex v. Then, we put the robber in G^+ on an arbitrary vertex of the clique $\phi(v)$, where v is the vertex on which the robber is in G. For simplicity of presentation, we do not move the cops at all in G during the first turn. Thus, the robber will move first.

Now, let us consider a robber's move in G, say from vertex u to vertex v. In G^+, the robber is on some vertex of $\phi(u)$. If $u = v$, we do not move the robber in G^+. Next, suppose $u \neq v$, and let $u'v'$ be the (unique) edge between $\phi(u)$ and $\phi(v)$, with $u' \in \phi(u)$ and $v' \in \phi(v)$. If in G^+, the robber is on u', we move it to v'. Otherwise, the robber is on another vertex of $\phi(u)$, and we move it first to u', then let the cops react to that move, and finally move the robber to v' (unless it has been caught). Once the robber is in its final position, we let the cops move in G^+. We refer to this sequence of 1 or 2 turns in G^+ as a *stage*.

Once a stage is finished in G^+, we translate the moves of the cops back to the graph G: Consider a cop in G^+. Let u and v be the vertices of G such that the cop was in the clique $\phi(u)$ at the end of the previous stage and in the clique $\phi(v)$ at the end of the current stage. Observe that, either $u = v$, or $uv \in E(G)$. We move the corresponding cop in G from u to v (or let it stay on u if $u = v$).

This describes our strategy for the cops in G. By our assumption, the robber will be caught during some stage in G^+. At the end of that stage, both a cop and the robber on the clique $\phi(v)$ for some vertex $v \in V(G)$. Hence, when the moves of cops from that stage are translated back to G, the corresponding cop in G will be on the same vertex as the robber. □

The *claw* is the complete bipartite graph with sides of size 1 and 3. The operation of clique substitution will be used to show that the cop number of claw-free graphs is unbounded.

Lemma 2.3. *The class of claw-free graphs has unbounded cop number.*

Proof. Let $\mathcal{G}$ be a class of graphs with unbounded cop number and $\mathcal{G}^+ := \{G^+ \mid G \in \mathcal{G}\}$. Notice that all graphs in $\mathcal{G}^+$ are claw-free. Applying Lemma 2.2, we see that the cop number of graphs in $\mathcal{G}^+$ is unbounded. □

The other graph operation needed for the proof of Theorem 1.1 is edge subdivision. Berarducci and Intrigila [4] proved the following lemma.

Lemma 2.4 ([4]). *Subdividing all edges of a graph an even number of times does not decrease the cop number.*

This leads to the following result. Recall that the *girth* of a graph is the length of its shortest cycle if it has one, $+\infty$ otherwise.

Lemma 2.5. *For every integer $\ell \geq 3$, the class of graphs with girth at least ℓ has unbounded cop number.*

Proof. Let $\mathcal{G}$ be an arbitrary class of graphs with unbounded cop number. For every $G \in \mathcal{G}$, let G' be a graph with girth at least ℓ obtained from G by subdividing all edges sufficiently often. Let $\mathcal{G}' := \{G' \mid G \in \mathcal{G}\}$. Applying Lemma 2.4, we see that the class $\mathcal{G}'$ has unbounded cop number. □

Now we are ready to complete the proof of Theorem 1.1.

Theorem 1.1. *The class of H-free graphs has bounded cop number if and only if every connected component of H is a path.*

Proof. The backward implication of the theorem follows from Proposition 1.2. Indeed, notice that if every connected component of H is a path, then H is a subgraph of the path on $|H| + p - 1$ vertices, where p is the number of connected components of H. Hence, the cop number of an H-free graph is bounded by $\max\{|H| + p - 3, 1\}$.

Now we will prove the forward implication of the theorem. Let H be a graph such that the class of H-free graphs has bounded cop number. Suppose that H contains a cycle and let ℓ be the length of the longest cycle of H. Clearly, the class of graphs with no induced cycle of length at most ℓ is contained in the class of H-free graphs. However, by Lemma 2.5 the class of graphs with no induced cycle of length at most ℓ has unbounded cop number; a contradiction. Hence, H is a forest.

