Contributions to Discrete Mathematics

Volume 4, Number 1, Pages 54–80
ISSN 1715-0868

WEAKLY PARTITIVE FAMILIES ON INFINITE SETS

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ABSTRACT. Given a finite or infinite set S and a positive integer k, a *binary structure* B of base S and of rank k is a function $(S \times S) \setminus \{(x,x) \ : \ x \in S\} \longrightarrow \{0, \ldots, k-1\}$. A subset X of S is an interval of B if for $a, b \in X$ and $x \in S \setminus X$, $B(a,x) = B(b,x)$ and $B(x,a) = B(x,b)$. The family of intervals of B satisfies the following: $\varnothing$, $\underline{B}$ and $\{x\}$, where $x \in \underline{B}$, are intervals of B; for every family $\mathcal{F}$ of intervals of B, the intersection of all the elements of $\mathcal{F}$ is an interval of B; given intervals X and Y of B, if $X \cap Y \neq \varnothing$, then $X \cup Y$ is an interval of B; given intervals X and Y of B, if $X \setminus Y \neq \varnothing$, then $Y \setminus X$ is an interval of B; for every up-directed family $\mathcal{F}$ of intervals of B, the union of all the elements of $\mathcal{F}$ is an interval of B. Given a set S, a family of subsets of S is weakly partitive if it satisfies the properties above. After suitably characterizing the elements of a weakly partitive family, we propose a new approach to establish the following [6]: given a weakly partitive family $\mathcal{I}$ on a set S, there is a binary structure of base S and of rank ≤ 3 whose intervals are exactly the elements of $\mathcal{I}$.

1. INTRODUCTION

Given a (finite or infinite) set S and a positive integer k, a *binary structure* is a function $B : (S \times S) \setminus \{(x,x) \ : \ x \in S\} \longrightarrow \{0, \ldots, k-1\}$. The set S is called the *base* of B. It is denoted by $\underline{B}$. The integer k is called the *rank* of B. It is denoted by $\mathrm{rk}(B)$. With each subset X of $\underline{B}$ associate the *binary substructure* $B[X]$ of B *induced* by X defined on $\underline{B[X]} = X$ by $B[X] = B_{|(X \times X) \setminus \{(x,x) \ : \ x \in X\}}$. Notice that $\mathrm{rk}(B[X]) = \mathrm{rk}(B)$. With each binary structure B associate its *dual* $B^\star$ defined on $\underline{B^\star} = \underline{B}$ by $B^\star(x,y) = B(y,x)$ for any $x \neq y \in \underline{B}$. Notice that $\mathrm{rk}(B^\star) = \mathrm{rk}(B)$.

A directed graph $D = (V(D), A(D))$ is defined by its *vertex* set $V(D)$ and by its *arc* set $A(D)$, where an arc of D is an ordered pair of distinct vertices of D. A *connected component* of a directed graph D is a subset X of $V(D)$ satisfying: for any $x \in X$ and $y \in V(D) \setminus X$, $(x,y) \notin A(D)$ and $(y,x) \notin A(D)$; for any $x \neq x' \in X$, there are $x = x_0, \ldots, x_n = x' \in X$ such that $(x_i, x_{i+1}) \in A(D)$ or $(x_{i+1}, x_i) \in A(D)$ for $0 \leq i \leq n-1$. A directed graph is *connected* if it possesses a unique connected component. A directed graph D may be identified with the binary structure B_D defined

Received by the editors February 5, 2008, and in revised form January 18, 2009.
2000 *Mathematics Subject Classification.* 05A18, 06A05, 06A06.
Key words and phrases. Interval, strong interval, limit strong interval, cut, zigzag, decomposition tree.

by $\underline{B_D} = V(D)$, $\mathrm{rk}(B_D) = 2$ and $(B_D)^{-1}(\{1\}) = A(D)$. So given a directed graph D, we denote by $D[X]$ the directed subgraph of D induced by $X \subseteq V(D)$. The *dual* of D is the directed graph $D^\star$ defined by $V(D^\star) = V(D)$ and $A(D^\star) = \{(x, y) : (y, x) \in A(D)\}$.

A *partial order* O is a directed graph satisfying: for any $x, y, z \in V(O)$, if $(x, y) \in A(O)$ and $(y, z) \in A(O)$, then $(x, z) \in A(O)$. Given a partial order O, $x < y$ modulo O means $(x, y) \in A(O)$, where $x, y \in V(O)$. A partial order O is a *total order* if for any $x \neq y \in V(O)$, either $x < y$ modulo O or $y < x$ modulo O. Lastly, a partial order O is a *tree* if it is connected and if for each $x \in V(O)$, $O[\{y \in V(O) : x < y \text{ modulo } O\}]$ is a total order. Given a tree τ, a *branch* of τ is a maximal subset under inclusion of $V(\tau)$ which induces a total order. We will use the following property of a branch b of a tree τ: for every $x \in b$, $\{y \in V(\tau) : x < y \text{ modulo } \tau\} \subseteq b$.

A binary structure B is *constant* if there is $i \in \{0, \ldots, \mathrm{rk}(B) - 1\}$ such that $B(x, y) = i$ for any $x \neq y \in \underline{B}$. Given a binary structure B and $i \neq j \in \{0, \ldots, \mathrm{rk}(B) - 1\}$, B is *totally ordered* by $\{i, j\}$ if the directed graph $(\underline{B}, B^{-1}(\{i\}))$ is a total order, the dual of which is $(\underline{B}, B^{-1}(\{j\}))$. More simply, a binary structure B is *totally ordered* if it is totally ordered by some unordered pair included in $\{0, \ldots, \mathrm{rk}(B) - 1\}$.

We use the following notation. Given sets X and Y, $X \subseteq Y$ means that X is a subset of Y whereas $X \subset Y$ means that X is a proper subset of Y. Now, consider a binary structure B. Given $X \subset \underline{B}$ and $u \in \underline{B} \setminus X$, $B(u, X) = i$, where $i \in \{0, \ldots, \mathrm{rk}(B) - 1\}$, means that $B(u, x) = i$ for every $x \in \underline{B}$. Given $X, Y \subseteq \underline{B}$ such that $X \cap Y = \varnothing$, $B(X, Y) = i$, where $i \in \{0, \ldots, \mathrm{rk}(B) - 1\}$, means that $B(x, Y) = i$ for every $x \in X$.

Given a binary structure B, a subset X of $\underline{B}$ is an *interval* ([3, Subsection 9.8] and [7]) or an *autonomous* subset [9] or a *homogeneous* subset [4, 10] or a *clan* [2, Subsection 3.2] of B if for any $a, b \in X$ and $x \in \underline{B} \setminus X$, we have $B(a, x) = B(b, x)$ and $B(x, a) = B(x, b)$. We denote by $\mathcal{I}(B)$ the family of the intervals of B. The following properties of the intervals of a binary structure are well known (see, for example, [2, Subsection 3.3]). Given a set S, recall that a family $\mathcal{F}$ of subsets of S is *up-directed* if for any $X, Y \in \mathcal{F}$, there is $Z \in \mathcal{F}$ such that $X \cup Y \subseteq Z$.

Proposition 1.1. *Given a binary structure B, the assertions below hold.*

- (A1) *$\varnothing$, $\underline{B}$ and $\{x\}$, where $x \in \underline{B}$, are intervals of B.*
- (A2) *For every family $\mathcal{F}$ of intervals of B, the intersection $\cap\mathcal{F}$ of all the elements of $\mathcal{F}$ is an interval of B. In particular, for any $X, Y \in \mathcal{I}(B)$, $X \cap Y \in \mathcal{I}(B)$.*
- (A3) *Given $X, Y \in \mathcal{I}(B)$, if $X \cap Y \neq \varnothing$, then $X \cup Y \in \mathcal{I}(B)$.*
- (A4) *Given $X, Y \in \mathcal{I}(B)$, if $X \setminus Y \neq \varnothing$, then $Y \setminus X \in \mathcal{I}(B)$.*
- (A5) *For every up-directed family $\mathcal{F}$ of intervals of B, the union $\cup\mathcal{F}$ of all the elements of $\mathcal{F}$ is an interval of B.*

Notice that if B is finite, that is $\underline{B}$ is finite, then Assertion A5 is always satisfied since an up-directed family of subsets of a finite set admits a largest element under inclusion. Given a set S, a family $\mathcal{F}$ of subsets of S satisfying Assertions A1,...,A5 is said to be *weakly partitive*. Such a family is also called *siba* (*semi-independent boolean algebra*) [2]. A family $\mathcal{F}$ of subsets of S is *partitive* [1] if $\mathcal{F}$ satisfies Assertions A1, A2, A3, A5 and the following: given $X, Y \in \mathcal{F}$, if $X \setminus Y \neq \varnothing$, $Y \setminus X \neq \varnothing$ and $X \cap Y \neq \varnothing$, then $(Y \setminus X) \cup (Y \setminus X) \in \mathcal{F}$. For instance, the family of the intervals of a binary structure B is partitive if $B = B^\star$. We examine weakly partitive families in order to establish the following theorem. It was obtained in [6] through a more complicated approach; for instance, the notion of strong interval was not utilized. For the finite case see, for example, [2, Theorem 5.7].

Theorem 1.2. *Given a weakly partitive family $\mathcal{I}$ on a set S, there exists a binary structure B such that $\underline{B} = S$, $\mathrm{rk}(B) \leq 3$ and $\mathcal{I}(B) = \mathcal{I}$.*

2. Decomposition of finite binary structures

Following Assertion A1, $\varnothing$, $\underline{B}$ and $\{x\}$, where $x \in \underline{B}$, are intervals of B called *trivial*. A binary structure all of whose intervals are trivial is *indecomposable* [7] or *prime* [9] or *primitive* [2]. Otherwise, it is *decomposable*. We recall further properties of intervals.

Proposition 2.1. *Given a binary structure B, the assertions below hold.*

- *Given a subset V of $\underline{B}$, if $X \in \mathcal{I}(B)$, then $X \cap V \in \mathcal{I}(B[V])$.*
- *Given $X \in \mathcal{I}(B)$, we have for every $Y \subseteq X$: $Y \in \mathcal{I}(B[X])$ if and only if $Y \in \mathcal{I}(B)$.*
- *For any $X, Y \in \mathcal{I}(B)$, if we have $X \cap Y = \varnothing$, then there exists $i \in \{0, \ldots, \mathrm{rk}(B) - 1\}$ such that $B(X, Y) = i$.*

Given a binary structure B, a partition P of $\underline{B}$ is an *interval partition* of B when all the elements of P are intervals of B. Using the last assertion of Proposition 2.1, for each interval partition P of B, we can define the *quotient* B/P of B by P on $\underline{B/P} = P$ as follows. For any $X \neq Y \in P$, $(B/P)(X, Y) = B(X, Y)$.

The following strengthening of the notion of interval is due to Gallai [4, 10]. It is used to decompose finite directed graphs in an intrinsic and unique way. Given a binary structure B, an interval X of B is *strong* if for every interval Y of B not disjoint from X, we have $X \subseteq Y$ or $Y \subseteq X$. We denote by $\mathcal{S}(B)$ the family of strong intervals of B. Properties analogous to those stated in Proposition 1.1 hold for strong intervals.

Proposition 2.2. *Given a binary structure B, the assertions below hold.*

(B1) *$\varnothing$, $\underline{B}$ and $\{x\}$, where $x \in \underline{B}$, are strong intervals of B.*
(B2) *For every family $\mathcal{F}$ of strong intervals of B, $\cap\mathcal{F} \in \mathcal{S}(B)$.*
(B3) *For every up-directed family $\mathcal{F}$ of strong intervals of B, $\cup\mathcal{F} \in \mathcal{S}(B)$.*
(B4) *Given $X \in \mathcal{I}(B)$, we have for every $Y \subset X$: Y is a strong interval of $B[X]$ if and only if Y is a strong interval of B.*
(B5) *Given $X \in \mathcal{S}(B)$, we have for every $Y \subseteq X$: $Y \in \mathcal{S}(B[X])$ if and only if $Y \in \mathcal{S}(B)$.*

For a proof of Assertion B4, we refer to [2, Lemma 3.11]. We denote the family of the maximal elements of $\mathcal{S}(B) \setminus \{\varnothing, \underline{B}\}$ under inclusion by $P(B)$. In the finite case, $P(B)$ yields the following decomposition theorem.

Theorem 2.3 (Gallai [4, 10], Ille [7])**.** *Given a finite binary structure B, with $|\underline{B}| \geq 2$, the family $P(B)$ realizes an interval partition of B. Furthermore, the corresponding quotient $B/P(B)$ is constant or totally ordered or indecomposable, with $|P(B)| \geq 3$.*

In fact, given a finite binary structure B, all the strong intervals of $B/P(B)$ are trivial. Thus, the main step in the proof of Theorem 2.3 is to establish the following

Theorem 2.4. *Given a finite binary structure B, all the strong intervals of B are trivial if and only if B is constant or totally ordered or indecomposable, with $|\underline{B}| \geq 3$.*

For a proof of this theorem see, for example, [8, Theorem 1]. Given Theorem 2.3, we label the finite binary structures as below: given a finite binary structure B,

- $\lambda(B) = \mathrm{c}$ if $B/P(B)$ is constant;
- $\lambda(B) = \mathrm{i}$ if $|P(B)| \geq 3$ and $B/P(B)$ is indecomposable;
- $\lambda(B) = \mathrm{t}$ if $B/P(B)$ is totally ordered.

Given a finite binary structure B, with $|\underline{B}| \geq 2$, the family $\mathcal{S}(B) \setminus \{\varnothing\}$ endowed with inclusion constitutes a tree which is called the *decomposition tree* of B.

3. Weakly partitive families defined on finite sets

An analogous study can be done from a weakly partitive family on a finite set without considering a binary structure.

3.1. **Preliminaries.** To commence, we recall the following result (see, for example, [6, Lemma 2.3])

Lemma 3.1. *Given a family $\mathcal{I}$ of subsets of a set S, if $\mathcal{I}$ satisfies Assertions A1–A4, then the following are equivalent.*

- (A5) *For every up-directed family $\mathcal{F} \subseteq \mathcal{I}$, $\cup\mathcal{F} \in \mathcal{I}$.*
- (A6) *For every $V \subseteq S$, $V \in \mathcal{I}$ if and only if for any $u, v \in V$ and $x \in S \setminus V$, there exists $X \in \mathcal{I}$ such that $u, v \in X$ and $x \notin X$.*
- (A7) *Given $\mathcal{F} \subseteq \mathcal{I}$, $\cup\mathcal{F} \in \mathcal{I}$ provided that for any $x \neq y \in \cup\mathcal{F}$, there is a sequence $x = x_0, \ldots, x_n = y \in S$ and a sequence $X_1, \ldots, X_n \in \mathcal{F}$ such that $x_{i-1}, x_i \in X_i$ for $1 \leq i \leq n$.*

We need the following notation. Given a set S, consider a family $\mathcal{F}$ of subsets of S. For each $V \subseteq S$, set

$$\mathcal{F}_{/\cap V} = \{U \cap V \ : \ U \in \mathcal{F}\},$$
$$\mathcal{F}_{/\subseteq V} = \{U \in \mathcal{F} \ : \ U \subseteq V\},$$
$$\mathcal{F}_{/\subset V} = \{U \in \mathcal{F} \ : \ U \subset V\},$$

and

$$\mathcal{F}_{/\supseteq V} = \{U \in \mathcal{F} \ : \ U \supseteq V\}.$$

For example, given a weakly partitive family $\mathcal{I}$, it follows from Assertion A2 that $\mathcal{I}_{/\cap X} = \mathcal{I}_{/\subseteq X}$ for every $X \in \mathcal{I}$.

Lemma 3.2. *Consider a weakly partitive family $\mathcal{I}$ on a set S. For each $V \subseteq S$, $\mathcal{I}_{/\cap V}$ is a weakly partitive family on V.*

Proof. Obviously, $\mathcal{F}_{/\cap V}$ satisfies Assertion A1.

For Assertion A2, consider $\mathcal{F} \subseteq \mathcal{I}_{/\cap V}$. For each $Y \in \mathcal{F}$, denote by $\mathcal{G}_Y$ the family of $X \in \mathcal{I}$ such that $Y = X \cap V$. Set $\mathcal{G} = \cup_{Y \in \mathcal{F}} \mathcal{G}_Y$. As $\mathcal{I}$ satisfies Assertion A2, we have $\cap \mathcal{G} \in \mathcal{I}$. Therefore, $\cap \mathcal{F} \in \mathcal{I}_{/\cap V}$ because $\cap \mathcal{F} = (\cap \mathcal{G}) \cap V$.

For Assertion A3, consider $Y, Y' \in \mathcal{I}$ such that $Y \cap Y' \neq \varnothing$. There exist $X, X' \in \mathcal{I}$ such that $Y = X \cap V$ and $Y' = X' \cap V$. We clearly have $X \cap X' \neq \varnothing$ so that $X \cup X' \in \mathcal{I}$ because $\mathcal{I}$ satisfies Assertion A3. Thus, $Y \cup Y' \in \mathcal{I}_{/\cap V}$ since $Y \cup Y' = (X \cap V) \cup (X' \cap V) = (X \cup X') \cap V$.