Now suppose that H contains a vertex of degree at least 3. Since H is a forest, it must contain a claw as an induced subgraph. Clearly, the class of claw-free graphs is contained in the class of H-free graphs. However, by Lemma 2.3 the class of claw-free graphs has unbounded cop number; a contradiction. Hence, H is a forest of maximum degree at most 2, that is, H is a disjoint union of paths. □

We note that in the second part of the proof (removing cycles) we could have used some known constructions which show that graphs simultaneously having an arbitrarily large girth and large cop number do exist; see for instance Andreae [2] and Frankl [10].

Some remarks about edge subdivisions. Lemma 2.4 by Berarducci and Intrigila [4] gives a bound on the cop number of graphs which result by uniformly subdividing all edges an even number of times. By modifying the proof of Lemma 2.2, the following can be shown.

Lemma 2.6. *Subdividing all edges of a graph once does not decrease the cop number.*

Combining Lemmas 2.4 and 2.6 we obtain the general result.

Corollary 2.7. *For every positive integer r, subdividing every edge of a graph r times does not decrease the cop number.*

Proof. Let G be a graph. The proof is by induction on r. The base case $r = 1$ is given by Lemma 2.6. For the inductive case, assume $r \geq 2$. If r is even, then the claim follows from Lemma 2.4. If r is odd, then by induction subdividing each edge of G $(r-1)/2$ times does not decrease the cop number. Subdividing once every edge of the resulting graph we obtain a subdivision of G where each edge has been subdivided $(r-1)/2 + ((r-1)/2 + 1) = r$ times, and its cop number is at least that of G by Lemma 2.6. □

Berarducci and Intrigila [4] noted that subdividing edges in a non-uniform manner can both increase and decrease the cop number. However, for uniform subdivisions it is possible to give an estimate.

Proposition 2.8. *Subdividing each edge r times increases the cop number by at most one.*

Sketch of proof. Denote by $\widetilde{G}$ the graph which results from the graph G by subdividing each edge r times. A winning strategy for $\text{cop}(G) + 1$ cops on $\widetilde{G}$ is the following. Let an auxiliary cop pursue the strategy described for the lead cop in the proof of Proposition 1.2. By this we make sure that the robber cannot change his direction or pass in the middle of a subdivided edge except for a finite number of times. The other $\text{cop}(G)$ cops simulate their winning strategy for G on $\widetilde{G}$. □

To further enlighten what happens if edges are subdivided, we propose the following construction. Let G be an arbitrary graph with n vertices and cop number at least 2. We construct a graph $\widehat{G}$ by adding paths of length n to G: every pair of non-adjacent vertices of G is joined by such a path. It is not difficult to see that $\text{cop}(\widehat{G}) = \text{cop}(G)$. But by subdividing edges of $\widehat{G}$, we can obtain a graph resulting from K_n by subdividing every edge n times. From Proposition 2.8 we know that the cop number of this graph is at most 2.

Considering this construction, it seems natural to propose the following conjecture, which implies the conjecture of Meyniel (see Frankl [10]) that $\text{cop}(G)$ is in $O(\sqrt{|G|})$.

Conjecture. *For graphs G obtained by subdividing edges of complete graphs K_n we have $\text{cop}(G)$ in $O(\sqrt{n})$.*

3. Forbidding (not necessarily induced) subgraphs

We now turn our attention to classes of graphs for which we forbid (not necessarily induced) subgraphs. One key ingredient for the proof of Theorem 1.4 is the fact that families of graphs with bounded circumference have bounded cop number. Although this already follows from Proposition 1.3, in this section we give a better upper bound based on an estimate on the cop number in terms of the tree-width, which we believe to be of interest in its own.