For Assertion A4, consider $Y, Y' \in \mathcal{I}$ such that $Y \setminus Y' \neq \varnothing$. There exist $X, X' \in \mathcal{I}$ such that $Y = X \cap V$ and $Y' = X' \cap V$. We have $X \setminus X' \neq \varnothing$ since $Y \setminus Y' = (X \setminus X') \cap V$. As $\mathcal{I}$ satisfies Assertion A4, $X' \setminus X \in \mathcal{I}$. Consequently, $Y' \setminus Y \in \mathcal{I}_{/\cap V}$ because $Y' \setminus Y = (X' \setminus X) \cap V$.

By the previous lemma, to show that $\mathcal{I}_{/\cap V}$ satisfies Assertion A5, it suffices to prove that it satisfies Assertion A7. So consider $\mathcal{F} \subseteq \mathcal{I}_{/\cap V}$ verifying the following. For any $u \neq v \in \cup \mathcal{F}$, there is a sequence $u = u_0, \ldots, u_n = v \in V$ and a sequence $Y_1, \ldots, Y_n \in \mathcal{F}$ such that $u_{i-1}, u_i \in Y_i$ for $1 \leq i \leq n$. Moreover, assume that the elements of $\mathcal{F}$ are non-empty. As for Assertion A2, set $\mathcal{G} = \cup_{Y \in \mathcal{F}} \mathcal{G}_Y$. Consider $x \neq x' \in \cup \mathcal{G}$. There are $Y, Y' \in \mathcal{F}$ such that $x \in \cup \mathcal{G}_Y$ and $x' \in \cup \mathcal{G}_{Y'}$. Thus, there are $X \in \mathcal{G}_Y$ and $X' \in \mathcal{G}_{Y'}$ such that $x \in X$ and $x' \in X'$. As $X \in \mathcal{G}_Y$ and $X' \in \mathcal{G}_{Y'}$, we have $Y = X \cap V$ and $Y' = X' \cap V$. Since $Y \neq \varnothing$ and $Y' \neq \varnothing$, consider $u \in Y$ and $u' \in Y'$. There exist a sequence $u = u_1, \ldots, u_n = u' \in V$ and a sequence $Y_2, \ldots, Y_n \in \mathcal{F}$ such that $u_{i-1}, u_i \in Y_i$ for $2 \leq i \leq n$. Now consider the sequence $x = u_0, u = u_1, \ldots, u_n = u', u_{n+1} = x' \in S$ and the sequence $X_1 = X, X_2, \ldots, X_n, X_{n+1} = X' \in \mathcal{G}$, where $X_i \in \mathcal{G}_{Y_i}$ for $2 \leq i \leq n$. They verify $u_{i-1}, u_i \in X_i$ for $1 \leq i \leq n+1$. Consequently, $\cup \mathcal{G} \in \mathcal{I}$ since $\mathcal{I}$ satisfies Assertion A7. Finally, $\cup \mathcal{F} \in \mathcal{I}_{/\cap V}$ because $\cup \mathcal{F} = (\cup \mathcal{G}) \cap V$. □

As for the strong intervals, we introduce the strong elements of a weakly partitive family in the following way. Given a weakly partitive family $\mathcal{I}$, an element X of $\mathcal{I}$ is *strong* provided that for every $Y \in \mathcal{I}$, we have: if

$X \cap Y \neq \varnothing$, then $X \subseteq Y$ or $Y \subseteq X$. We denote by $\mathcal{S}(\mathcal{I})$ the family of strong elements of $\mathcal{I}$. Properties analogous to those stated in Proposition 2.2 hold for strong elements of a weakly partitive family.

Proposition 3.3. *Given a weakly partitive family $\mathcal{I}$ on a set S, the assertions below hold.*

(B1) *$\varnothing$, S and $\{x\}$, where $x \in S$, are strong elements of $\mathcal{I}$.*
(B2) *For every $\mathcal{F} \subseteq \mathcal{S}(\mathcal{I})$, $\cap\mathcal{F} \in \mathcal{S}(\mathcal{I})$.*
(B3) *For every up-directed family $\mathcal{F}$ of elements of $\mathcal{S}(\mathcal{I})$, $\cup\mathcal{F} \in \mathcal{S}(\mathcal{I})$.*
(B4) *For every $X \in \mathcal{I}$, $\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\} = \mathcal{S}(\mathcal{I})_{/\subseteq X} \setminus \{X\}$.*
(B5) *For every $X \in \mathcal{S}(\mathcal{I})$, $\mathcal{S}(\mathcal{I}_{/\subseteq X}) = \mathcal{S}(\mathcal{I})_{/\subseteq X}$.*

Proof. Clearly, $\varnothing, S \in \mathcal{S}(\mathcal{I})$ and $\{x\} \in \mathcal{S}(\mathcal{I})$ for every $x \in S$.

For Assertion B2, consider $\mathcal{F} \subseteq \mathcal{S}(\mathcal{I})$. Let $Y \in \mathcal{I}$ such that $Y \cap (\cap\mathcal{F}) \neq \varnothing$. For every $X \in \mathcal{F}$, we have $X \cap Y \neq \varnothing$ so that $X \subseteq Y$ or $Y \subseteq X$ because $X \in \mathcal{S}(\mathcal{I})$. If there is $X \in \mathcal{F}$ such that $X \subseteq Y$, then $\cap\mathcal{F} \subseteq Y$. Otherwise, $Y \subseteq X$ for every $X \in \mathcal{F}$ and hence $Y \subseteq \cap\mathcal{F}$.

For Assertion B3, consider an up-directed family $\mathcal{F}$ of elements of $\mathcal{S}(\mathcal{I})$. Let $Y \in \mathcal{I}$ such that $Y \cap (\cup\mathcal{F}) \neq \varnothing$. Set $\mathcal{G} = \{X \in \mathcal{F} : X \cap Y \neq \varnothing\}$. Obviously, $\mathcal{G} \neq \varnothing$ because $Y \cap (\cup\mathcal{F}) \neq \varnothing$. For every $X \in \mathcal{G}$, we have $X \subseteq Y$ or $Y \subseteq X$ because $X \in \mathcal{S}(\mathcal{I})$. If there is $X \in \mathcal{G}$ such that $Y \subseteq X$, then $Y \subseteq \cup\mathcal{F}$. Otherwise, $X \subseteq Y$ for every $X \in \mathcal{G}$. Consider $X \in \mathcal{G}$. For every $X' \in \mathcal{F}$, there exists $X'' \in \mathcal{F}$ such that $X \cup X' \subseteq X''$. We have $X'' \in \mathcal{G}$ because $X \in \mathcal{G}$. Therefore $X'' \subseteq Y$ and thus $X' \subseteq Y$. Consequently $\cup\mathcal{F} \subseteq Y$.

For Assertion B4, we first verify that $\mathcal{S}(\mathcal{I})_{/\subseteq X} \setminus \{X\} \subseteq \mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\}$. Let $Y \in \mathcal{S}(\mathcal{I})_{/\subseteq X} \setminus \{X\}$. Consider $Z \in \mathcal{I}_{/\subseteq X}$ such that $Y \cap Z \neq \varnothing$. As $Y \in \mathcal{S}(\mathcal{I})$ and $Z \in \mathcal{I}$, we have $Y \subseteq Z$ or $Z \subseteq Y$. Second, we establish that $\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\} \subseteq \mathcal{S}(\mathcal{I})_{/\subseteq X} \setminus \{X\}$. Let $Y \in \mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\}$. Observe that $Y \in \mathcal{I}$ because $Y \in \mathcal{I}_{/\subseteq X}$. Now consider $Z \in \mathcal{I}$ such that $Y \cap Z \neq \varnothing$. Clearly, $Z \cap X \in \mathcal{I}_{/\subseteq X}$ and $(Z \cap X) \cap Y \neq \varnothing$ because $(Z \cap X) \cap Y = Y \cap Z$. As $Y \in \mathcal{S}(\mathcal{I}_{/\subseteq X})$, we have either $Y \subseteq Z \cap X$ or $Z \cap X \subset Y$. In the first instance, we have $Y \subseteq Z$. Therefore, assume that $Z \cap X \subset Y$. For a contradiction, suppose that $Z \setminus X \neq \varnothing$. Then $X \setminus Z \in \mathcal{I}_{/\subseteq X}$. Since $Z \cap X \subset Y$, we have $(X \setminus Z) \cap Y \neq \varnothing$. As $Y \in \mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\}$, we get $Y \subseteq X \setminus Z$ or $X \setminus Z \subseteq Y$. In the first instance, $Y \cap Z$ would be empty. In the second, Y would equal X because $Z \cap X \subset Y$. Consequently $Z \setminus X = \varnothing$. We obtain $Z \subset Y$ because $Z \cap X \subset Y$.

Assertion B5 is an immediate consequence of Assertion B4 because $X \in \mathcal{S}(\mathcal{I}_{/\subseteq X}) \cap \mathcal{S}(\mathcal{I})_{/\subseteq X}$. □

Given a set S, consider a family $\mathcal{F}$ of subsets of S. A partition of S, all the elements of which belong to $\mathcal{F}$, is called an $\mathcal{F}$-partition. Given such a partition P, the *quotient* $\mathcal{F}/P$ of $\mathcal{F}$ by P is the family of the subsets Q of P such that $\cup Q \in \mathcal{F}$.

Lemma 3.4. *Consider a weakly partitive family $\mathcal{I}$ on a set S. For each $\mathcal{I}$-partition P, the quotient $\mathcal{I}/P$ is a weakly partitive family on P.*

Proof. The quotient $\mathcal{I}/P$ satisfies Assertion A1 because $\cup\varnothing = \varnothing$, $\cup P = S$ and $\cup\{X\} = X$ for every $X \in P$.

For Assertion A2, consider $\mathcal{Q} \subseteq \mathcal{I}/P$. We easily verify that $\cup(\cap\mathcal{Q}) = \cap\{\cup Q : Q \in \mathcal{Q}\}$. As $\mathcal{I}$ satisfies Assertion A2 and as $\cup Q \in \mathcal{I}$ for every $Q \in \mathcal{Q}$, we have $\cap\{\cup Q : Q \in \mathcal{Q}\} \in \mathcal{I}$ and hence $\cap\mathcal{Q} \in \mathcal{I}/P$.

For Assertion A3, consider $Q, R \in \mathcal{I}/P$ such that $Q \cap R \neq \varnothing$. By the definition of $\mathcal{I}/P$, we have $\cup Q, \cup R \in \mathcal{I}$. Obviously, $(\cup Q) \cap (\cup R) = \cup(Q \cap R)$. Therefore, $(\cup Q) \cap (\cup R) \neq \varnothing$ and hence $(\cup Q) \cup (\cup R) \in \mathcal{I}$ because $\mathcal{I}$ satisfies Assertion A3. As $(\cup Q) \cup (\cup R) = \cup(Q \cup R)$, we obtain that $Q \cup R \in \mathcal{I}/P$.

For Assertion A4, consider $Q, R \in \mathcal{I}/P$ such that $Q \setminus R \neq \varnothing$. By the definition of $\mathcal{I}/P$, we have $\cup Q, \cup R \in \mathcal{I}$. Obviously, $\cup(Q \setminus R) = (\cup Q) \setminus (\cup R)$. Therefore, $(\cup Q) \setminus (\cup R) \neq \varnothing$ and hence $(\cup R) \setminus (\cup Q) \in \mathcal{I}$ because $\mathcal{I}$ satisfies Assertion A4. Since $\cup(R \setminus Q) = (\cup R) \setminus (\cup Q)$, we have $\cup(R \setminus Q) \in \mathcal{I}$ so that $R \setminus Q \in \mathcal{I}/P$.

For Assertion A5, consider an up-directed family $\mathcal{Q}$ of elements of $\mathcal{I}/P$. Set $\mathcal{F} = \{\cup Q : Q \in \mathcal{Q}\}$. Clearly, $\mathcal{F}$ is an up-directed family of elements of $\mathcal{I}$. As $\mathcal{I}$ satisfies Assertion A5, we obtain that $\cup\mathcal{F} \in \mathcal{I}$. We have $\cup\mathcal{Q} \in \mathcal{I}/P$ because $\cup\mathcal{F} = \cup(\cup\mathcal{Q})$. □

Lemma 3.5. *Consider a weakly partitive family $\mathcal{I}$ on a set S. For each $\mathcal{S}(\mathcal{I})$-partition P, we have $\mathcal{S}(\mathcal{I}/P) = \mathcal{S}(\mathcal{I})/P$.*

Proof. To begin, we verify that $\mathcal{S}(\mathcal{I})/P \subseteq \mathcal{S}(\mathcal{I}/P)$. Let $Q \in \mathcal{S}(\mathcal{I})/P$. We have $\cup Q \in \mathcal{S}(\mathcal{I})$. Consider $R \in \mathcal{I}/P$ such that $Q \cap R \neq \varnothing$. We have $\cup R \in \mathcal{I}$. Furthermore, $(\cup Q) \cap (\cup R) \neq \varnothing$ because $(\cup Q) \cap (\cup R) = \cup(Q \cap R)$. Since $\cup Q \in \mathcal{S}(\mathcal{I})$, we obtain that $\cup Q \subseteq \cup R$ or $\cup R \subseteq \cup Q$, which is equivalent to $Q \subseteq R$ or $R \subseteq Q$. It follows that $Q \in \mathcal{S}(\mathcal{I}/P)$.

Conversely, we establish that $\mathcal{S}(\mathcal{I}/P) \subseteq \mathcal{S}(\mathcal{I})/P$. Let $Q \in \mathcal{S}(\mathcal{I}/P)$. We have to prove that $\cup Q \in \mathcal{S}(\mathcal{I})$. So consider $Y \in \mathcal{I}$ such that $(\cup Q) \cap Y \neq \varnothing$. Set $R = \{X \in P : X \cap Y \neq \varnothing\}$. If there is $X \in P$ such that $Y \subseteq X$, then $X \in Q$ and hence $Y \subseteq X \subseteq \cup Q$. Otherwise, $|R| \geq 2$. As $R \subseteq P \subseteq \mathcal{S}(\mathcal{I})$, we have $X \subseteq Y$ or $Y \subseteq X$ for every $X \in R$. Since $|R| \geq 2$, we obtain that $X \subseteq Y$ for every $X \in R$. Therefore $Y = \cup R$ and hence $R \in \mathcal{I}/P$. As $(\cup Q) \cap Y \neq \varnothing$, we have $Q \cap R \neq \varnothing$. Since $Q \in \mathcal{S}(\mathcal{I}/P)$, we obtain that $Q \subseteq R$ or $R \subseteq Q$, which implies that $\cup Q \subseteq \cup R$ or $\cup R \subseteq \cup Q$. Consequently $\cup Q \in \mathcal{S}(\mathcal{I})$. □

Consider a weakly partitive family $\mathcal{I}$ on a set S. We denote by $P(\mathcal{I})$ the family constituted by the maximal elements under inclusion of $\mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$. Notice that if $|S| \leq 1$, then $P(\mathcal{I}) = \varnothing$. There are other cases when S is infinite. For example, on the set of integers $\mathbb{Z}$, consider the family

$$\mathcal{I} = \{\varnothing, \mathbb{Z}\} \cup \{\{n\} \ : \ n \in \mathbb{Z}\} \cup \{(-\infty, n] \ : \ n \in \mathbb{Z}\}.$$

We easily verify that $\mathcal{I}$ is a weakly partitive family on $\mathbb{Z}$ and that $\mathcal{S}(\mathcal{I}) = \mathcal{I}$. Therefore $P(\mathcal{I}) = \varnothing$. So we say that a weakly partitive family $\mathcal{I}$ is *a limit* if $P(\mathcal{I}) = \varnothing$. For convenience, given a weakly partitive family $\mathcal{I}$, we denote by $\mathcal{L}(\mathcal{I})$ the family of $X \in \mathcal{S}(\mathcal{I})$ such that $\mathcal{I}_{/\subseteq X}$ is a limit.

The next lemma is known when the weakly partitive family considered is the family of the intervals of a binary structure (see, for example, [5, Theorem 4.2]).

Lemma 3.6. *Consider a weakly partitive family $\mathcal{I}$ on a set S. If $\mathcal{I}$ is not a limit, then $P(\mathcal{I})$ realizes an $\mathcal{S}(\mathcal{I})$-partition of S. Moreover, for every $X \in \mathcal{S}(\mathcal{I}) \setminus \{S\}$, there is $Y \in P(\mathcal{I})$ such that $X \subseteq Y$.*

Proof. The elements of $P(\mathcal{I})$ are pairwise disjoint. Indeed, consider $Y, Z \in P(\mathcal{I})$ such that $Y \cap Z \neq \varnothing$. As $Y, Z \in \mathcal{S}(\mathcal{I})$, we have $Y \subseteq Z$ or $Z \subseteq Y$. Since Y and Z are maximal elements under inclusion of $\mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$, we have $Y = Z$. Therefore, it suffices to verify that $\cup P(\mathcal{I}) = S$ when $P(\mathcal{I}) \neq \varnothing$. Let $X \in P(\mathcal{I})$. For each $x \in S \setminus X$, denote by $\mathcal{F}_x$ the family of the strong elements of $\mathcal{I}$ which are distinct from S and which contain x. Notice that $\mathcal{F}_x \neq \varnothing$ because $\{x\} \in \mathcal{F}_x$. Consider $x \in S \setminus X$. Since $\mathcal{F}_x \subseteq \mathcal{S}(\mathcal{I})$ and since $x \in \cap\mathcal{F}_x$, $\mathcal{F}_x$ is a total order under inclusion. By Assertion B3 of Proposition 3.3, we have $\cup\mathcal{F}_x \in \mathcal{S}(\mathcal{I})$. Furthermore $X \cap (\cup\mathcal{F}_x) = \varnothing$. Otherwise, there is $Y_x \in \mathcal{F}_x$ such that $X \cap Y_x \neq \varnothing$. As $X \in \mathcal{S}(\mathcal{I})$ and as $x \in Y_x \setminus X$, we obtain that $X \subset Y_x$, which is not possible because $X \in P(\mathcal{I})$ and $Y_x \in \mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$. In particular, we proved that $\cup\mathcal{F}_x \in \mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$. To conclude, it is sufficient to show that $P(\mathcal{I}) \setminus \{X\} = \{\cup\mathcal{F}_x \ : \ x \in S \setminus X\}$. First, consider $Y \in P(\mathcal{I}) \setminus \{X\}$. Given $y \in Y$, we have $Y \in \mathcal{F}_y$ and hence $Y \subseteq \cup\mathcal{F}_y$. Since $\cup\mathcal{F}_y \in \mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$, we obtain $Y = \cup\mathcal{F}_y$. Second, consider $x \in S \setminus X$ and $Y \in \mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$ such that $\cup\mathcal{F}_x \subseteq Y$. Clearly, $Y \in \mathcal{F}_x$ and hence $Y = \cup\mathcal{F}_x$. Thus $\cup\mathcal{F}_x \in P(\mathcal{I})$. Consequently, $P(\mathcal{I})$ is an $\mathcal{S}(\mathcal{I})$-partition of S. Lastly, consider $X \in \mathcal{S}(\mathcal{I}) \setminus \{S\}$. There is $Y \in P(\mathcal{I})$ such that $Y \cap X \neq \varnothing$. We have either $Y \subset X$ or $X \subseteq Y$. As Y is a maximal element under inclusion of $\mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$, we get $X \subseteq Y$. □

To state the analogue of Theorem 2.4 for weakly partitive families, we introduce the following. Consider a family $\mathcal{F}$ of subsets of a set S. The family $\mathcal{F}$ is *trivial* if $\mathcal{F} = \{\varnothing, S\} \cup \{\{x\} \ : \ x \in S\}$. It is said to be *complete* if $\mathcal{F} = 2^S$. Given a total order T such that $V(T) = S$, $\mathcal{F}$ is *totally ordered* by $\{T, T^\star\}$ if $\mathcal{F}$ coincides with the family of the intervals of T (or of $T^\star$). Notice that such a total order is unique up to duality. So we simply say that $\mathcal{F}$ is *totally ordered* when such a total order T exists. As consequence of Lemmas 3.5 and 3.6, we obtain

Corollary 3.7. *Consider a weakly partitive family $\mathcal{I}$ on a set S. If $\mathcal{I}$ is not a limit, then $\mathcal{S}(\mathcal{I}/P(\mathcal{I}))$ is trivial.*

Proof. By Lemma 3.6, $P(\mathcal{I})$ is an $\mathcal{S}(\mathcal{I})$-partition of S. It follows from Lemma 3.5 that $\mathcal{S}(\mathcal{I}/P(\mathcal{I})) = \mathcal{S}(\mathcal{I})/P(\mathcal{I})$. Now consider $Q \in \mathcal{S}(\mathcal{I})/P(\mathcal{I})$ such that $|Q| \geq 2$. We have to show that $Q = P(\mathcal{I})$. Since $Q \in \mathcal{S}(\mathcal{I})/P(\mathcal{I})$, we have $\cup Q \in \mathcal{S}(\mathcal{I})$. Given $X \in Q$, we have $X \subset \cup Q$ because $|Q| \geq 2$. As

X is a maximal element under inclusion of $\mathcal{S}(\mathcal{I}) \setminus \{\varnothing, S\}$, we obtain $\cup Q = S$ or, equivalently, $Q = P(\mathcal{I})$. □

To study a weakly partitive family, we will demonstrate the following result. It constitutes the main part of our study.

Theorem 3.8. *Given a weakly partitive family $\mathcal{I}$ on a set S, $\mathcal{S}(\mathcal{I})$ is trivial if and only if $\mathcal{I}$ is trivial or complete or totally ordered.*

3.2. **Theorem 3.8 in the finite case.** Theorem 3.8 is known in the finite case. For instance, it is easy to adapt the proof of [2, Theorem 5.3] or of [8, Theorem 1]. Theorem 3.8 allows a description of the elements of a weakly partitive family as follows.

Consider a weakly partitive family $\mathcal{I}$ on a finite set S, with $|S| \geq 2$. We will localize and decompose the elements of $\mathcal{I}$ according to the tree constituted by the non-empty strong elements of $\mathcal{I}$. Since S is finite, the family $\mathcal{I}$ is not a limit. By Lemma 3.6, $P(\mathcal{I})$ realizes an $\mathcal{S}(\mathcal{I})$-partition of S such that $\mathcal{S}(\mathcal{I}/P(\mathcal{I})) = \mathcal{S}(\mathcal{I})/P(\mathcal{I})$ by Lemma 3.5. Furthermore, by Corollary 3.7, the family $\mathcal{S}(\mathcal{I}/P(\mathcal{I}))$ of the strong elements of the corresponding quotient is trivial. Consequently, it follows from Theorem 3.8 that $\mathcal{I}/P(\mathcal{I})$ is trivial or complete or totally ordered. In the last instance, there is a total order $T(\mathcal{I})$ defined on $P(\mathcal{I})$ such that $\mathcal{I}/P(\mathcal{I})$ is totally ordered by $\{T(\mathcal{I}), T(\mathcal{I})^\star\}$. For convenience, we label $\mathcal{I}$ as

- $\lambda(\mathcal{I}) = \mathrm{c}$ if $\mathcal{I}/P(\mathcal{I})$ is complete;
- $\lambda(\mathcal{I}) = \mathrm{i}$ if $|P(\mathcal{I})| \geq 3$ and $\mathcal{I}/P(\mathcal{I})$ is trivial;
- $\lambda(\mathcal{I}) = \mathrm{t}$ if $\mathcal{I}/P(\mathcal{I})$ is totally ordered.

For each $X \in \mathcal{S}(\mathcal{I})$, with $|X| \geq 2$, we carry out the same study to obtain with the corresponding labeling $\lambda(\mathcal{I}_{/\subseteq X})$ that $(\mathcal{I}_{/\subseteq X})/P(\mathcal{I}_{/\subseteq X})$ is trivial or complete or totally ordered.

For each non-empty subset V of S, the family $\mathcal{S}(\mathcal{I})_{/\supseteq V}$ endowed with inclusion is a total order. Its smallest element $\cap(\mathcal{S}(\mathcal{I})_{/\supseteq V})$ belongs to $\mathcal{S}(\mathcal{I})$ by Assertion B2 of Proposition 3.3. It is denoted by $\overline{V}^{\mathcal{I}}$ or simply $\overline{V}$.

Finally, every $X \in \mathcal{I}$, with $|X| \geq 2$, is decomposed as follows. Clearly, $X \in \mathcal{I}_{/\subseteq \overline{X}}$. Denote by Q_X the family of $Y \in P(\mathcal{I}_{/\subseteq \overline{X}})$ such that $Y \cap X \neq \varnothing$. It follows from the definition of $\overline{X}$ that $|Q_X| \geq 2$. As $P(\mathcal{I}_{/\subseteq \overline{X}}) \subseteq \mathcal{S}(\mathcal{I}_{/\subseteq \overline{X}})$, we obtain that $X = \cup Q_X$ and hence $Q_X \in (\mathcal{I}_{/\subseteq \overline{X}})/P(\mathcal{I}_{/\subseteq \overline{X}})$. In addition, it follows from the definition of $\lambda(\mathcal{I}_{/\subseteq \overline{X}})$ that $Q_X = P(\mathcal{I}_{/\subseteq \overline{X}})$ if $\lambda(\mathcal{I}_{/\subseteq \overline{X}}) = \mathrm{i}$ and that Q_X is an interval of $T(\mathcal{I}_{/\subseteq \overline{X}})$ if $\lambda(\mathcal{I}_{/\subseteq \overline{X}}) = \mathrm{t}$.

Conversely, consider a subset V of S, with $|V| \geq 2$, such that there is $Q_V \subseteq P(\mathcal{I}_{/\subseteq \overline{V}})$ satisfying $V = \cup Q_V$. Furthermore, assume that $Q_V = P(\mathcal{I}_{/\subseteq \overline{V}})$ if $\lambda(\mathcal{I}_{/\subseteq \overline{V}}) = \mathrm{i}$ and that Q_V is an interval of $T(\mathcal{I}_{/\subseteq \overline{V}})$ if $\lambda(\mathcal{I}_{/\subseteq \overline{V}}) = \mathrm{t}$. Whatever $\lambda(\mathcal{I}_{/\subseteq \overline{V}})$, we obtain that $Q_V \in (\mathcal{I}_{/\subseteq \overline{V}})/P(\mathcal{I}_{/\subseteq \overline{V}})$. Consequently, $V = \cup Q_V \in \mathcal{I}_{/\subseteq \overline{V}}$ and hence $V \in \mathcal{I}$.

We summarize the previous discussion in the following theorem.

Theorem 3.9. *Consider a weakly partitive family $\mathcal{I}$ on a finite set S, with $|S| \geq 2$. For every $V \subseteq S$, we have $V \in \mathcal{I}$ if and only if either $|V| \leq 1$ or $|V| \geq 2$ and there is $Q_V \subseteq P(\mathcal{I}_{/\subseteq \overline{V}})$ such that $V = \cup Q_V$ and satisfying*

- *if $\lambda(\mathcal{I}_{/\subseteq \overline{V}}) = \mathrm{i}$, then $Q_V = P(\mathcal{I}_{/\subseteq \overline{V}})$;*
- *if $\lambda(\mathcal{I}_{/\subseteq \overline{V}}) = \mathrm{t}$, then Q_V is an interval of $T(\mathcal{I}_{/\subseteq \overline{V}})$.*

Consequently, the elements of a weakly partitive family $\mathcal{I}$ on a finite set S, with $|S| \geq 2$, are decomposed into a union of elements of $\mathcal{S}(\mathcal{I}) \setminus \{\varnothing\}$. The family $\mathcal{S}(\mathcal{I}) \setminus \{\varnothing\}$ endowed with inclusion is a tree called the *decomposition tree* of $\mathcal{I}$ and denoted by $\mathcal{D}(\mathcal{I})$.

3.3. **The zigzag.** Consider a weakly partitive family $\mathcal{I}$ on a set S. Given $a \neq b \in S$ and $c \neq d \in S$, $(a,b) \vee_{\mathcal{I}} (c,d)$ signifies that one of the following holds

- $a = c$ and there is $X \in \mathcal{I}$ such that $b, d \in X$ and $a \in S \setminus X$;
- $b = d$ and there is $X \in \mathcal{I}$ such that $a, c \in X$ and $b \in S \setminus X$.

Notice that $(a,b) \vee_{\mathcal{I}} (a,b)$ and that $(a,b) \vee_{\mathcal{I}} (c,d)$ if and only if $(c,d) \vee_{\mathcal{I}} (a,b)$.

Given $a \in S$ and $b, b', b'' \in S \setminus \{a\}$, if $(a,b) \vee_{\mathcal{I}} (a,b')$ and $(a,b') \vee_{\mathcal{I}} (a,b'')$, then there are $X, X' \in \mathcal{I}$ such that $b, b' \in X$, $a \in S \setminus X$, $b', b'' \in X'$ and $a \in S \setminus X'$. As $b' \in X \cap X'$, $X \cup X' \in \mathcal{I}$ by Assertion A3. Since $b, b'' \in X \cup X'$ and $a \in S \setminus (X \cup X')$, we get $(a,b) \vee_{\mathcal{I}} (a,b'')$. Thus, when we consider the transitive closure of $\vee_{\mathcal{I}}$, we can return to a sequence where pivots alternate. So a sequence $(a_0, b_0), \ldots, (a_n, b_n)$ of ordered pairs of distinct elements of S is called a *zigzag* modulo $\mathcal{I}$ between (a_0, b_0) and (a_n, b_n) if $(a_i, b_i) \vee_{\mathcal{I}} (a_{i+1}, b_{i+1})$for $0 \leq i \leq n-1$. A subset of S is a *support* of this zigzag modulo $\mathcal{I}$ if it contains $a_0, b_0, \ldots, a_n, b_n$. Given $a \neq b \in S$ and $c \neq d \in S$, $(a,b) \leftrightsquigarrow_{\mathcal{I}} (c,d)$ means that there is a zigzag modulo $\mathcal{I}$ between (a,b) and (c,d). Clearly, $\leftrightsquigarrow_{\mathcal{I}}$ constitutes an equivalence relation on $(S \times S) \setminus \{(x,x) \ : \ x \in S\}$. For $a \neq b \in S$, $[(a,b)]_{\mathcal{I}}$ denotes the equivalence class of (a,b) modulo $\leftrightsquigarrow_{\mathcal{I}}$. Given $a \neq b \in S$ and $c \neq d \in S$, notice that $(a,b) \leftrightsquigarrow_{\mathcal{I}} (c,d)$ if and only if $(b,a) \leftrightsquigarrow_{\mathcal{I}} (d,c)$. When S is finite, we obtain the following characterization of the equivalence classes of $\leftrightsquigarrow_{\mathcal{I}}$.

Proposition 3.10. *Consider a weakly partitive family $\mathcal{I}$ on a finite set S, with $|S| \geq 2$. Given $a \neq b \in S$, the equivalence class $[(a,b)]_{\mathcal{I}}$ satisfies one of the following, where for $x \in \overline{\{a,b\}}$, $\overline{\{a,b\}}_x$ denotes the element of $P(\mathcal{I}_{/\subseteq \overline{\{a,b\}}})$ which contains x.*

- *If $\lambda(\mathcal{I}_{/\subseteq \overline{\{a,b\}}}) = \mathrm{i}$, then $[(a,b)]_{\mathcal{I}} = \overline{\{a,b\}}_a \times \overline{\{a,b\}}_b$.*
- *If $\lambda(\mathcal{I}_{/\subseteq \overline{\{a,b\}}}) = \mathrm{c}$, then*

$$[(a,b)]_{\mathcal{I}} = \{(x,y) \in \overline{\{a,b\}} \times \overline{\{a,b\}} : \ \overline{\{a,b\}}_x \neq \overline{\{a,b\}}_y\}.$$

- *If $\lambda(\mathcal{I}_{/\subseteq \overline{\{a,b\}}}) = \mathrm{t}$, then*

$$[(a,b)]_{\mathcal{I}} = \{(x,y) \in \overline{\{a,b\}} \times \overline{\{a,b\}} : \ \overline{\{a,b\}}_x < \overline{\{a,b\}}_y \text{ modulo } T_{\{a,b\}}\},$$

where $T_{\{a,b\}}$ *is either* $T(\mathcal{I}_{/\subseteq\overline{\{a,b\}}})$ *or* $(T(\mathcal{I}_{/\subseteq\overline{\{a,b\}}}))^\star$ *chosen so that* $\overline{\{a,b\}}_a < \overline{\{a,b\}}_b$ modulo $T_{\{a,b\}}$.

Proof. Let $c \neq d \in S$ such that $(a,b) \vee_{\mathcal{I}} (c,d)$. For instance, assume that $a = c$. Then, there is $X \in \mathcal{I}$ such that $b, d \in X$ and $a \in S \setminus X$. As $\overline{\{a,b\}} \in \mathcal{S}(\mathcal{I})$ and as $b \in X \cap \overline{\{a,b\}}$ and $a \in \overline{\{a,b\}} \setminus X$, we have $X \subset \overline{\{a,b\}}$ so that $d \in \overline{\{a,b\}}$ and $\overline{\{a,d\}} \subseteq \overline{\{a,b\}}$. By interchanging (a,b) and (a,d), we obtain $\overline{\{a,b\}} \subseteq \overline{\{a,d\}}$ and hence $\overline{\{a,b\}} = \overline{\{a,d\}}$. In particular, $\overline{\{a,b\}}_a \neq \overline{\{a,b\}}_d$. Furthermore, observe that if $\overline{\{a,b\}}_b \neq \overline{\{a,b\}}_d$, then $\overline{\{a,b\}}_b \cup \overline{\{a,b\}}_d \subseteq X$ because $\overline{\{a,b\}}_b$ and $\overline{\{a,b\}}_d$ are strong elements of $\mathcal{I}$ intersected by X. Since $X \subset \overline{\{a,b\}}$, we have $\overline{X} = \overline{\{a,b\}}$. Consequently, if $\overline{\{a,b\}}_b \neq \overline{\{a,b\}}_d$, then $\lambda(\mathcal{I}_{/\subseteq\overline{\{a,b\}}}) \neq \mathrm{i}$ and there is $Q_X \subset P(\mathcal{I}_{/\subseteq\overline{\{a,b\}}})$, with $|Q_X| \geq 2$, such that $X = \cup Q_X$. Now we distinguish the three cases below.

Case 1: $\lambda(\mathcal{I}_{/\subseteq\overline{\{a,b\}}}) = \mathrm{i}$.