Let us first briefly recall the definition of the tree-width of a graph. A *tree decomposition* of a graph G is a pair $(T, \{W_x \mid x \in V(T)\})$ where T is a tree, and $\{W_x \mid x \in V(T)\}$ a family of subsets of $V(G)$ (called "bags") such that

- $\bigcup_{x \in V(T)} W_x = V(G)$;
- for every edge $uv \in E(G)$, there exists $x \in V(T)$ with $u, v \in W_x$, and
- for every vertex $u \in V(G)$, the set $\{x \in V(T) \mid u \in W_x\}$ induces a subtree of T.

The *width* of tree decomposition $(T, \{W_x \mid x \in V(T)\})$ is $\max\{|W_x| - 1 \mid x \in V(T)\}$. The *tree-width* $\mathrm{tw}(G)$ of G is the minimum width among all tree decompositions of G. We refer the reader to Diestel's book [7] for an introduction to the theory around tree-width.

Our proof of Proposition 1.5 relies on a well-known strategy for the cops and robber game: guarding a shortest path. Assume that P is a shortest uv-path, for two distinct vertices u, v of a graph G, and that a cop is sitting at the beginning on some vertex of P. The cop's strategy consists in moving along P in such a way that his distance to u is as close as possible to the robber's distance to u. It is easily seen that, after a finite number of initial moves, when it is the robber's turn to play, the cop's distance to u will be the same as the robber's distance to u when the latter is no more than $|P|$. This ensures that the robber cannot go on any vertex of P without being caught. (This strategy has been first used by Aigner and Fromme [1], in their proof that the cop number of planar graphs is at most 3.)

Proposition 1.5. *The cop number of a graph G is at most* $\mathrm{tw}(G)/2 + 1$.

Proof. Let us consider an optimal tree decomposition of G. Since the tree-width of G equals the maximum tree-width of its connected components, we may assume without loss of generality that G is connected. For a bag $X \subseteq V(G)$ of the tree decomposition, we denote by t_X the vertex of T corresponding to X.

At the beginning, an arbitrary bag $B \subseteq V(G)$ of the tree decomposition is selected, and all its vertices are guarded in the following way: Letting $b_1, b_2, \ldots, b_k$ denote the vertices in B, we let the ith cop $(1 \leq i \leq \lfloor k/2 \rfloor)$ guard a shortest $b_{2i-1}b_{2i}$-path in G, and, if k is odd, we put an additional cop on vertex b_k. This ensures that, after a finite number of moves, the robber cannot go on any vertex in B, and hence is confined to (the subgraph corresponding to) some tree T' of $T \setminus t_B$. (We may assume $B \neq V(G)$, as otherwise the robber is trivially caught.)

Let $B' \subseteq V(G)$ be the unique bag of the tree decomposition that is adjacent to B in T with $B' \cap C \neq \varnothing$. Observe that $B \cap B'$ is a cutset of the graph G. We show that the cops can move in such a way that the vertices of $B \cap B'$ remains guarded, and after a finite number of moves all the vertices of B' (instead of B) are guarded.

Consider each cop. Suppose first that the cop sits on a vertex of $B \setminus B'$ or guards a shortest path between two vertices in $B \setminus B'$. Then we send him to guard a shortest path between two unguarded vertices in $B' \setminus B$ (or to sit on the last unguarded vertex if there is only one such vertex). Assume now that the cop sits on a vertex of $B \cap B'$ or guards a shortest $b_i b_j$-path with $b_i \in B \cap B'$ and $b_j \in B \setminus B'$. Then the cop first goes to b_i (if he is not already there) along the path he keeps. The he starts guarding an arbitrary $b_i b'_{j'}$-path, where $b'_{j'}$ is any unguarded vertex of $B' \setminus B$. Notice that, while it may take some moves before all the vertices of the path are safely guarded, at least the vertex b_i is guarded at every time. Suppose finally that the cop guards a shortest $b_i b_j$-path with $b_i, b_j \in B \cap B'$. In this case, the cop does not modify his strategy, and keeps guarding his path.