By the preceding observation, we have $\overline{\{a,b\}}_b = \overline{\{a,b\}}_d$. Therefore $(a,d) \in \overline{\{a,b\}}_a \times \overline{\{a,b\}}_b$. Now consider any zigzag $(a_0,b_0) = (a,b), \ldots, (a_n,b_n)$ modulo $\mathcal{I}$. We similarly obtain by induction on $0 \leq i \leq n$ that $(a_i,b_i) \in \overline{\{a,b\}}_a \times \overline{\{a,b\}}_b$. Consequently $[(a,b)]_{\mathcal{I}} \subseteq \overline{\{a,b\}}_a \times \overline{\{a,b\}}_b$. The opposite inclusion is clear.

Case 2: $\lambda(\mathcal{I}_{/\subseteq\overline{\{a,b\}}}) = \mathrm{t}$.

By the preceding observation, if $\overline{\{a,b\}}_b \neq \overline{\{a,b\}}_d$, then there is $Q_X \subset P(\mathcal{I}_{/\subseteq\overline{\{a,b\}}})$, with $|Q_X| \geq 2$, such that $X = \cup Q_X$. By Theorem 3.9, Q_X is an interval of $T_{\{a,b\}}$. Since $\overline{\{a,b\}}_b, \overline{\{a,b\}}_d \in Q_X$ and $\overline{\{a,b\}}_a \in P(\mathcal{I}_{/\subseteq\overline{\{a,b\}}}) \setminus Q_X$ and since $\overline{\{a,b\}}_a < \overline{\{a,b\}}_b$ modulo $T_{\{a,b\}}$, we have $\overline{\{a,b\}}_a < \overline{\{a,b\}}_d$ modulo $T_{\{a,b\}}$. By using an induction as in the first case, we obtain that $[(a,b)]_{\mathcal{I}} \subseteq \{(x,y) \in \overline{\{a,b\}} \times \overline{\{a,b\}} : \overline{\{a,b\}}_x < \overline{\{a,b\}}_y \text{ modulo } T_{\{a,b\}}\}$. The opposite inclusion is easily verified.

Case 3: $\lambda(\mathcal{I}_{/\subseteq\overline{\{a,b\}}}) = \mathrm{c}$.

For any $x \neq y \in \overline{\{a,b\}}$ such that $\overline{\{a,b\}}_x \neq \overline{\{a,b\}}_y$, we clearly have $(a,b) \leftrightsquigarrow_{\mathcal{I}} (x,y)$. Then, the conclusion follows from the previous observation.

□

Lemma 3.11. *Consider a weakly partitive family* $\mathcal{I}$ *on a set* S. *Given* $V \subseteq S$, *for any* $(a,b),(c,d) \in (V \times V) \setminus \{(x,x) : x \in V\}$, *if* $(a,b) \leftrightsquigarrow_{\mathcal{I}_{/\cap V}} (c,d)$, *then* $(a,b) \leftrightsquigarrow_{\mathcal{I}} (c,d)$.

Proof. It suffices to verify that for any $(a,b),(c,d) \in (V \times V) \setminus \{(x,x) : x \in V\}$, if $(a,b) \vee_{\mathcal{I}_{/\cap V}} (c,d)$, then $(a,b) \vee_{\mathcal{I}} (c,d)$. For instance, assume that $a = c$. Then, there exists $X \in \mathcal{I}$ such that $b,d \in X \cap V$ and $a \in V \setminus (X \cap V)$. Obviously, $b,d \in X$ and $a \in S \setminus X$. Thus $(a,b) \vee_{\mathcal{I}} (c,d)$. □

Lemma 3.12. *Consider a weakly partitive family $\mathcal{I}$ on a set S. For any $(a,b),(c,d) \in (S \times S) \setminus \{(x,x) : x \in S\}$, if $(a,b) \leftrightsquigarrow_{\mathcal{I}} (c,d)$ and if $V \subseteq S$ is a support of a zigzag modulo $\mathcal{I}$ between (a,b) and (c,d), then $(a,b) \leftrightsquigarrow_{\mathcal{I}_{/\cap V}} (c,d)$.*

Proof. It suffices to verify that for any $(a,b),(c,d) \in (S \times S) \setminus \{(x,x) : x \in S\}$, if $(a,b) \vee_{\mathcal{I}} (c,d)$ and if a subset V of S contains a, b, c and d, then $(a,b) \vee_{\mathcal{I}_{/\cap V}} (c,d)$. For instance, assume that $a = c$. Then, there exists $X \in \mathcal{I}$ such that $b, d \in X$ and $a \in S \setminus X$. Obviously, $b, d \in X \cap V$ and $a \in V \setminus (X \cap V)$. Thus $(a,b) \vee_{\mathcal{I}_{/\cap V}} (c,d)$. □

Corollary 3.13. *Consider a weakly partitive family $\mathcal{I}$ on a set S. For any distinct elements a, b and c of S, if $(a,c) \leftrightsquigarrow_{\mathcal{I}} (b,c)$, then $(a,c) \vee_{\mathcal{I}} (b,c)$.*

Proof. Since $(a,c) \leftrightsquigarrow_{\mathcal{I}} (b,c)$, there is a finite support F of a zigzag modulo $\mathcal{I}$ between (a,c) and (b,c). By Lemma 3.2, $\mathcal{I}_{/\cap F}$ is a weakly partitive family on F and $(a,c) \leftrightsquigarrow_{\mathcal{I}_{/\cap F}} (b,c)$ by Lemma 3.12. We distinguish the three cases below according to Proposition 3.10. For convenience, denote $\mathcal{I}_{/\cap F}$ by $\mathcal{J}$ and then denote $\overline{\{a,c\}}^{\mathcal{J}}$ by X. So $X \in \mathcal{S}(\mathcal{J})$. Furthermore, for $u \in X$, denote by X_u the element of $P(\mathcal{J}_{/\subseteq X})$ containing u. Given $u \in X$, we have $X_u \in \mathcal{S}(\mathcal{J}_{/\subseteq X})$. Thus $X_u \in \mathcal{J}_{/\subseteq X}$ and hence $X_u \in \mathcal{J}$.

Case 1: $\lambda(\mathcal{J}_{/\subseteq X}) = \text{i}$.
By Proposition 3.10, $[(a,c)]_{\mathcal{J}} = X_a \times X_c$. Thus $b \in X_a$ and hence $(a,c) \vee_{\mathcal{J}} (b,c)$ because $a, b \in X_a$ and $c \in F \setminus X_a$.

Case 2: $\lambda(\mathcal{J}_{/\subseteq X}) = \text{c}$.
We have $[(a,c)]_{\mathcal{J}} = \{(u,v) \in X \times X : X_u \neq X_v\}$. We obtain that $X_c \neq X_a$ and $X_c \neq X_b$. Moreover, as $\lambda(\mathcal{J}_{/\subseteq X}) = \text{c}$, we have $X_a \cup X_b \in \mathcal{J}_{/\subseteq X}$. Therefore $X_a \cup X_b \in \mathcal{J}$, with $a, b \in X_a \cup X_b$ and $c \in F \setminus (X_a \cup X_b)$.

Case 3: $\lambda(\mathcal{J}_{/\subseteq X}) = \text{t}$.
Let $T_X = T(\mathcal{J}_{/\subseteq X})$ or $(T(\mathcal{J}_{/\subseteq X}))^\star$ such that $X_a < X_c$ modulo T_X. We obtain that $X_b < X_c$ modulo T_X as well. For example, assume that $X_a < X_b$ modulo T_X and denote by $[X_a, X_b]$ the intersection of all the intervals of T_X which contain X_a and X_b. Clearly, $[X_a, X_b]$ is an interval of T_X and hence $\cup[X_a, X_b] \in \mathcal{J}_{/\subseteq X}$. Once again, we get $\cup[X_a, X_b] \in \mathcal{J}$, with $a, b \in \cup[X_a, X_b]$ and $c \in F \setminus (\cup[X_a, X_b])$.

In the three cases above, we obtain $(a,c) \vee_{\mathcal{J}} (b,c)$, that is, $(a,c) \vee_{\mathcal{I}_{/\cap F}} (b,c)$. As observed in the proof of Lemma 3.11, we get $(a,c) \vee_{\mathcal{I}} (b,c)$. □

4. Theorem 3.8 in the infinite case

We commence with some results on weakly partitive families defined on infinite sets.

Lemma 4.1. *Given a weakly partitive family $\mathcal{I}$ on a set S, if $X_1, \ldots, X_n$ are pairwise disjoint elements of $\mathcal{I}$, where $n \geq 2$, then $\overline{X_1 \cup \cdots \cup X_n} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$.*

Proof. We already observed that $\overline{X_1 \cup \cdots \cup X_n} \in \mathcal{S}(\mathcal{I})$. Choose $a_1 \in X_1, \ldots, a_n \in X_n$. We have $\overline{\{a_1, \ldots, a_n\}} \in \mathcal{S}(\mathcal{I})$ as well. Clearly, $\overline{\{a_1, \ldots, a_n\}} \subseteq \overline{X_1 \cup \cdots \cup X_n}$. As $\overline{\{a_1, \ldots, a_n\}} \cap X_1 \neq \varnothing, \ldots, \overline{\{a_1, \ldots, a_n\}} \cap X_n \neq \varnothing$ and the X_i are pairwise disjoint, we obtain that $X_1 \cup \cdots \cup X_n \subseteq \overline{\{a_1, \ldots, a_n\}}$. Therefore $\overline{\{a_1, \ldots, a_n\}} = \overline{X_1 \cup \cdots \cup X_n}$. Denote by $\mathcal{F}$ the family of the elements of $\mathcal{S}(\mathcal{I})_{/\subset\overline{\{a_1,\ldots,a_n\}}}$ which contains a_1. Since $\{a_1\} \in \mathcal{F}$, $\mathcal{F} \neq \varnothing$. As $\mathcal{F} \subseteq \mathcal{S}(\mathcal{I})$, we obtain that $\mathcal{F}$ endowed with inclusion is a total order so that $\overline{\cup\mathcal{F}} \in \mathcal{S}(\mathcal{I})$ by Assertion B3 of Proposition 3.3. Furthermore, $\overline{\cup\mathcal{F}} \subseteq \overline{\{a_1, \ldots, a_n\}}$ because $\mathcal{F} \subseteq \mathcal{S}(\mathcal{I})_{/\subset\overline{\{a_1,\ldots,a_n\}}}$. For every $X \in \mathcal{F}$, we have $\{a_1, \ldots, a_n\} \setminus X \neq \varnothing$ because $X \subset \overline{\{a_1, \ldots, a_n\}}$. It follows that $\{a_1, \ldots, a_n\} \setminus \overline{\cup\mathcal{F}} \neq \varnothing$ and hence $\overline{\cup\mathcal{F}} \subset \overline{\{a_1, \ldots, a_n\}}$. By Assertion B5 of Proposition 3.3, we have $\mathcal{S}(\mathcal{I})_{/\subseteq\overline{\{a_1,\ldots,a_n\}}} = \mathcal{S}(\mathcal{I}_{/\subseteq\overline{\{a_1,\ldots,a_n\}}})$. In particular, $\overline{\cup\mathcal{F}} \in \mathcal{S}(\mathcal{I}_{/\subseteq\overline{\{a_1,\ldots,a_n\}}})$. Lastly, consider $Y \in \mathcal{S}(\mathcal{I}_{/\subseteq\overline{\{a_1,\ldots,a_n\}}})$ such that $\overline{\cup\mathcal{F}} \subset Y \subseteq \overline{\{a_1, \ldots, a_n\}}$. As $a_1 \in Y$ and as $Y \notin \mathcal{F}$, we obtain that $Y = \overline{\{a_1, \ldots, a_n\}}$. Consequently, $\overline{\cup\mathcal{F}} \in P(\mathcal{I}_{/\subseteq\overline{\{a_1,\ldots,a_n\}}})$ and hence $\mathcal{I}_{/\subseteq\overline{\{a_1,\ldots,a_n\}}}$ is not a limit. □

Corollary 4.2. *Given a weakly partitive family $\mathcal{I}$ on a set S, with $|S| \geq 2$, the next assertions are equivalent.*

(1) *$\mathcal{I}$ is a limit.*
(2) *$\mathcal{S}(\mathcal{I})\setminus\{S\}$ is up-directed.*
(3) *$(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}) = S$.*

Proof. Assume that $\mathcal{I}$ is a limit and consider $X, Y \in \mathcal{S}(\mathcal{I})\setminus\{S\}$. If $X \cap Y \neq \varnothing$, then one of these contains the other. If $X \cap Y = \varnothing$, it follows from Lemma 4.1 that $\overline{X \cup Y}$ is not a limit. Therefore $\overline{X \cup Y} \in \mathcal{S}(\mathcal{I})\setminus\{S\}$. Consequently, $\mathcal{S}(\mathcal{I})\setminus\{S\}$ is up-directed. Conversely, assume that $\mathcal{S}(\mathcal{I})\setminus\{S\}$ is up-directed and consider $X \in \mathcal{S}(\mathcal{I})\setminus\{\varnothing, S\}$. Given $x \in S \setminus X$, as $\{x\} \in \mathcal{S}(\mathcal{I})\setminus\{S\}$, there exists $Y \in \mathcal{S}(\mathcal{I})\setminus\{S\}$ such that $X \cup \{x\} \subseteq Y$ and hence $X \subset Y$. Consequently $P(\mathcal{I}) = \varnothing$.

Assume that $\mathcal{I}$ is a limit or equivalently that $\mathcal{S}(\mathcal{I})\setminus\{S\}$ is up-directed. We have $\cup(\mathcal{S}(\mathcal{I})\setminus\{S\}) = S$ because $\{x\} \in \mathcal{S}(\mathcal{I})\setminus\{S\}$ for each $x \in S$. Therefore, to establish that $(\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I}))\setminus\{S\}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I}))\setminus\{S\}) = S$, it is sufficient to establish that for every $X \in \mathcal{S}(\mathcal{I})\setminus\{S\}$, there is $Y \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}$ such that $X \subseteq Y$. In fact, by the previous lemma, for every $x \in S \setminus X$, we have $\overline{X \cup \{x\}} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. Since $S \in \mathcal{L}(\mathcal{I})$, $\overline{X \cup \{x\}} \neq S$. Conversely, assume that $(\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I}))\setminus\{S\}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}) = S$. Consider $X \in \mathcal{S}(\mathcal{I})\setminus\{\varnothing, S\}$. For $x \in X$ and $y \in S \setminus X$, there are $Y, Y' \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}$ such that $x \in Y$ and $y \in Y'$ because $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}) = S$. As $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}$ is up-directed, there exists $Z \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))\setminus\{S\}$ such that $Y \cup Y' \subseteq Z$. Since $x \in X \cap Z$ and $y \in Z \setminus X$, we obtain that $X \subset Z$. Therefore $P(\mathcal{I}) = \varnothing$. □

Corollary 4.2 is also formulated as

Corollary 4.3. *Consider a weakly partitive family $\mathcal{I}$ on a set S. For every $X \in \mathcal{S}(\mathcal{I})$, with $|X| \geq 2$, the following assertions are equivalent.*

(1) $X \in \mathcal{L}(\mathcal{I})$.
(2) $\mathcal{S}(\mathcal{I})_{/\subset X}$ *is up-directed.*
(3) $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}$ *is up-directed and* $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}) = X$.

Consequently, if $X \in \mathcal{L}(\mathcal{I})$, then $\cup(\mathcal{S}(\mathcal{I})_{/\subset X}) = \cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}) = X$.

Proof. By applying the previous corollary to $\mathcal{I}_{/\subseteq X}$, we obtain that the following assertions are equivalent.

- $\mathcal{I}_{/\subseteq X}$ is a limit, that is, $X \in \mathcal{L}(\mathcal{I})$.
- $\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\}$ is up-directed.
- $(\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \mathcal{L}(\mathcal{I}_{/\subseteq X})) \setminus \{X\}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \mathcal{L}(\mathcal{I}_{/\subseteq X})) \setminus \{X\}) = X$.

As $X \in \mathcal{S}(\mathcal{I})$, it follows from Assertion B5 of Proposition 3.3 that $\mathcal{S}(\mathcal{I}_{/\subseteq X}) = \mathcal{S}(\mathcal{I})_{/\subseteq X}$. Thus $\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{X\} = \mathcal{S}(\mathcal{I})_{/\subset X}$. Furthermore, by definition, $\mathcal{L}(\mathcal{I}_{/\subseteq X})$ is the family of $Y \subseteq X$ such that $Y \in \mathcal{S}(\mathcal{I}_{/\subseteq X})$ and $(\mathcal{I}_{/\subseteq X})_{/\subseteq Y}$ is a limit. Since $\mathcal{S}(\mathcal{I}_{/\subseteq X}) = \mathcal{S}(\mathcal{I})_{/\subseteq X}$ and since $(\mathcal{I}_{/\subseteq X})_{/\subseteq Y} = \mathcal{I}_{/\subseteq Y}$, we obtain that $\mathcal{L}(\mathcal{I}_{/\subseteq X}) = \mathcal{L}(\mathcal{I})_{/\subseteq X}$. Therefore $(\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \mathcal{L}(\mathcal{I}_{/\subseteq X})) \setminus \{X\} = (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}$. □

Proposition 4.4. *Given a weakly partitive family $\mathcal{I}$ on a set S, $\mathcal{I}$ is not a limit and $\mathcal{I}/P(\mathcal{I})$ is non-trivial if and only if there exists a non-empty proper subset C of S such that $\{C, S \setminus C\}$ is an $\mathcal{I}$-partition and not an $\mathcal{S}(\mathcal{I})$-partition.*

Proof. Assume that $\mathcal{I}$ is a non-limit and $\mathcal{I}/P(\mathcal{I})$ is non-trivial. Consider $Q \in \mathcal{I}/P(\mathcal{I})$ such that $|Q| \geq 2$ and $Q \neq P(\mathcal{I})$. Let $X \in P(\mathcal{I}) \setminus Q$ and denote by $\mathcal{Q}$ the family of $R \in \mathcal{I}/P(\mathcal{I})$ such that $Q \subseteq R$ and $X \notin R$. By Lemmas 3.1 and 3.4, $\mathcal{I}/P(\mathcal{I})$ satisfies Assertion A7 so that $\cup\mathcal{Q} \in \mathcal{I}/P(\mathcal{I})$. Now let $\mathcal{R}$ be the family of $R \in \mathcal{I}/P(\mathcal{I})$ such that $(\cup\mathcal{Q}) \cap R = \varnothing$ and $X \in R$. By Assertion A7, $\cup\mathcal{R} \in \mathcal{I}/P(\mathcal{I})$. Since $Q \subseteq \cup\mathcal{Q}$ and $X \notin \cup\mathcal{Q}$, $\cup\mathcal{Q}$ is not strong by Corollary 3.7. Therefore, there is $Q' \in \mathcal{I}/P(\mathcal{I})$ such that $Q' \cap (\cup\mathcal{Q}) \neq \varnothing$, $Q' \setminus (\cup\mathcal{Q}) \neq \varnothing$ and $(\cup\mathcal{Q}) \setminus Q' \neq \varnothing$. We obtain that $Q' \cup (\cup\mathcal{Q}) \in \mathcal{I}/P(\mathcal{I})$ and $\cup\mathcal{Q} \subset Q' \cup (\cup\mathcal{Q})$. Thus $Q' \cup (\cup\mathcal{Q}) \notin \mathcal{Q}$ and hence $X \in Q' \setminus (\cup\mathcal{Q})$. As $(\cup\mathcal{Q}) \setminus Q' \neq \varnothing$, we have $Q' \setminus (\cup\mathcal{Q}) \in \mathcal{I}$. Therefore $Q' \setminus (\cup\mathcal{Q}) \in \mathcal{R}$ and $Q' \subseteq (\cup\mathcal{Q}) \cup (\cup\mathcal{R})$. Since $Q' \cup (\cup\mathcal{Q}) \in \mathcal{I}$ and $X \in Q' \cup (\cup\mathcal{Q})$, we get $(Q' \cup (\cup\mathcal{Q})) \cup (\cup\mathcal{R}) \in \mathcal{I}$, that is, $(\cup\mathcal{Q}) \cup (\cup\mathcal{R}) \in \mathcal{I}$. Suppose for a contradiction that $(\cup\mathcal{Q}) \cup (\cup\mathcal{R}) \neq P(\mathcal{I})$. As previously for $\cup\mathcal{Q}$, there is $Q' \in \mathcal{I}/P(\mathcal{I})$ such that $Q' \cap ((\cup\mathcal{Q}) \cup (\cup\mathcal{R})) \neq \varnothing$, $Q' \setminus ((\cup\mathcal{Q}) \cup (\cup\mathcal{R})) \neq \varnothing$ and $((\cup\mathcal{Q}) \cup (\cup\mathcal{R})) \setminus Q' \neq \varnothing$. We have $Q' \cap (\cup\mathcal{R}) \neq \varnothing$; otherwise $Q' \cup (\cup\mathcal{Q}) \in \mathcal{I}/P(\mathcal{I})$, with $\cup\mathcal{Q} \subset Q' \cup (\cup\mathcal{Q})$ and $X \notin Q' \cup (\cup\mathcal{Q})$. Similarly, we have $Q' \cap (\cup\mathcal{Q}) \neq \varnothing$; otherwise $Q' \cup (\cup\mathcal{R}) \in \mathcal{I}/P(\mathcal{I})$, with $\cup\mathcal{R} \subset Q' \cup (\cup\mathcal{R})$ and $(\cup\mathcal{Q}) \cap (Q' \cup (\cup\mathcal{R})) = \varnothing$. But, since $((\cup\mathcal{Q}) \cup (\cup\mathcal{R})) \setminus Q' \neq \varnothing$, we get $(\cup\mathcal{R}) \setminus Q' \neq \varnothing$ or $(\cup\mathcal{Q}) \setminus Q' \neq \varnothing$. In the first instance, $Q' \setminus (\cup\mathcal{R}) \in \mathcal{I}/P(\mathcal{I})$. As $Q' \cap (\cup\mathcal{Q}) \neq \varnothing$, $(Q' \setminus (\cup\mathcal{R})) \cap (\cup\mathcal{Q}) \neq \varnothing$; which leads to the following contradiction: $(Q' \setminus (\cup\mathcal{R})) \cup (\cup\mathcal{Q}) \in \mathcal{I}/P(\mathcal{I})$, with $\cup\mathcal{Q} \subset (Q' \setminus (\cup\mathcal{R})) \cup (\cup\mathcal{Q})$ and $X \notin (Q' \setminus (\cup\mathcal{R})) \cup (\cup\mathcal{Q})$. In the second instance, we also obtain a contradiction in a similar way. Consequently $(\cup\mathcal{Q}) \cup (\cup\mathcal{R}) = P(\mathcal{I})$. Finally, $\cup(\cup\mathcal{Q})$ and $\cup(\cup\mathcal{R})$ are non-empty elements of $\mathcal{I}$ such that $(\cup(\cup\mathcal{Q})) \cup (\cup(\cup\mathcal{R})) = S$. Furthermore, since $|Q| \geq 2$, we have $|\cup\mathcal{Q}| \geq 2$. As $\cup\mathcal{Q} \in \mathcal{I}/P(\mathcal{I})$ and $\cup\mathcal{Q} \neq P(\mathcal{I})$, we obtain that $\lambda(\mathcal{I}/P(\mathcal{I})) \neq \mathrm{i}$. Consider $Y \neq Z \in \cup\mathcal{Q}$. Firstly, if $\lambda(\mathcal{I}/P(\mathcal{I})) = \mathrm{c}$, then $X \cup Y \in \mathcal{I}$ such that $X \subseteq (X \cup Y) \setminus (\cup(\cup\mathcal{Q}))$, $Y \subseteq (X \cup Y) \cap (\cup(\cup\mathcal{Q}))$ and $Z \subseteq (\cup(\cup\mathcal{Q})) \setminus (X \cup Y)$. Secondly, if $\lambda(\mathcal{I}/P(\mathcal{I})) = \mathrm{t}$, then assume that $Y < Z$ modulo $T(\mathcal{I}/P(\mathcal{I}))$. As $\cup\mathcal{Q} \in \mathcal{I}/P(\mathcal{I})$, $\cup\mathcal{Q}$ is an interval of $T(\mathcal{I}/P(\mathcal{I}))$ and hence either $X < Y < Z$ modulo $T(\mathcal{I}/P(\mathcal{I}))$ or $Y < Z < X$ modulo $T(\mathcal{I}/P(\mathcal{I}))$. For example, assume that the first instance holds. Denote by $[X, Y]$ the intersection of the elements of $(\mathcal{I}/P(\mathcal{I}))_{/\supseteq\{X,Y\}}$. By Assertion A2, $[X, Y] \in \mathcal{I}/P(\mathcal{I})$. Moreover, $X \subseteq [X, Y] \setminus (\cup(\cup\mathcal{Q}))$, $Y \subseteq [X, Y] \cap (\cup(\cup\mathcal{Q}))$ and $Z \subseteq (\cup(\cup\mathcal{Q})) \setminus [X, Y]$. In both cases, we conclude that $\cup(\cup\mathcal{Q}) \notin \mathcal{S}(\mathcal{I})$.

Conversely, assume that there exists a non-empty proper subset C of S such that $\{C, S \setminus C\}$ is an $\mathcal{I}$-partition and not an $\mathcal{S}(\mathcal{I})$-partition. By Lemma 4.1 applied to C and $S \setminus C$, $\overline{C \cup (S \setminus C)} = S$ is not a limit, that is, $\mathcal{I}$ is not a limit. Without loss of generality, assume that $C \notin \mathcal{S}(\mathcal{I})$. There is $Y \in \mathcal{I}$ such that $C \cap Y \neq \varnothing$, $C \setminus Y \neq \varnothing$ and $Y \setminus C \neq \varnothing$. Furthermore, for each $Z \in P(\mathcal{I})$, either $Z \cap C = \varnothing$ or $Z \subseteq C$. Thus $C = \cup(P(\mathcal{I})_{/\subseteq C})$. Therefore $P(\mathcal{I})_{/\subseteq C} \neq P(\mathcal{I})$ and $P(\mathcal{I})_{/\subseteq C} \in \mathcal{I}/P(\mathcal{I})$. Lastly, there are $Z, Z' \in P(\mathcal{I})$ such that $Z \cap (C \cap Y) \neq \varnothing$ and $Z' \cap (C \setminus Y) \neq \varnothing$. Suppose for a contradiction that $Z = Z'$. Since $Z \subseteq C \cap Y$ or $C \cap Y \subseteq Z$ and since $Z \cap (C \setminus Y) \neq \varnothing$, we have $C \cap Y \subseteq Z$. Moreover, as $Y \setminus C \neq \varnothing$, we get $C \setminus Y \in \mathcal{I}$. Since $C \setminus Y \subseteq Z$ or $Z \subseteq C \setminus Y$, and since $Z \cap (C \cap Y) \neq \varnothing$, we obtain that $C \setminus Y \subseteq Z$. Thus $C \subseteq Z$. As previously observed, either $Z \cap C = \varnothing$ or $Z \subseteq C$. It would follow that $C = Z$, which contradicts $C \notin \mathcal{S}(\mathcal{I})$. Consequently, $Z \neq Z'$ and, by the previous observation, $Z \subseteq C$ and $Z' \subseteq C$. It follows that $|P(\mathcal{I})_{/\subseteq C}| \geq 2$ and hence $\mathcal{I}/P(\mathcal{I})$ is not trivial. □

Proposition 4.4 leads us to the following definition. Given a weakly partitive family $\mathcal{I}$ on a set S, $X \subseteq S$ is a *cut* of $\mathcal{I}$ if $X \in \mathcal{I}$ and $S \setminus X \in \mathcal{I}$. For convenience, the family of the cuts of $\mathcal{I}$ is denoted by $\mathcal{C}(\mathcal{I})$. We introduce the following equivalence relation on S. Given $x, y \in S$, $x \sim_{\mathcal{C}(\mathcal{I})} y$ if for each $C \in \mathcal{C}(\mathcal{I})$, either $x, y \in C$ or $x, y \in S \setminus C$.

Proposition 4.5. *Given a weakly partitive family $\mathcal{I}$ on a set S, each equivalence class of $\sim_{\mathcal{C}(\mathcal{I})}$ is a strong element of $\mathcal{I}$.*

Proof. Let E be an equivalence class of $\sim_{\mathcal{C}(\mathcal{I})}$. Given $e \in E$, since E is the intersection of the cuts containing e, $E \in \mathcal{I}$. For a contradiction, suppose that there exists $X \in \mathcal{I}$ such that there are $a \in E \cap X$, $b \in E \setminus X$ and $x \in X \setminus E$. As a and x are not equivalent modulo $\sim_{\mathcal{C}(\mathcal{I})}$, there exists $C \in \mathcal{C}(\mathcal{I})$ such that $x \in C$ and $a \in S \setminus C$. We have $E \cap C = \varnothing$ because E is an equivalence class of $\sim_{\mathcal{C}(\mathcal{I})}$. To obtain a contradiction, it suffices to prove that $C \cup X \in \mathcal{C}(\mathcal{I})$ because we would then have $a \in C \cup X$ and

$b \in S \setminus (C \cup X)$. We have $C \cap X \neq \varnothing$ and $X \setminus (S \setminus C) \neq \varnothing$ because $x \in C \cap X$. Thus $C \cup X \in \mathcal{I}$ and $(S \setminus C) \setminus X = S \setminus (C \cup X) \in \mathcal{I}$. Therefore $(S \setminus C) \setminus X = S \setminus (C \cup X) \in \mathcal{I}$. □

Lemma 4.6. *Consider a weakly partitive family $\mathcal{I}$ on a set S, with $|S| \geq 2$, such that $\mathcal{I}$ is not trivial and $\mathcal{S}(\mathcal{I})$ is trivial. Let $a \neq b \in S$. For any $x \neq y \in S$, we have $(x, y) \leftrightsquigarrow_{\mathcal{I}} (a, b)$ or $(y, x) \leftrightsquigarrow_{\mathcal{I}} (a, b)$.*

Proof. Since $|S| \geq 2$ and $\mathcal{S}(\mathcal{I})$ is trivial, we have $P(\mathcal{I}) = \{\{x\} : x \in S\}$, and hence $\mathcal{I}$ is not a limit. Therefore, $\mathcal{I}/P(\mathcal{I})$ is not trivial because $\mathcal{I}$ is not. By Proposition 4.4, there exists $C \in \mathcal{C}(\mathcal{I})$ such that $C \neq \varnothing$ and $C \neq S$. Consequently, for every equivalence class E of $\sim_{\mathcal{C}(\mathcal{I})}$, $E \neq S$. Furthermore, by Proposition 4.5, E is a strong element of $\mathcal{I}$. Since $\mathcal{S}(\mathcal{I})$ is trivial, $|E| = 1$. It follows that there is $C \in \mathcal{C}(\mathcal{I})$ such that $a \in C$ and $b \in S \setminus C$. Similarly, for any $x \neq y \in S$, there is $D \in \mathcal{C}(\mathcal{I})$ such that $x \in D$ and $y \in S \setminus D$. If $x \in C$ and $y \in S \setminus C$, then $(x, y) \leftrightsquigarrow_{\mathcal{I}} (a, b)$. Similarly, if $y \in C$ and $x \in S \setminus C$, then $(y, x) \leftrightsquigarrow_{\mathcal{I}} (a, b)$. So assume that either $x, y \in C$ or $x, y \in S \setminus C$. In the same way, assume that $a, b \in D$ or $a, b \in S \setminus D$. For instance, assume that $x, y \in C$ and $a, b \in D$. As $b, x \in D$ and $y \in S \setminus D$, $(y, x) \vee_{\mathcal{I}} (y, b)$. As $a, y \in C$ and $b \in S \setminus C$, $(y, b) \vee_{\mathcal{I}} (a, b)$. For the other three cases, we proceed in the same manner by interchanging a and b and by interchanging C and $S \setminus C$, and similarly for x, y and $D, S \setminus D$ if necessary. □

Consider a weakly partitive family $\mathcal{I}$ on a set S. Given $a \neq b \in S$, $D_{(a,b)}$ denotes the directed graph $(S, [(a, b)]_{\mathcal{I}})$. Given distinct elements $a_1, \ldots, a_n$ of S, where $n \geq 2$, recall that the sequence $(a_1, \ldots, a_n, a_{n+1} = a_1)$ is a *circuit* of $D_{(a,b)}$ of length n when $(a_1, a_2), \ldots, (a_n, a_1) \in [(a, b)]_{\mathcal{I}}$.

Proposition 4.7. *Consider a weakly partitive family $\mathcal{I}$ on a set S, with $|S| \geq 2$, such that $\mathcal{I}$ is not trivial and $\mathcal{S}(\mathcal{I})$ is trivial. The following assertions are equivalent.*

(1) *$\mathcal{I}$ is complete.*
(2) *$\leftrightsquigarrow_{\mathcal{I}}$ admits a unique equivalence class.*
(3) *There are $a \neq b \in S$ such that $D_{(a,b)}$ contains a circuit.*

Proof. Obviously, the first assertion implies the second. Conversely, consider any $V \subseteq S$. By Assertion A6, it suffices to verify that for any $a, b \in V$ and $x \in S \setminus V$, there is $X \in \mathcal{I}$ such that $a, b \in X$ and $x \in S \setminus X$, that is, $(a, x) \vee_{\mathcal{I}} (b, x)$. Since $(a, x) \leftrightsquigarrow_{\mathcal{I}} (b, x)$, apply Corollary 3.13.