After a finite number of moves all the vertices in B' are guarded, and the robber did not have, at any time, the opportunity to go on a vertex in $B \cap B'$ without being caught. Moreover, the number of necessary cops is at most $\lceil |B'|/2 \rceil \leq \mathrm{tw}(G)/2 + 1$, and the robber is reduced to stay in (the subgraph corresponding to) some tree of $T \setminus t_{B'}$ which is a proper subtree of T'. Therefore, by repeating this operation a finite number of times the robber will eventually be caught. This completes the proof. □

We remark that the bound given in Proposition 1.5 is best possible for small values of the tree-width: For every $k = 1, 2, \ldots, 5$, there are graphs with tree-width k and cop number $\lfloor k/2 \rfloor + 1$ (this is easily seen for $k = 1, 2, 3$, and the Petersen graph and the graph which is the disjoint union of the Petersen graph and a complete graph on 6 vertices are such examples for $k = 4$ and 5, respectively). On the other hand, we do not know whether there exists a constant $c > 0$ and an infinite family of graphs such that $\mathrm{cop}(G) \geq c \cdot \mathrm{tw}(G)$ holds for every graph G in the family.

Let us recall that the *circumference* of a graph is the length of its longest cycle if it has one, $+\infty$ otherwise.

Corollary 3.1. *The cop number of a graph is less than or equal to half its circumference.*

Proof. It is a well-known fact that $\mathrm{tw}(G) \leq \mathrm{circum}(G) - 1$ holds for every graph G, where $\mathrm{circum}(G)$ denotes the circumference of G (see for instance Exercise 12.18 in Diestel's book [7]). With Proposition 1.5, we conclude $\mathrm{cop}(G) \leq \mathrm{circum}(G)/2$. □

Theorem 1.4. *The class of H-subgraph-free graphs has bounded cop number if and only if every connected component of H is a tree with at most three leaves.*

Proof. We first show that the requirements in the statement of the theorem are necessary. Let H be a graph such that the family $\mathscr{F}$ of connected graphs not containing H has bounded cop number.

First, suppose that H contains a cycle, and let ℓ be the length of a longest cycle in H. Then $\mathscr{F}$ contains the family of connected graphs with girth at least $\ell + 1$. However, by Lemma 2.5, the cop number of this family is unbounded. Hence, H is a forest.

Second, suppose that H has a vertex of degree at least 4. This implies that $\mathscr{F}$ contains all connected graphs with maximum degree 3, but Andreae [2] proved that there exists a family of 3-regular graphs on which the cop number is unbounded. Hence, H has maximum degree at most 3.

Third, suppose that there is a tree in H which has two vertices of degree 3. Let ℓ denote the distance between these two vertices in H. Now $\mathscr{F}$ contains the family of all those connected graphs in which every two vertices of degree 3 or more have distance at least $\ell + 1$. Starting from an arbitrary family of graphs on which the cop number is unbounded, a family with this property can be constructed by subdividing every edge ℓ times, as follows from Corollary 2.7. Thus, each connected component of H contains at most one vertex of degree 3.

We now show that any H meeting the conditions in the theorem yields a family of graphs with bounded cop number. The proof will be by induction on the number of connected components of H. For a single component, by Proposition 1.2, we may assume that a vertex of degree 3 does in fact exist. We will prove the following claim.

Claim. *Let H be a tree with maximum degree 3 which has precisely one vertex v of degree 3. Denote by r the maximum distance of a vertex from v. If G does not contain H, then* $\operatorname{cop}(G) \le 2r$.

Before we prove the claim, let us complete the induction. The start of the induction is settled. Let T be a connected component of H, and assume that $\operatorname{cop}(G') \le k$ for every graph G' not containing $H \setminus V(T)$. Let G be a graph not containing H. If G does not contain T, we are done by the claim and the remark preceding it. Otherwise, let T' be a subgraph of G isomorphic to T. We place $|T|$ cops on the vertices of T'. This corners the robber in a connected component of $G \setminus V(T')$. Noting that $G \setminus V(T')$ does not contain $H \setminus V(T)$, by induction, by restricting to the connected component containing the robber, we can catch the robber in $G \setminus V(T')$ using k cops. This bounds the cop number of G by $k + |T|$ and concludes the induction.