Clearly, if $\leftrightsquigarrow_{\mathcal{I}}$ admits a unique equivalence class, then $D_{(a,b)}$ contains the circuit (a, b, a) for any $a \neq b \in S$. Conversely, assume that $D_{(a,b)}$ contains a circuit $(a_1, \ldots, a_n, a_{n+1} = a_1)$. Consider a finite set F which is a support of a zigzag modulo $\mathcal{I}$ between (a_i, a_{i+1}) and (a_{i+1}, a_{i+2}) for $1 \leq i \leq n - 1$. By Lemma 3.12, $(a_i, a_{i+1}) \leftrightsquigarrow_{\mathcal{I}_{/\cap F}} (a_{i+1}, a_{i+2})$ for $1 \leq i \leq n - 1$. For convenience, denote $\mathcal{I}_{/\cap F}$ by $\mathcal{J}$ and then denote $\overline{\{a_1, a_2\}}^{\mathcal{J}}$ by X; then $X \in \mathcal{S}(\mathcal{J})$. Furthermore, for $u \in X$, denote by X_u the element of $P(\mathcal{J}_{/\subseteq X})$ containing u. We have $X_u \in S(\mathcal{J}_{/\subseteq X})$. Thus $X_u \in \mathcal{J}_{/\subseteq X}$ and hence $X_u \in \mathcal{J}$. By Proposition 3.10, $a_1, \ldots, a_n \in X$ and $X_{a_i} \neq X_{a_{i+1}}$ for $1 \leq i \leq n-1$. Suppose for a first contradiction that $\lambda(\mathcal{J}_{/\subseteq X}) = \text{i}$. By Proposition 3.10, we then have $X_{a_1} \times X_{a_2} = X_{a_2} \times X_{a_3}$, which implies that $X_{a_1} = X_{a_2}$. Suppose for a second contradiction that $\lambda(\mathcal{J}_{/\subseteq X}) = \text{t}$. Let $T_X = T(\mathcal{J}_{/\subseteq X})$ or $(T(\mathcal{J}_{/\subseteq X}))^\star$ selected so that $X_{a_1} < X_{a_2}$ modulo T_X. By Proposition 3.10, we obtain

$$X_{a_1} < X_{a_2} < \cdots < X_{a_n} < X_{a_{n+1}} = X_{a_1} \text{ modulo } T_X.$$

Consequently $\lambda(\mathcal{J}_{/\subseteq X}) = \text{c}$. For any $u, v \in X$, we have $X_u \cup X_v \in \mathcal{J}_{/\subseteq X}$ and hence $X_u \cup X_v \in \mathcal{J}$. Therefore, we have $(a_1, a_2) \vee_{\mathcal{J}} (a_3, a_2)$ because $a_1, a_3 \in X_{a_1} \cup X_{a_3}$ and $a_2 \in F \setminus (X_{a_1} \cup X_{a_3})$. If $a_3 \in X_{a_1}$, then $(a_3, a_2) \vee_{\mathcal{J}} (a_1, a_2)$ because $a_1, a_3 \in X_{a_1}$ and $a_2 \in F \setminus X_{a_1}$. If $a_3 \in X_{a_1}$, then $(a_3, a_2) \vee_{\mathcal{J}} (a_3, a_1)$ because $a_1, a_2 \in X_{a_1} \cup X_{a_2}$ and $a_3 \in F \setminus (X_{a_1} \cup X_{a_2})$. Furthermore, $(a_3, a_1) \vee_{\mathcal{J}} (a_2, a_1)$ because $a_2, a_3 \in X_{a_2} \cup X_{a_3}$ and $a_1 \in F \setminus (X_{a_2} \cup X_{a_3})$. Consequently, we get $(a_1, a_2) \leftrightsquigarrow_{\mathcal{J}} (a_2, a_1)$, that is, $(a_1, a_2) \leftrightsquigarrow_{\mathcal{I}_{/\cap F}} (a_2, a_1)$. By Lemma 3.11, we have $(a_1, a_2) \leftrightsquigarrow_{\mathcal{I}} (a_2, a_1)$. It follows from Lemma 4.6 that $\leftrightsquigarrow_{\mathcal{I}}$ admits a unique equivalence class. □

Proof of Theorem 3.8 in the infinite case. Let $\mathcal{I}$ be a weakly partitive family on an infinite set S. Obviously, if $\mathcal{I}$ is complete, trivial or totally ordered, then $\mathcal{S}(\mathcal{I})$ is trivial. Conversely, we will prove the following: if $\mathcal{I}$ is not trivial and if $\mathcal{S}(\mathcal{I})$ is trivial, then $\mathcal{I}$ is complete or totally ordered. Consider $a \neq b \in S$. By Proposition 4.7, if $D_{(a,b)}$ contains a circuit, then $\mathcal{I}$ is complete. Otherwise, it follows from Lemma 4.6 that $D_{(a,b)}$ is a total order. Let I be an interval of $D_{(a,b)}$. By Assertion A6, to prove that $I \in \mathcal{I}$, it suffices to verify that for any $u, v \in I$ and $x \in S \setminus I$, there is $X \in \mathcal{I}$ such that $u, v \in X$ and $x \in S \setminus X$, that is, $(u, x) \vee_{\mathcal{I}} (v, x)$. As I is an interval of $D_{(a,b)}$, we obtain that $(u, x) \leftrightsquigarrow_{\mathcal{I}} (v, x)$, and we conclude by Corollary 3.13. Conversely, let $X \in \mathcal{I}$. Consider any $u, v \in X$ and $x \in S \setminus X$. We have $(u, x) \vee_{\mathcal{I}} (v, x)$, and hence either $u < x$ and $v < x$ modulo $D_{(a,b)}$ when $(u, x) \in [(a, b)]_{\mathcal{I}}$, or $x < u$ and $x < v$ modulo $D_{(a,b)}$ when $(x, u) \in [(a, b)]_{\mathcal{I}}$. Consequently, $\mathcal{I}$ is totally ordered by $\{D_{(a,b)}, (D_{(a,b)})^\star\}$. □

Given a weakly partitive family $\mathcal{I}$ on an infinite set S, we define $\lambda(\mathcal{I})$ as in the finite case when $\mathcal{I}$ is not a limit. Furthermore, when $\lambda(\mathcal{I}) = \text{t}$, $T(\mathcal{I})$ still denotes the unique total order up to duality defined on $P(\mathcal{I})$ such that $\mathcal{I}$ is totally ordered by $\{T(\mathcal{I}), (T(\mathcal{I}))^\star\}$.

In the infinite case, Theorem 3.9 becomes

Theorem 4.8. *Consider a weakly partitive family $\mathcal{I}$ on an infinite set S. For every $V \subseteq S$, we have $V \in \mathcal{I}$ if and only if one of the following holds:*

- $V = \varnothing$;
- $V = \{x\}$, *where* $x \in S$;
- $|V| \geq 2$, $\overline{V} \in \mathcal{L}(\mathcal{I})$ *and* $V = \overline{V}$;
- $|V| \geq 2$, $\overline{V} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$ *and there is* $Q_V \subseteq P(\mathcal{I}_{/\subseteq \overline{V}})$ *such that* $V = \cup Q_V$, *and furthermore*

– *if* $\lambda(\mathcal{I}_{/\subseteq\overline{V}}) = \mathrm{i}$, *then* $Q_V = P(\mathcal{I}_{/\subseteq\overline{V}})$*;*
– *if* $\lambda(\mathcal{I}_{/\subseteq\overline{V}}) = \mathrm{t}$, *then* Q_V *is an interval of* $T(\mathcal{I}_{/\subseteq\overline{V}})$*.*

Proof. To begin, consider $X \in \mathcal{I}$ such that $|X| \geq 2$. First, assume that $\overline{X} \in \mathcal{L}(\mathcal{I})$. By Corollary 4.3, $\mathcal{S}(\mathcal{I})_{/\subset\overline{X}}$ is up-directed. Given $a \in X$, for every $Y \in \mathcal{S}(\mathcal{I})_{/\subset\overline{X}}$, there is $Z \in \mathcal{S}(\mathcal{I})_{/\subset\overline{X}}$ such that $Y \cup \{a\} \subseteq Z$. As $Z \in \mathcal{S}(\mathcal{I})$, with $a \in X \cap Z$, we get either $X \subset Z$ or $Z \subseteq X$. Since $Z \subset \overline{X}$, we have $Z \subseteq X$ and hence $Y \subseteq X$. Therefore $\cup(\mathcal{S}(\mathcal{I})_{/\subset\overline{X}}) \subseteq X$. As $\{x\} \in \mathcal{S}(\mathcal{I})_{/\subset\overline{X}}$ for each $x \in \overline{X}$, we obtain that $X = \overline{X}$. Secondly, assume that $\overline{X} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. Denote by Q_X the family of $Y \in P(\mathcal{I}_{/\subseteq\overline{X}})$ such that $Y \cap X \neq \varnothing$. Given $Y \in Q_X$, since $Y \in \mathcal{S}(\mathcal{I}_{/\subseteq\overline{X}})$ and since $X \in \mathcal{I}_{/\subseteq\overline{X}}$, we have either $X \subseteq Y$ or $Y \subset X$. As $Y \subset \overline{X}$, we get $Y \subset X$. Therefore $|Q_X| \geq 2$, $X = \cup Q_X$ and hence $Q_X \in (\mathcal{I}_{/\subseteq\overline{X}})/P(\mathcal{I}_{/\subseteq\overline{X}})$.

Conversely, consider $V \subseteq S$ such $|V| \geq 2$. Obviously, if $V = \overline{V}$, then $V \in \mathcal{I}$. So assume that the last assertion holds. We obtain that $Q_V \in (\mathcal{I}_{/\subseteq\overline{V}})/P(\mathcal{I}_{/\subseteq\overline{V}})$. Thus $V = \cup Q_V \in \mathcal{I}_{/\subseteq\overline{V}}$ and hence $V \in \mathcal{I}$. □

Given a weakly partitive family $\mathcal{I}$ on an infinite set S, it follows from this theorem that the elements of $\mathcal{I}$ are decomposed into a union of elements of

$$\mathcal{D}(\mathcal{I}) = \bigcup_{X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})} \{X\} \cup P(\mathcal{I}_{/\subseteq X}) .$$

Clearly, $\mathcal{D}(\mathcal{I})$ endowed with inclusion constitutes a tree, called the *decomposition tree* of $\mathcal{I}$. The following corollary of Theorem 4.8 ends this section.

Corollary 4.9. *Given weakly partitive families $\mathcal{I}$ and $\mathcal{J}$ on the same infinite set S, we have $\mathcal{I} = \mathcal{J}$ if and only if $\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}) = \mathcal{S}(\mathcal{J}) \setminus \mathcal{L}(\mathcal{J})$ and for each $X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$, $P(\mathcal{I}_{/\subseteq X}) = P(\mathcal{J}_{/\subseteq X})$, $\lambda(\mathcal{I}_{/\subseteq X}) = \lambda(\mathcal{J}_{/\subseteq X})$ and $\{T(\mathcal{I}_{/\subseteq X}), (T(\mathcal{I}_{/\subseteq X}))^\star\} = \{T(\mathcal{J}_{/\subseteq X}), (T(\mathcal{J}_{/\subseteq X}))^\star\}$ when $\lambda(\mathcal{I}_{/\subseteq X}) = \lambda(\mathcal{J}_{/\subseteq X}) = \mathrm{t}$.*

Proof. Consider $I \in \mathcal{I}$ such that $|I| \geq 2$. First, assume that $\overline{I}^{\mathcal{I}} \in \mathcal{L}(\mathcal{I})$. By Theorem 4.8 applied to $\mathcal{I}$, we have $I = \overline{I}^{\mathcal{I}}$ and hence $I \in \mathcal{L}(\mathcal{I})$. It follows from Corollary 4.3, applied to $I \in \mathcal{L}(\mathcal{I})$, that $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset I}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset I}) = I$. As $\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}) = \mathcal{S}(\mathcal{J}) \setminus \mathcal{L}(\mathcal{J})$, we have $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset I} = (\mathcal{S}(\mathcal{J}) \setminus \mathcal{L}(\mathcal{J}))_{/\subset I}$ and hence $(\mathcal{S}(\mathcal{J}) \setminus \mathcal{L}(\mathcal{J}))_{/\subset I}$ is up-directed. By Assertion A5, $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset I}) = I$ belongs to $\mathcal{J}$. Secondly, assume that $\overline{I}^{\mathcal{I}} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. So we have $\overline{I}^{\mathcal{I}} \in \mathcal{S}(\mathcal{J}) \setminus \mathcal{L}(\mathcal{J})$ and $P(\mathcal{I}_{/\subseteq\overline{I}^{\mathcal{I}}}) = P(\mathcal{J}_{/\subseteq\overline{I}^{\mathcal{I}}})$. By Theorem 4.8 applied to $\mathcal{I}$, there is $Q_I \subseteq P(\mathcal{I}_{/\subseteq\overline{I}^{\mathcal{I}}})$ such that $I = \cup Q_I$. By definition of $\overline{I}^{\mathcal{I}}$, $|Q_I| \geq 2$. Recall that $P(\mathcal{J}_{/\subseteq\overline{I}^{\mathcal{I}}})$ is constituted by the maximal elements under inclusion of $\mathcal{S}(\mathcal{J}_{/\subseteq\overline{I}^{\mathcal{I}}}) \setminus \{\varnothing, \overline{I}^{\mathcal{I}}\}$.

Moreover $\mathcal{S}(\mathcal{J}_{/\subseteq \overline{I}^{\mathcal{I}}}) \setminus \{\varnothing, \overline{I}^{\mathcal{I}}\} = \mathcal{S}(\mathcal{J})_{/\subseteq \overline{I}^{\mathcal{I}}} \setminus \{\varnothing, \overline{I}^{\mathcal{I}}\}$. Consequently $\overline{I}^{\mathcal{J}} = \overline{I}^{\mathcal{I}}$. To obtain that $I \in \mathcal{J}$, it suffices to apply the preceding theorem to $\mathcal{J}$ by using the facts that $\lambda(\mathcal{I}_{/\subseteq \overline{I}^{\mathcal{J}}}) = \lambda(\mathcal{J}_{/\subseteq \overline{I}^{\mathcal{J}}})$ and that Q_I is an interval of $T(\mathcal{I}_{/\subseteq \overline{I}^{\mathcal{J}}})$, and hence of $T(\mathcal{J}_{/\subseteq \overline{I}^{\mathcal{J}}})$ when $\lambda(\mathcal{I}_{/\subseteq \overline{I}^{\mathcal{J}}}) = \lambda(\mathcal{J}_{/\subseteq \overline{I}^{\mathcal{J}}}) = \mathrm{t}$. It follows that $\mathcal{I} \subseteq \mathcal{J}$. The opposite inclusion is obtained by interchanging $\mathcal{I}$ and $\mathcal{J}$ in what precedes. $\square$

Theorem 4.8 also allows the extension of Proposition 3.10 to the infinite case.

5. Theorem 1.2 in the infinite case

We say that a binary structure B is a *limit* if $P(B) = \varnothing$. For convenience, denote by $\mathcal{L}(B)$ the family of the strong intervals X of B such that $B[X]$ is a limit.

Observation 5.1. *Consider a binary structure B. Clearly, $\mathcal{S}(B) = \mathcal{S}(\mathcal{I}(B))$. Let $X \in \mathcal{S}(B)$. By Assertion B5 of Proposition 2.2, we have $\mathcal{S}(B[X]) = \mathcal{S}(B)_{/\subseteq X}$. As $\mathcal{S}(B) = \mathcal{S}(\mathcal{I}(B))$, we get $\mathcal{S}(B)_{/\subseteq X} = \mathcal{S}(\mathcal{I}(B))_{/\subseteq X}$. But, by Assertion B5 of Proposition 2.2, we have $\mathcal{S}(\mathcal{I}(B))_{/\subseteq X} = \mathcal{S}(\mathcal{I}(B)_{/\subseteq X})$. It follows that for each $X \in \mathcal{S}(B)$, $P(B[X]) = P(\mathcal{I}(B)_{/\subseteq X})$. Thus $\mathcal{L}(B) = \mathcal{L}(\mathcal{I}(B))$. Lastly, let $X \in \mathcal{S}(B) \setminus \mathcal{L}(B)$. For every $Q \subseteq P(B[X])$, it is easy to verify that Q is an interval of the quotient $B[X]/P(B[X])$ if and only if $\cup Q$ is an interval of $B[X]$. In other words,*

$$\mathcal{I}(B[X]/P(B[X])) = \mathcal{I}(B[X])/P(B[X]).$$

By Proposition 2.1, $\mathcal{I}(B[X]) = \mathcal{I}(B)_{/\subseteq X}$. As $P(B[X]) = P(\mathcal{I}(B)_{/\subseteq X})$, we obtain that $\mathcal{I}(B[X]/P(B[X])) = (\mathcal{I}(B)_{/\subseteq X})/P(\mathcal{I}(B)_{/\subseteq X})$. Therefore, we clearly have that $B[X]/P(B[X])$ is:

- *indecomposable if and only if $(\mathcal{I}(B)_{/\subseteq X})/P(\mathcal{I}(B)_{/\subseteq X})$ is trivial;*
- *constant if and only if $(\mathcal{I}(B)_{/\subseteq X})/P(\mathcal{I}(B)_{/\subseteq X})$ is complete;*
- *totally ordered if and only if $(\mathcal{I}(B)_{/\subseteq X})/P(\mathcal{I}(B)_{/\subseteq X})$ is totally ordered.*

Consequently $\lambda(B[X]) = \lambda(\mathcal{I}(B)_{/\subseteq X})$.

We utilize the following to demonstrate Theorem 1.2 in the infinite case.

Let O be a partial order. A *bicoloring* of O is a function $\mathscr{C} : V(O) \longrightarrow \{0, 1\}$. A subset X of $V(O)$ is *monochromatic* if there is $i \in \{0, 1\}$ such that $\mathscr{C}(x) = i$ for every $x \in X$. With each bicoloring $\mathscr{C}$ of O associate its *complement* $\overline{\mathscr{C}}$ defined by $\overline{\mathscr{C}}(x) = 1 - \mathscr{C}(x)$ for each $x \in V(O)$. A bicoloring $\mathscr{C}$ of O is *dense* provided that for any $x \neq y \in V(O)$, if $x < y$ modulo O and if $\mathscr{C}(x) = \mathscr{C}(y)$, then there is $z \in V(O)$ such that $x < z < y$ modulo O and $\mathscr{C}(z) \neq \mathscr{C}(x)$. For a total order T, we then have: a bicoloring $\mathscr{C}$ of T is *dense* if the only monochromatic intervals of T are the empty set and the singletons.

Proposition 5.2 ([6])**.** (Axiom of Choice) *Every total order admits a dense bicoloring.*

This result easily extends to trees.

Corollary 5.3. (Axiom of Choice) *Every tree admits a dense bicoloring.*

Proof. Consider a tree τ. Using the Axiom of Choice, there exists an ordinal α and an ordinal sequence $(b_\beta)_{\beta<\alpha}$ of all the branches of τ. We will define by transfinite induction a sequence $(\mathscr{C}_\beta)_{\beta<\alpha}$ of bicolorings such that $\mathscr{C}_\beta$ is a dense bicoloring of $\tau[\cup_{\delta\leq\beta} b_\delta]$ for $\beta<\alpha$ and $\mathscr{C}_\gamma$ is a restriction of $\mathscr{C}_\beta$ for $\gamma<\beta<\alpha$. Once again, we use the Axiom of Choice as follows. For each $\beta<\alpha$, associate a dense bicoloring $\mathscr{D}_\beta$ of $\tau[b_\beta]$. Set $\mathscr{C}_0=\mathscr{D}_0$. Now, given $0<\beta<\alpha$, assume that the bicolorings $(\mathscr{C}_\gamma)_{\gamma<\beta}$ are well defined. The bicolorings $(\mathscr{C}_\gamma)_{\gamma<\beta}$ admit a common extension, denoted by $\cup_{\gamma<\beta}\mathscr{C}_\gamma$, which is a dense bicoloring of $\tau[\cup_{\gamma<\beta} b_\gamma]$. If $b_\beta\subseteq\cup_{\gamma<\beta} b_\gamma$, then set $\mathscr{C}_\beta=\cup_{\gamma\ <\beta}\mathscr{C}_\gamma$. Now assume that $b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma)\neq\varnothing$. As τ is connected, consider a shortest sequence $x_0,\dots,x_n$ of vertices of τ satisfying $x_0\in\cup_{\gamma<\beta} b_\gamma$, $x_n\in b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma)$, and for $0\leq i\leq n-1$, either $(x_i,x_{i+1})\in A(\tau)$ or $(x_{i+1},x_i)\in A(\tau)$. For a contradiction, suppose that $n\geq 2$. Let $0\leq i\leq n-2$. Since n is the smallest for such a sequence, we have $(x_i,x_{i+2})\notin A(\tau)$. As τ is a tree, we get $x_i<x_{i+1}$ and $x_{i+2}<x_{i+1}$ modulo τ. It follows that $n=2$ and $x_0<x_1$ and $x_2<x_1$ modulo τ. But, since $x_0\in\cup_{\gamma<\beta} b_\gamma$, we have $x_1\in\cup_{\gamma<\beta} b_\gamma$. So we could have considered the sequence (x_1,x_2) instead of (x_0,x_1,x_2). Consequently $n=1$. As previously observed, if $x_0<x_1$ modulo τ, then $x_1\in\cup_{\gamma<\beta} b_\gamma$. Thus $x_1<x_0$ and hence $x_0\in b_\beta\cap(\cup_{\gamma<\beta} b_\gamma)$. For $x\in b_\beta\cap(\cup_{\gamma<\beta} b_\gamma)$ and $y\in V(\tau)$, if $x<y$ modulo τ, then $y\in b_\beta\cap(\cup_{\gamma<\beta} b_\gamma)$. Therefore, for $x\in b_\beta\cap(\cup_{\gamma<\beta} b_\gamma)$ and $y\in b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma)$, we have $y<x$ modulo τ. If $\tau[b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma)]$ does not possess a biggest element or if $\tau[b_\beta\cap(\cup_{\gamma<\beta} b_\gamma)]$ does not possess a smallest element, then choose for $\mathscr{C}_\beta$ the common extension of $\cup_{\gamma<\beta}\mathscr{C}_\gamma$ and of $\mathscr{D}_{\beta|(b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma))}$. So assume that $\tau[b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma)]$ admits a biggest element denoted by M and $\tau[b_\beta\cap(\cup_{\gamma<\beta} b_\gamma)]$ admits a smallest element denoted by m. If $(\cup_{\gamma<\beta}\mathscr{C}_\gamma)(m)\neq\mathscr{D}_{\beta|(b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma))}(M)$, then we choose for $\mathscr{C}_\beta$ the common extension of $\cup_{\gamma<\beta}\mathscr{C}_\gamma$ and of $\mathscr{D}_{\beta|(b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma))}$ as well. If $(\cup_{\gamma<\beta}\mathscr{C}_\gamma)(m)=\mathscr{D}_{\beta|(b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma))}(M)$, then $\mathscr{C}_\beta$ is the common extension of $\cup_{\gamma<\beta}\mathscr{C}_\gamma$ and of $\overline{\mathscr{D}_\beta}_{|(b_\beta\setminus(\cup_{\gamma<\beta} b_\gamma))}$. In this manner, we complete the definition of the required sequence of bicolorings $(\mathscr{C}_\beta)_{\beta<\alpha}$. Their common extension $\cup_{\beta<\alpha}\mathscr{C}_\beta$ realizes a dense bicoloring of τ. □

The following is deduced from Proposition 5.2 as well.

Corollary 5.4. (Axiom of Choice) *For every set S, there exists a binary structure B such that $\underline{B}=S$, $\mathrm{rk}(B)=3$ and B is indecomposable.*

Proof. By the Ultrafilter Axiom, there exists a total order T defined on $V(T)=S$. By Proposition 5.2, T admits a dense coloring $\mathscr{C}$. Define B as follows. Given $x\neq y\in S$, $B(x,y)=0$ if $x<y$ modulo T and if $\mathscr{C}(x)=\mathscr{C}(y)$; $B(x,y)=1$ if $x<y$ modulo T and if $\mathscr{C}(x)\neq\mathscr{C}(y)$; otherwise,

$B(x,y)=2$. For every proper subset X of S, we have X is an interval of B if and only if X is a monochromatic interval of T. Since $\mathscr{C}$ is dense, B does not have a non-trivial interval. □

Proof of Theorem 1.2 in the general case. (Using the Axiom of Choice)

Consider a weakly partitive family $\mathcal{I}$ on a set S, with $|S|\geq 2$. The family $\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$ endowed with inclusion constitutes a tree. By Corollary 5.3, it admits a dense bicoloring $\mathscr{C}$. Let $X\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$. We associate with X the binary structure B_X defined on $P(\mathcal{I}_{/\subseteq X})$ by distinguishing the three cases below.

Case 1: $\lambda(\mathcal{I}_{/\subseteq X})=$ c.
The binary structure B_X is constant, and is defined by: for any $Y\neq Z\in P(\mathcal{I}_{/\subseteq X})$, $B_X(Y,Z)=\mathscr{C}(X)$.

Case 2: $\lambda(\mathcal{I}_{/\subseteq X})=$ i.
Using the preceding corollary, we choose for B_X an indecomposable binary structure defined on $P(\mathcal{I}_{/\subseteq X})$ of rank 3.

Case 3: $\lambda(\mathcal{I}_{/\subseteq X})=$ t.
There is a total order $T(\mathcal{I}_{/\subseteq X})$ such that $(\mathcal{I}_{/\subseteq X})/P(\mathcal{I}_{/\subseteq X})$ is totally ordered by $\{T(\mathcal{I}_{/\subseteq X}),(T(\mathcal{I}_{/\subseteq X}))^\star\}$. Recall that $T(\mathcal{I}_{/\subseteq X})$ is identified with the binary structure $B_{T(\mathcal{I}_{/\subseteq X})}$ of rank 2 defined on $P(\mathcal{I}_{/\subseteq X})$. We define B_X as follows: for any $Y\neq Z\in P(\mathcal{I}_{/\subseteq X})$, $B_X(Y,Z)=B_{T(\mathcal{I}_{/\subseteq X})}(Y,Z)+\mathscr{C}(X)$. Thus, B_X is totally ordered by $\{0,1\}$ if $\mathscr{C}(X)=0$ and by $\{1,2\}$ if $\mathscr{C}(X)=1$.

Now we define a binary structure B of rank 3 on S as follows. Let $a\neq b\in S$. By Lemma 4.1 applied to $\{a\}$ and $\{b\}$, $\overline{\{a,b\}}^{\mathcal{I}}\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$. For $x\in\overline{\{a,b\}}^{\mathcal{I}}$, denote by $\overline{\{a,b\}}^{\mathcal{I}}_x$ the element of $P(\mathcal{I}_{/\subseteq\overline{\{a,b\}}^{\mathcal{I}}})$ which contains x. Lastly, set $B(a,b)=B_{(\overline{\{a,b\}}^{\mathcal{I}})}(\overline{\{a,b\}}^{\mathcal{I}}_a,\overline{\{a,b\}}^{\mathcal{I}}_b)$. To prove that $\mathcal{I}(B)=\mathcal{I}$, we establish the next claims. The first one follows directly from the definition of B_X, where $X\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$, and of B.

Claim 5.5. *Let $X\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$.*

- $\mathcal{I}(B_X)=(\mathcal{I}_{/\subseteq X})/P(\mathcal{I}_{/\subseteq X})$.
- *For every $I\in\mathcal{I}(B[X])$, we have*
$$\{Y\in P(\mathcal{I}_{/\subseteq X}):\ Y\cap I\neq\varnothing\}\in\mathcal{I}(B_X).$$
- *For every $Q\in\mathcal{I}(B_X)$, $\cup Q\in\mathcal{I}(B[X])$.*

Claim 5.6. *$\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})\subseteq\mathcal{I}(B)$ and for every $X\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$, we have $P(\mathcal{I}_{/\subseteq X})\subseteq\mathcal{I}(B)$.*

Proof. Given $X\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$, consider $a,b\in X$ and $x\in S\setminus X$. Clearly, $X\subset\overline{\{a,b,x\}}^{\mathcal{I}}$, and $\overline{\{a,b,x\}}^{\mathcal{I}}\in\mathcal{S}(\mathcal{I})\setminus\mathcal{L}(\mathcal{I})$ by Lemma 4.1. By Assertion B5 of Proposition 3.3, $X\in\mathcal{S}(\mathcal{I}_{/\subseteq\overline{\{a,b,x\}}^{\mathcal{I}}})\setminus\{\overline{\{a,b,x\}}^{\mathcal{I}}\}$. It follows from Lemma 3.6

applied to $\mathcal{I}_{/\subseteq\overline{\{a,b,x\}}^{\mathcal{I}}}$ that there is $Y \in P(\mathcal{I}_{/\subseteq\overline{\{a,b,x\}}^{\mathcal{I}}})$ such that $X \subseteq Y$. As $a, b \in Y$ and $Y \subset \overline{\{a,b,x\}}^{\mathcal{I}}$, we have $x \notin Y$. Thus, there is $Z \in P(\mathcal{I}_{/\subseteq\overline{\{a,b,x\}}^{\mathcal{I}}}) \setminus \{Y\}$ such that $x \in Z$. We also deduced that $\overline{\{a,x\}}^{\mathcal{I}} = \overline{\{b,x\}}^{\mathcal{I}} = \overline{\{a,b,x\}}^{\mathcal{I}}$. So we have

$$B(a,x) = B_{(\overline{\{a,b,x\}}^{\mathcal{I}})}(Y,Z) = B(b,x)$$

and

$$B(x,a) = B_{(\overline{\{a,b,x\}}^{\mathcal{I}})}(Z,Y) = B(x,b).$$

Consequently $X \in \mathcal{I}(B)$.

Given $X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$, consider $Y \in P(\mathcal{I}_{/\subseteq X})$. Since $X \in \mathcal{I}(B)$, it follows from Proposition 2.1 that it suffices to verify that Y is an interval of $B[X]$. So consider $a, b \in Y$ and $x \in X \setminus Y$. As Y is a maximal element under inclusion of $\mathcal{S}(\mathcal{I}_{/\subseteq X}) \setminus \{\varnothing, X\}$ and as $\mathcal{S}(\mathcal{I}_{/\subseteq X}) = \mathcal{S}(\mathcal{I})_{/\subseteq X}$ by Assertion B5 of Proposition 3.3, we have $\overline{\{a,x\}}^{\mathcal{I}} = \overline{\{b,x\}}^{\mathcal{I}} = X$. Denote by Z the element of $P(\mathcal{I}_{/\subseteq X}) \setminus \{Y\}$ which contains x. We get $B(a,x) = B_X(Y,Z) = B(b,x)$ and $B(x,a) = B_X(Z,Y) = B(x,b)$. Consequently Y is an interval of $B[X]$. □

Claim 5.7. *Let $X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. Given an interval I of $B[X]$, denote by Q_I the family of $Y \in P(\mathcal{I}_{/\subseteq X})$ such that $Y \cap I \neq \varnothing$. If $|Q_I| \geq 2$, then $I = \cup Q_I$.*

Proof. By contradiction.

Suppose that there is $Y \in Q_I$ such that $Y \setminus I \neq \varnothing$. Consider $Z \in Q_I \setminus \{Y\}$ and elements $a \in I \cap Y$, $b \in I \cap Z$ and $y \in Y \setminus I$. Clearly, $\overline{\{y,b\}}^{\mathcal{I}} = X$ and hence $B(y,b) = B_X(Y,Z)$ and $B(b,y) = B_X(Z,Y)$. Moreover, by Claim 5.5, Q_I is an interval of B_X. Thus, if $Q_I \neq P(\mathcal{I}_{/\subseteq X})$, then $\lambda(\mathcal{I}_{/\subseteq X}) \neq$ i. So assume that $Q_I = P(\mathcal{I}_{/\subseteq X})$. By Claim 5.6, Y is an interval of $B[X]$. Since $Y \setminus I \neq \varnothing$, we have $I \setminus Y$ is an interval of $B[X]$. By Claim 5.5, $P(\mathcal{I}_{/\subseteq X}) \setminus \{Y\}$ is an interval of B_X and hence $\lambda(\mathcal{I}_{/\subseteq X}) \neq$ i as well.

Clearly, $\overline{\{a,y\}}^{\mathcal{I}} \subseteq Y$ and, by Lemma 4.1, $\overline{\{a,y\}}^{\mathcal{I}} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. We will prove that for every $U \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$, if $\overline{\{a,y\}}^{\mathcal{I}} \subseteq U \subset X$, then $\lambda(\mathcal{I}_{/\subseteq U}) \neq$ i. By Lemma 3.6, $U \subseteq Y$. For every $u \in U$, denote by U_u the element of $P(\mathcal{I}_{/\subseteq U})$ which contains u. We have $U_a \in \{Y' \in P(\mathcal{I}_{/\subseteq U}) :\ Y' \cap I \neq \varnothing\}$. By Claim 5.6, U is an interval of B and hence $U \cap I$ is an interval of $B[U]$. By Claim 5.5 applied to U, $\{Y' \in P(\mathcal{I}_{/\subseteq U}) :\ Y' \cap I \neq \varnothing\}$ is an interval of B_U. Thus, if $|\{Y' \in P(\mathcal{I}_{/\subseteq U}) :\ Y' \cap I \neq \varnothing\}| \geq 2$ and if $\{Y' \in P(\mathcal{I}_{/\subseteq U}) :\ Y' \cap I \neq \varnothing\} \neq P(\mathcal{I}_{/\subseteq U})$, then $\lambda(\mathcal{I}_{/\subseteq U}) \neq$ i. By distinguishing the two cases below, we will show that we always have $\lambda(\mathcal{I}_{/\subseteq U}) \neq$ i. First, assume that $|\{Y' \in P(\mathcal{I}_{/\subseteq U}) :\ Y' \cap I \neq \varnothing\}| = 1$, that is, $\{Y' \in P(\mathcal{I}_{/\subseteq U}) :\ Y' \cap I \neq \varnothing\} = \{U_a\}$. By Claim 5.6, U_a is an interval of B and hence $I \cup U_a$ is an interval of $B[X]$. Since $b \in (I \cup U_a) \setminus U$, we obtain that $U \setminus (I \cup U_a) = U \setminus U_a$ is an interval of

$B[U]$. It follows from Claim 5.5 that $P(\mathcal{I}_{/\subseteq U}) \setminus \{U_a\}$ is an interval of B_U. Therefore $\lambda(\mathcal{I}_{/\subseteq U}) \neq$ i. Second, assume that $\{Y' \in P(\mathcal{I}_{/\subseteq U}) : Y' \cap I \neq \varnothing\} = P(\mathcal{I}_{/\subseteq U})$. As previously observed, $U \cap I$ is an interval of $B[U]$. Since $y \in U_y \setminus (U \cap I)$, we obtain that $(U \cap I) \setminus U_y$ is an interval of $B[U]$. By Claim 5.5, $P(\mathcal{I}_{/\subseteq U}) \setminus \{U_y\}$ is an interval of B_U and hence $\lambda(\mathcal{I}_{/\subseteq U}) \neq$ i.