Proof of Claim. We prove the claim in the case when each leaf of H has distance exactly r from v. The general case follows easily from this.

By Proposition 1.2, we may assume that G contains a path P on $2r$ vertices, because otherwise we have $\text{cop}(G) \leq 2r$. We guard the path by placing r cops on every other vertex of P, and show that what remains of G has cop number at most r. Assume that $G \setminus V(P)$ contains a cycle C of length at least $2r+1$. Then, since G is connected, we can identify a subgraph isomorphic to H choosing v to be a vertex on C which has minimum distance to a vertex in P, while two of the three branches of the tree are wound around C, the other extends to P. Hence, $G \setminus V(P)$ contains no such cycle. By invoking Corollary 3.1 for the connected component of $G \setminus V(P)$ containing the robber, we see that the cop number of $G \setminus V(P)$ is at most r. □

This completes the proof of Theorem 1.4. □

References

[1] M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Appl. Math. **8** (1984), 1–12.

[2] T. Andreae, *Note on a pursuit game played on graphs*, Discrete Appl. Math. **9** (1984), 111–115.

[3] ______, *On a pursuit game played on graphs for which a minor is excluded*, J. Combin. Th., Ser. B **41** (1986), 37–47.

[4] A. Berarducci and B. Intrigila, *On the cop number of a graph*, Adv. Appl. Math. **14** (1993), 389–403.

[5] B. Bollobás, G. Kun, and I. Leader, *Cops and robbers in a random graph*, preprint.

[6] N. E. Clarke, *Constrained cops and robber*, Ph.D. thesis, Dalhousie University, 2002.

[7] R. Diestel, *Graph Theory*, Springer-Verlag, New York, 2006.

[8] F. V. Fomin, P. A. Golovach, and J. Kratochvíl, *On tractability of cops and robbers game*, Fifth IFIP International Conference On Theoretical Computer Science (IFIP TCS), vol. 273, Springer, 2008, pp. 171–185.

[9] F. V. Fomin and D. M. Thilikos, *An annotated bibliography on guaranteed graph searching*, Theoret. Comput. Sci. **399** (2008), no. 3, 236–245.

[10] P. Frankl, *Cops and robbers in graphs with large girth and Cayley graphs*, Discrete Appl. Math. **17** (1987), 301–305.

[11] A. S. Goldstein and E. M. Reingold, *The complexity of pursuit on a graph*, Theoret. Comput. Sci. **143** (1995), 93–112.

[12] G. Hahn, *Cops, robbers and graphs*, Tatra Mt. Math. Publ. **36** (2007), 163–176.

[13] G. Hahn and G. MacGillivray, *A note on k-cop, l-robber games on graphs*, Discrete Math **306** (2006), 2492–2497.

[14] R. J. Nowakowski and P. Winkler, *Vertex to vertex pursuit in a graph*, Discrete Math. **43** (1983), 23–29.

[15] A. Quilliot, *Jeux et points fixes sur les graphes*, Thèse de 3ème cycle, Université de Paris VI, 1978, pp. 131–145.

[16] ______, *Problèmes de jeux, de point fixe, de connectivité et de représentation sur des graphes, des ensembles ordonnés et des hypergraphes*, Thèse d'etat, Université de Paris VI, 1983.

[17] ______, *A short note about pursuit games played on a graph with a given genus*, J. Combin. Th., Ser. B **38** (1985), 89–92.

[18] B. S. W. Schroeder, *The copnumber of a graph is bounded by* $\lfloor \frac{3}{2} genus(g) \rfloor + 3$, Categorical perspectives (Kent, OH, 1998), Trends Math., Birkhäuser Boston, Boston, MA, 2001, pp. 243–263.

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