Now, we will establish that $\lambda(\mathcal{I}_{/\subseteq \overline{\{a,y\}}^{\mathcal{I}}}) = \lambda(\mathcal{I}_{/\subseteq X})$ and $\mathscr{C}(\overline{\{a,y\}}^{\mathcal{I}}) = \mathscr{C}(X)$. For $U = X$ or $\overline{\{a,y\}}^{\mathcal{I}}$, we proved that B_U is either constant or totally ordered. Thus, given $W \neq W' \in P(\mathcal{I}_{/\subseteq U})$, we have B_U is constant if and only if $B_U(W, W') = B_U(W', W)$. We also have B_U is totally ordered if and only if $B_U(W, W') \neq B_U(W', W)$. Furthermore, if B_U is constant then $\mathscr{C}(U) = B_U(W, W')$, and if B_U is totally ordered, then $\mathscr{C}(U) = \min(B_U(W, W'), B_U(W', W))$. Consequently, it suffices to find $X' \neq X'' \in P(\mathcal{I}_{/\subseteq X})$ and $Y' \neq Y'' \in P(\mathcal{I}_{/\subseteq \overline{\{a,y\}}^{\mathcal{I}}})$ such that

$$\{B_X(X', X''), B_X(X'', X')\} = \{B_{\overline{\{a,y\}}^{\mathcal{I}}}(Y', Y''), B_{\overline{\{a,y\}}^{\mathcal{I}}}(Y'', Y')\}.$$

We already obtained that $B(y, b) = B_X(Y, Z)$ and $B(b, y) = B_X(Z, Y)$. Since I is an interval of $B[X]$, $B(y, a) = B(y, b)$ and $B(a, y) = B(b, y)$. But $B(y, a) = B_{\overline{\{a,y\}}^{\mathcal{I}}}(\overline{\{a,y\}}^{\mathcal{I}}_y, \overline{\{a,y\}}^{\mathcal{I}}_a)$ and $B(a, y) = B_{\overline{\{a,y\}}^{\mathcal{I}}}(\overline{\{a,y\}}^{\mathcal{I}}_a, \overline{\{a,y\}}^{\mathcal{I}}_y)$. Therefore

$$\{B_X(Y, Z), B_X(Z, Y)\} = \{B_{\overline{\{a,y\}}^{\mathcal{I}}}(\overline{\{a,y\}}^{\mathcal{I}}_y, \overline{\{a,y\}}^{\mathcal{I}}_a), B_{\overline{\{a,y\}}^{\mathcal{I}}}(\overline{\{a,y\}}^{\mathcal{I}}_a, \overline{\{a,y\}}^{\mathcal{I}}_y)\}.$$

Finally, to obtain a contradiction, we will show that the bicoloring $\mathscr{C}$ is not dense. In fact, we will verify that for every $U \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$, if $\overline{\{a,y\}}^{\mathcal{I}} \subseteq U \subseteq X$, then $\mathscr{C}(U) = \mathscr{C}(X)$. Let $U \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$ be such that $\overline{\{a,y\}}^{\mathcal{I}} \subset U \subset X$. For every $u \in U$, denote by U_u the element of $P(\mathcal{I}_{/\subseteq U})$ which contains u. By Lemma 3.6, $\overline{\{a,y\}}^{\mathcal{I}} \subseteq U_a = U_y$ and $U \subseteq Y$. Let $u \in U \setminus U_a$. First, assume that $u \in I$. Since I is an interval of $B[X]$, we have $B(u, y) = B(b, y)$ and $B(y, u) = B(y, b)$. Thus

$$\{B_U(U_u, U_y), B_U(U_y, U_u)\} = \{B_X(Y, Z), B_X(Z, Y)\}.$$

Second, assume that $u \in U \setminus I$. As I is an interval of $B[X]$, we have $B(u, a) = B(u, b)$ and $B(a, u) = B(b, u)$. Moreover, since $U \subseteq Y$, we have $B(u, b) = B_X(Y, Z)$ and $B(b, u) = B_X(Z, Y)$. So

$$\{B_U(U_u, U_a), B_U(U_a, U_u)\} = \{B_X(Y, Z), B_X(Z, Y)\}.$$

□

Claim 5.8. *$\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}) \subseteq \mathcal{S}(B)$ and for every $X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$, we have $P(\mathcal{I}_{/\subseteq X}) \subseteq \mathcal{S}(B)$.*

Proof. Let $X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. By Claim 5.6, X is an interval of B. So consider an interval Y of B such that $Y \setminus X \neq \varnothing$ and $Y \cap X \neq \varnothing$. We have to show that $X \subseteq Y$. Let $a \in Y \setminus X$ and $b \in Y \cap X$. By Lemma 4.1, $\overline{\{a,b\}}^{\mathcal{I}} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. Since $b \in X \cap \overline{\{a,b\}}^{\mathcal{I}}$, we have either $X \subset \overline{\{a,b\}}^{\mathcal{I}}$ or $\overline{\{a,b\}}^{\mathcal{I}} \subseteq X$. We get $X \subset \overline{\{a,b\}}^{\mathcal{I}}$ because $a \notin X$. By Lemma 3.6, there is $Z_b \in P(\mathcal{I}_{/\subseteq \overline{\{a,b\}}^{\mathcal{I}}})$ such that $X \subseteq Z_b$. As $b \in Z_b$ and $Z_b \subset \overline{\{a,b\}}^{\mathcal{I}}$, $a \notin Z_b$. Denote by Z_a the element of $P(\mathcal{I}_{/\subseteq \overline{\{a,b\}}^{\mathcal{I}}})$ which contains a. By Claim 5.6, $\overline{\{a,b\}}^{\mathcal{I}}$ is an interval of B and hence $Y \cap \overline{\{a,b\}}^{\mathcal{I}}$ is an interval of $B[\overline{\{a,b\}}^{\mathcal{I}}]$. Denote by Q the family of elements Z of $P(\mathcal{I}_{/\subseteq \overline{\{a,b\}}^{\mathcal{I}}})$ such that $Z \cap (Y \cap \overline{\{a,b\}}^{\mathcal{I}}) \neq \varnothing$. We have $|Q| \geq 2$ because $Z_a \neq Z_b \in Q$. It follows from the preceding claim that $Y \cap \overline{\{a,b\}}^{\mathcal{I}} = \cup Q$. Consequently $X \subseteq Z_b \subseteq Y \cap \overline{\{a,b\}}^{\mathcal{I}} \subseteq Y$.

Let $X \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. We showed that $X \in \mathcal{S}(B)$. Thus, it follows from Assertion B5 of Proposition 2.2 that $P(\mathcal{I}_{/\subseteq X}) \subseteq \mathcal{S}(B)$ if and only if $P(\mathcal{I}_{/\subseteq X}) \subseteq \mathcal{S}(B[X])$. But, by Claim 5.7, we have $P(\mathcal{I}_{/\subseteq X}) \subseteq \mathcal{S}(B[X])$. □

Claim 5.9. $\mathcal{I} \subseteq \mathcal{I}(B)$.

Proof. Let $X \in \mathcal{I}$. Firstly, assume that $\overline{X}^{\mathcal{I}} \in \mathcal{L}(\mathcal{I})$. By Theorem 4.8, $X = \overline{X}^{\mathcal{I}}$. By Corollary 4.3, $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}) = X$. By the previous claim, $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X} \subseteq \mathcal{S}(B)$ and $X = \cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset X}) \in \mathcal{S}(B)$ by Assertion B3 of Proposition 2.2.

Secondly, assume that $\overline{X}^{\mathcal{I}} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. By Theorem 4.8, there is $Q_X \in (\mathcal{I}_{/\subseteq \overline{X}^{\mathcal{I}}})/P(\mathcal{I}_{/\subseteq \overline{X}^{\mathcal{I}}})$ such that $X = \cup Q_X$. By Claim 5.5, $Q_X \in \mathcal{I}(B_{(\overline{X}^{\mathcal{I}})})$ and $X = \cup Q_X \in \mathcal{I}(B[\overline{X}^{\mathcal{I}}])$. By Claim 5.6, $\overline{X}^{\mathcal{I}} \in \mathcal{I}(B)$ and hence $X \in \mathcal{I}(B)$ by Proposition 2.1. □

Claim 5.10. $\mathcal{I}(B) \subseteq \mathcal{I}$.

Proof. Let $X \in \mathcal{I}(B)$. To begin, assume that $\overline{X}^{\mathcal{I}} \in \mathcal{L}(\mathcal{I})$. By Corollary 4.3, $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}}) = \overline{X}^{\mathcal{I}}$. By Claim 5.8, $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}} \subseteq \mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}$. We verify that $\mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}$ is up-directed. Indeed, let $Y, Z \in \mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}$. If $Y \cap Z \neq \varnothing$, then $Y \subseteq Z$ or $Z \subseteq Y$. Thus, assume that $Y \cap Z = \varnothing$ and consider $y \in Y$ and $z \in Z$. As $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}}$ is up-directed and $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}}) = \overline{X}^{\mathcal{I}}$, there is $U \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}}$ such that $y, z \in U$. By Claim 5.8, $U \in \mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}$ and hence $X \cup Y \subseteq U$ because $y \in Y \cap U$, $z \in U \setminus Y$, $z \in Z \cap U$ and $y \in U \setminus Z$. Consequently, $\mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}$ is up-directed. By Assertion B3 of Proposition 2.2, $\cup(\mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}) \in \mathcal{S}(B)$. Since $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}} \subseteq \mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}$, we obtain that $\overline{X}^{\mathcal{I}} = \cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset \overline{X}^{\mathcal{I}}}) \subseteq \cup(\mathcal{S}(B)_{/\subset \overline{X}^{\mathcal{I}}}) \subseteq \overline{X}^{\mathcal{I}}$. Therefore $\overline{X}^{\mathcal{I}} \in \mathcal{S}(B)$.

Lastly, we show that $\overline{X}^{\mathcal{I}} = X$. As $\cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset\overline{X}^{\mathcal{I}}}) = \overline{X}^{\mathcal{I}}$, there is $U_0 \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset\overline{X}^{\mathcal{I}}}$ such that $U_0 \cap X \neq \varnothing$. For each $U \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset\overline{X}^{\mathcal{I}}}$, there exists $U' \in (\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset\overline{X}^{\mathcal{I}}}$ such that $U_0 \cup U \subseteq U'$ because $(\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset\overline{X}^{\mathcal{I}}}$ is up-directed. Obviously, $U' \cap X \neq \varnothing$. By Claim 5.8, $U' \in \mathcal{S}(B)_{/\subset\overline{X}^{\mathcal{I}}}$. Since $X \in \mathcal{I}(B)$, we get either $X \subset U'$ or $U' \subseteq X$. In the first instance, we obtain that $X \subset U' \subset \overline{X}^{\mathcal{I}}$, which is impossible because $U' \in \mathcal{S}(\mathcal{I})$. Thus $U' \subseteq X$ and hence $U \subseteq X$. It follows that $\overline{X}^{\mathcal{I}} = \cup((\mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I}))_{/\subset\overline{X}^{\mathcal{I}}}) \subseteq X$ so that $\overline{X}^{\mathcal{I}} = X$.

To finish, assume that $\overline{X}^{\mathcal{I}} \in \mathcal{S}(\mathcal{I}) \setminus \mathcal{L}(\mathcal{I})$. By Claim 5.8, $\overline{X}^{\mathcal{I}} \in \mathcal{S}(B)$ and hence $X \in \mathcal{I}(B[\overline{X}^{\mathcal{I}}])$ by Proposition 2.1. Denote by Q_X the elements Y of $P(\mathcal{I}_{/\subseteq\overline{X}^{\mathcal{I}}})$ such that $Y \cap X \neq \varnothing$. By definition of $\overline{X}^{\mathcal{I}}$, $|Q_X| \geq 2$. By Claim 5.8, $Q_X \subseteq \mathcal{S}(B)$ so that $X = \cup Q_X$. By Claim 5.5, as $\cup Q_X \in \mathcal{I}(B)$, we have $Q_X \in \mathcal{I}(B_{\overline{X}^{\mathcal{I}}})$ and $Q_X \in (\mathcal{I}_{/\subseteq\overline{X}^{\mathcal{I}}})/P(\mathcal{I}_{/\subseteq\overline{X}^{\mathcal{I}}})$ as well. Thus $X = \cup Q_X \in \mathcal{I}_{/\subseteq\overline{X}^{\mathcal{I}}}$. □

6. Another proof of Theorem 3.8 using the Axiom of Choice

We will use the following lemma.

Lemma 6.1 ([6]). *Consider a weakly partitive family $\mathcal{I}$ on a set S. Assume that there is a total order T defined on S such that all the intervals of T belong to $\mathcal{I}$. Then, either $\mathcal{I}$ is complete or $\mathcal{I}$ is totally ordered by $\{T, T^\star\}$.*

Another proof of Theorem 3.8. Let $\mathcal{I}$ be a weakly partitive family on a set S such that $|S| \geq 2$. As in Section 4, we will prove the following: if $\mathcal{I}$ is not trivial and if $\mathcal{S}(\mathcal{I})$ is trivial, then $\mathcal{I}$ is complete or totally ordered. Using Zorn's lemma, consider a maximal family $\mathcal{M}$ under inclusion among the families of cuts of $\mathcal{I}$ which are total orders under inclusion. As $\mathcal{M}$ is maximal, we have $\varnothing, S \in \mathcal{M}$. Furthermore, as seen at the beginning of the proof of Lemma 4.6, there is $C \in \mathcal{C}(I)$ such that $C \neq \varnothing$ and $C \neq S$. Thus $|\mathcal{M}| \geq 3$.

Consider $C \in \mathcal{M}$. Denote $\cup(\mathcal{M}_{/\subset C})$ by C^-. Since $\mathcal{M}_{/\subset C}$ is a total order under inclusion, $C^- \in \mathcal{I}$ by Assertion A5. We have $S \setminus C^- = \cap\{S \setminus D : D \in \mathcal{M}_{/\subset C}\}$ belongs to $\mathcal{I}$ by Assertion A2. Therefore $C^- \in \mathcal{C}(\mathcal{I})$. Clearly, $\mathcal{M} \cup \{C^-\}$ endowed with inclusion is a total order and, $\mathcal{M}$ being maximal for this property, we get $C^- \in \mathcal{M}$. By Assertion A2, $C \setminus C^- \in \mathcal{I}$ because $C \setminus C^- = C \cap (S \setminus C^-)$. Now we show that $C \setminus C^- \in \mathcal{S}(\mathcal{I})$. For a contradiction, suppose that there is $X \in \mathcal{I}$ such that $X \cap (C \setminus C^-) \neq \varnothing$, $X \setminus (C \setminus C^-) \neq \varnothing$ and $(C \setminus C^-) \setminus X \neq \varnothing$. Notice that $\{S \setminus D : D \in \mathcal{M}\}$ is also maximal under inclusion among the families of cuts of $\mathcal{I}$ which are totally ordered by inclusion. Furthermore

$$\cup(\{S \setminus D :\ D \in \mathcal{M}\}_{/\subset (S\setminus C^-)}) = \cap(\{E \in \mathcal{M} :\ C^- \subset E\})$$
$$= \cap(\{E \in \mathcal{M} :\ C \subseteq E\}) = C.$$

By interchanging $\mathcal{M}$ and $\{S \setminus D :\ D \in \mathcal{M}\}$, we can assume that $X \cap C^- \neq \varnothing$. By Assertion A2, $X \cap C \in \mathcal{I}$ and, since $(X \cap C) \cap C^- = X \cap C^-$, $C^- \cup (X \cap C) \in \mathcal{I}$ by Assertion A3. Clearly, $C^- \neq \varnothing$ because $X \cap C^- \neq \varnothing$. By Assertion A4, as $S \setminus C^- \in \mathcal{I}$ and as $(C^- \cup (X \cap C)) \setminus (S \setminus C^-) = C^-$, we have $(S \setminus C^-) \setminus (C^- \cup (X \cap C)) = S \setminus (C^- \cup (X \cap C)) \in \mathcal{I}$. Consequently, $C^- \cup (X \cap C) \in \mathcal{C}(\mathcal{I})$, which is impossible because $C^- \subset C^- \cup (X \cap C) \subset C$. It follows that $C \setminus C^- \in \mathcal{S}(\mathcal{I})$. As $\mathcal{S}(\mathcal{I})$ is trivial and as $\mathcal{M} \setminus \{\varnothing, S\} \neq \varnothing$, we obtain that $|C \setminus C^-| \leq 1$.

For each $x \in S$, set $C_x = \cap(\mathcal{M}_{/\supseteq \{x\}})$. It follows from Assertions A2 and A5 that $C_x \in \mathcal{C}(\mathcal{I})$. Given $C \in \mathcal{M}$, either $x \in C$ and $C_x \subseteq C$ or $x \notin C$. In the last instance, we have $C \subset D$ for every $D \in \mathcal{M}_{/\supseteq\{x\}}$ and hence $C \subseteq C_x$. Therefore, $\mathcal{M} \cup \{C_x\}$ endowed with inclusion is a total order, so that $C_x \in \mathcal{M}$. Furthermore, for every $C \in \mathcal{M}$ such that $C \subset C_x$, we have $x \notin C$. As $(C_x)^- = \cup(\mathcal{M}_{/\subset C_x})$, $x \in C_x \setminus (C_x)^-$. Consequently $C_x \setminus (C_x)^- = \{x\}$. Finally, we define an order T on S as follows. Given $x \neq y \in S$, $x < y$ modulo T if $C_x \subset C_y$. Given an interval I of T, we use Assertion A6 to verify that $I \in \mathcal{I}$. For $x \in S \setminus I$ and for $a \neq b \in I$, with $a < b$ modulo T, we have either $x < a$ or $b < x$. In the first case, $a, b \in S \setminus (C_a)^-$ and $x \notin S \setminus (C_a)^-$. In the second, $a, b \in C_b$ and $x \notin C_b$. To conclude, it suffices to apply Lemma 6.1. □

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